Mathematical methods and its applications
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Lecture – 02
Linear dependence, independence and Wronskian of functions

Hello friends. Welcome to the lecture on the Linear Dependence, Independence and Wronskian of Functions. First of all we will define what do we mean by homogeneous and non-homogeneous linear differential equations. The nth order linear differential equation is given by a naught x into y n x, y n x is the nth order of derivative of y with respect to x, plus a 1 x by n minus 1 x and so on.

A n x into y x equal to r x, where a i x, i equal to 0 1 2 and so on up to n are continuous functions and an interval i, and the coefficient of y n x which is a naught x is assumed to be naught 0 for every x in i. So, this equation the nth order linear differential equation given in 1 will be called a homogenous linear differential equation provided r x is a identically 0.

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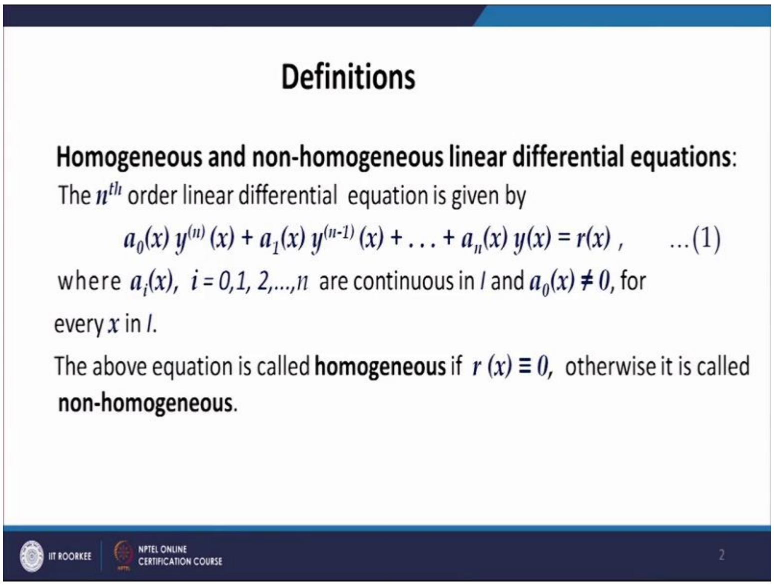

And otherwise it will be called a non homogenous linear differential equation. Regarding the homogenous linear differential equation, there is very nice theorem which we call as the superposition principle for homogenous functions.

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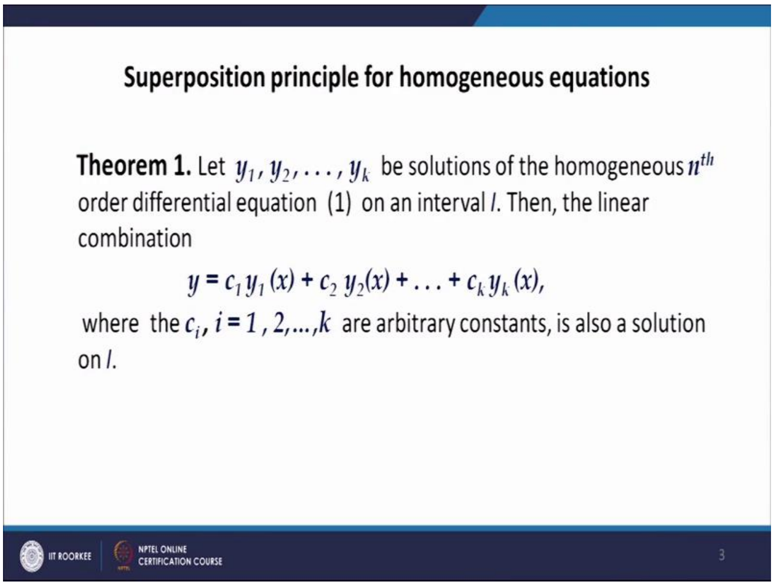

It says that let y 1, y 2, y k be k a solutions of the homogenous nth order differential equation one; we are considering homogenous equation here which means that we are taking r x to be identically 0.

So, in the case of homogenous linear differential equation given by 1, if we have k solutions y 1, y 2, y k then their linear combination; linear combination means the function c 1 into y 1 x plus c 2 into y 2 x and so on c k into y k x, they are c 1, c 2, c k are arbitrary constants is also a solution on i, it is very simple we can easily prove this.

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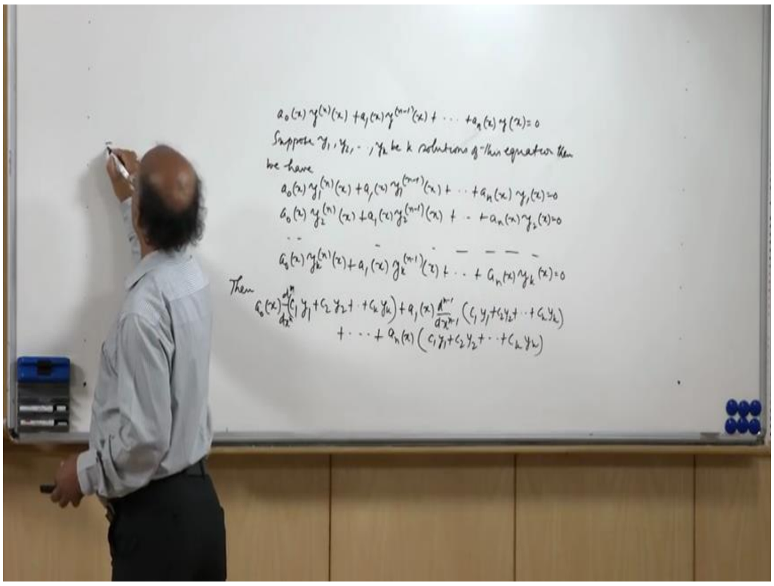

So, suppose let us see how we can prove this our homogenous differential equation is a naught x into y n x, plus a 1 x into y n minus 1 x and so on; a naught a n x into y x equal to 0, this is our homogenous linear differential equation. So, let suppose y 1, y 2 and so on y k be k a solution of this equation, this equation then we will have a naught x, into y 1 n x, plus a 1 x y 1 n minus 1 x and so on a n x into y 1 x equal to 0. Similarly for the function y 2 x we will have as we can write in a similar manner the kth equation a naught x, y k n x plus a 1 x into y k n minus 1 x and so on, a n x, y k n, y k x equal to 0. From these k equations then we can show that c 1 y 1 plus c 2 y 2 and so on c k y k is also solution of this homogenous equation.

So, then a naught x into c 1, y 1 plus c 2, y 2 and so on c k y k the kth derivative of this that is d k; no we have here nth derivative. So, nth derivative of this a naught x into nth derivative of c 1, y 1 plus c 2 y 2 and so on c k y k plus a 1 x into n minus oneth derivative of c 1, y 1 plus c 2, y 2 and so on c k, y k. And the last term a n x into c 1, y 1 plus c 2, y 2 and so on c k, y k; this is equal to a naught x.

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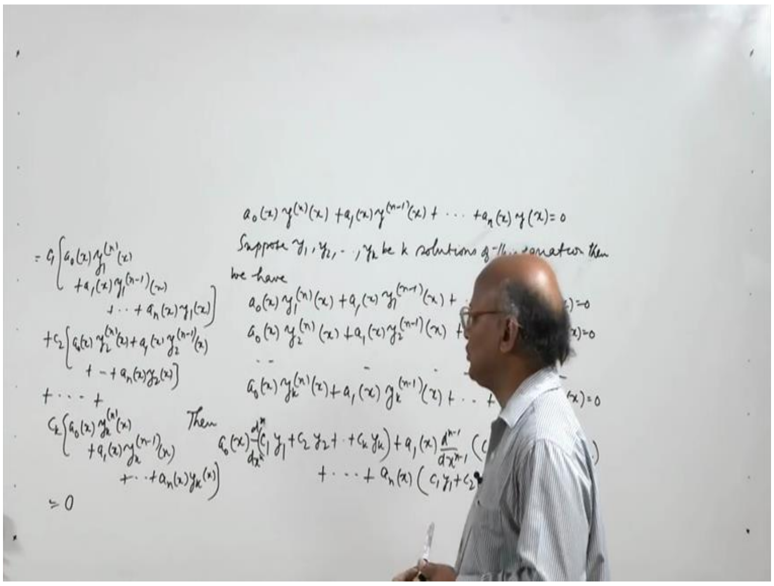

Now another derivative of c 1, y 1 plus c 2, y 2 plus c k, y k will be c 1 times nth derivative of y 1, plus c 2 times nth derivative of y 2, plus c k times nth derivative of y k similarly here n minus 1 of the derivative of this linear combination will be c 1 times n minus derivative of y 1, plus c 2 times n minus 1 derivative of y 2 and so on c k times n minus oneth derivative of y k.

So, we can write it as equal to c 1 times a naught x, n minus nth derivative of y 1, this plus c 2 times a naught x, into n nth derivative of y 2, plus a 1 x n minus oneth derivative of y 2 and so on; a n x, y 2 x and so on we can write c k times a naught x, y k n x plus a 1 x, y k n minus 1 x and so on a n x, y k x. Now since y 1 y 2 y k are solutions of the given equation homogenous equation, so this is 0, this is 0 and this is 0 and therefore, we have 0. So, c 1, y 1 plus c 2, y 2 and so on c k, y k is also a solution of the homogenous linear differential equation; so this known as the superposition principle for homogenous equations.

Now, let us define linear dependence independence of functions, it is a very important concept in the solution of differential equations.

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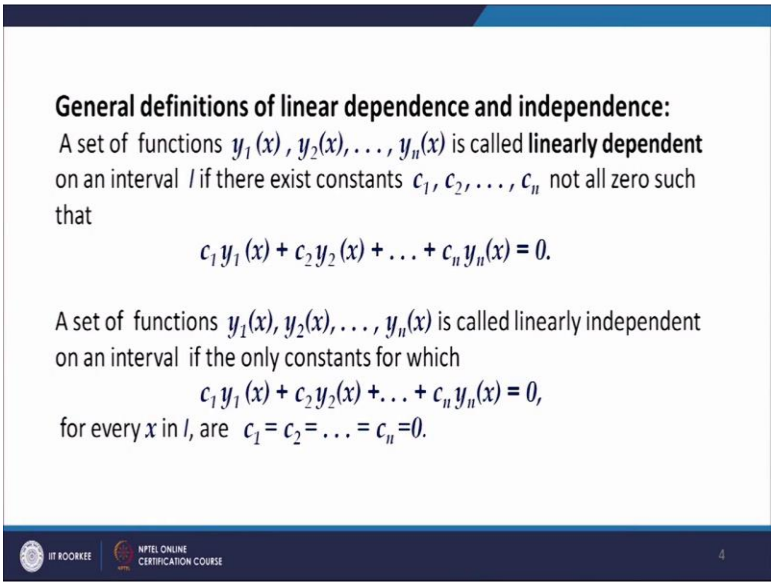

So, a set of functions y 1, y 2, y n will be called linearly dependent on an interval i, if we can find some constants c 1, c 2, c n not all 0; that means, at least one constant here must be a non-zero constant such that c 1, y 1 plus c 2, y 2 and so on c n, y n equal to 0. So, if the n functions are not linearly dependent then they will be called linearly independent. And for that we have the definition that c 1, y 1 plus c 2, y 2 and so on c n, y n is equal to 0 will always imply that c 1 equal to 0, c 2 equal to 0 and so on c l, c n equal to 0; that means, we cannot find any set of n constant c 1, c 2, c n not all 0 such that c 1, y 1 plus c 2, y 2 plus c n, y n equal to 0, which further which in other words it means that the n functions y 1, y 2, y n will be called linearly independent if an if c 1 whenever c 1, y 1 plus c 2, y 2 plus c n, y n is 0, it will always imply that c 1, c 2, c n all are zeroes.

Now, let us see how we can check the linear dependence independence of functions. From the definition of linear dependence it follows that you take n equal to 2 here; that means, if you take 2 functions y 1, x and y 2, x then they will be linearly dependent provided c 1, y 1 plus c 2, y 2 equal to 0. There both c 1 and c 2 where c 1, c 2 both are not 0; that means, at least one of them is non 0. So, suppose c 1 is nonzero then c 1, y 1 plus c 2, y 2 equal to 0 can be written as.

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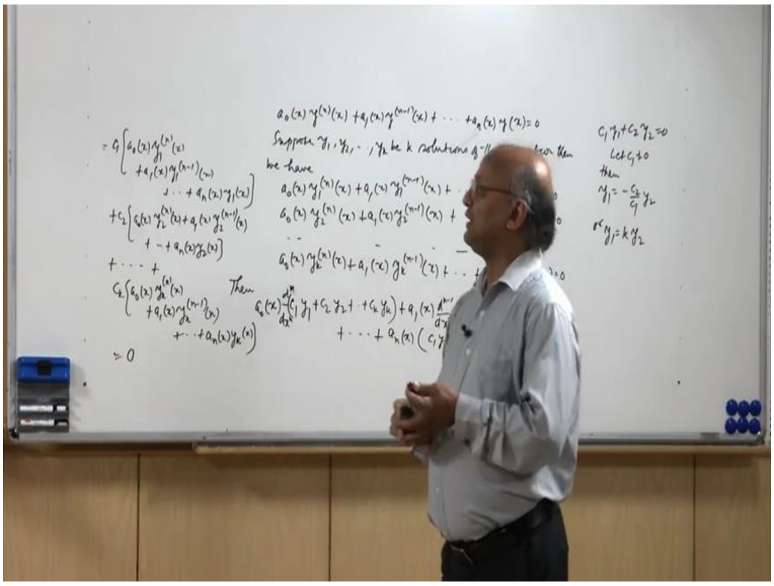

So, let c 1 be non-zero then we can divide y c 1 and write y 1 equal to minus c 2 by c 1, into y 2. So, c 2 by c 1 minus c 2 by c 1 is a constant so we can write r y 1 equal to some constant k times by 2, which will mean that y 1 is a scalar multiple of y 2.

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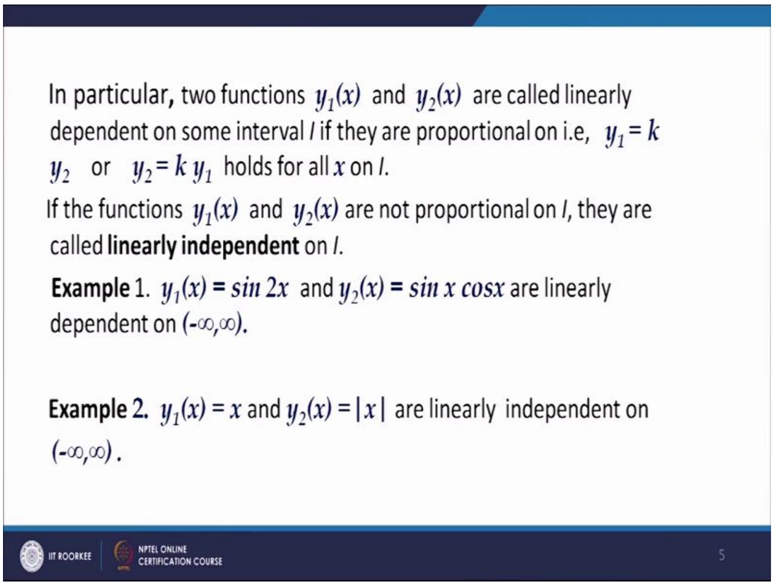

In particular the 2 functions let us say y 1 and y 2, they will be called linearly dependent on some interval i if they are proportional. That is y 1 is k times y 2 or y 2 is equal to k times y 1 because when we take n equal to 2 in this definition, then if y 1 and y 2 are linearly dependent then c 1, y 1 plus c 2, y 2 will be equal to 0 where c 1 and c 2 are not both zeros.

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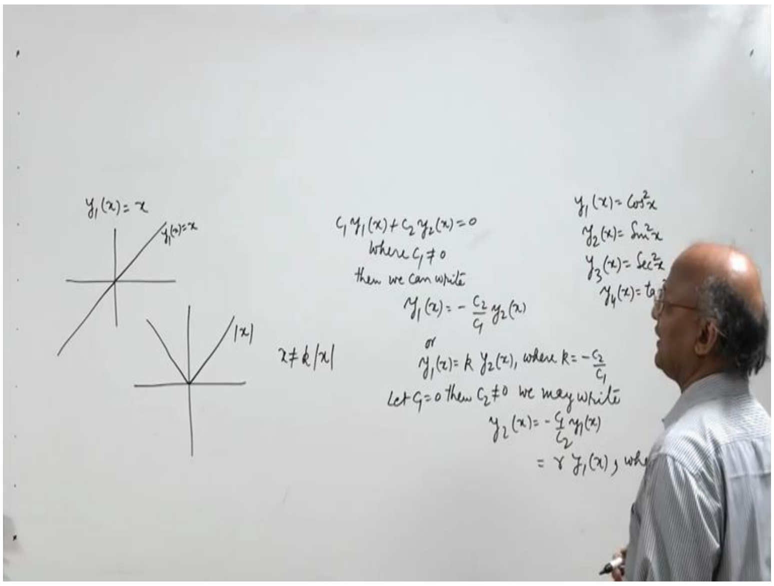

So, suppose c 1, y 1 x plus c 2, y 2 x equal to 0; where let us say c 1 is not equal to 0, then we can write y 1 x equal to minus c 2 over c 1 into y 2 x. Minus c 2 over c 1 is some constant. So, we can write or y 1 x equal to some constant k times y 2 x, where k is equal to minus c 2 over c 1. So, y 1 is a constant multiple y 2, if instead of c 1 let us say c 2 is equal to 0, c 2 is not equal to 0. Let us say c 1 is equal to 0 then c 2 must be not equal to 0. So, if c 2 is not equal to 0, then we can divide this equation by c 2, and write we may write y 2 is equal to minus c 1 by c 2 into y 1 x. So, this will be some constant let us say some constant r, so r into y 1 x, where r is equal to minus c 1 by c 2. So, in either case y 1 and y 2 will be proportional to each other; one will be a constant multiple of the other.

Now, if y 1, y 2 are not proportional on i then they will be linearly independent on i. For example, if you take the function y 1 x equal to sin 2 x, and y 2 x equal to sin x into cos x then we know that sin 2 x is 2 sin x cos x. So, we can write y 1 x to be equal to 2 times y 2 x, which means that y 1 is a scalar multiple of y 2 and so y 1 and y 2 will be linearly dependent on the interval minus infinity. Now in example 2; suppose we take the function y 1 x equal to x and y 2 x equal to mod of x, then we can see from the graph that they are not linearly dependent on the interval minus infinity to infinity. Y 1 x is equal to x the graph of y 1 x equal to x is this line, this is y 1 x is equal to x and the graph of y 2 x equal to mod of x, this is y mod of x graph. So, we can see that x is not a scalar multiple of mod of x. If x is a suppose if x is equal to mod of x then x is not a equal to some constant times mod of x, because mod of x is equal to x when x is positive, and minus x when x is negative. So, x and mod of x cannot be equal for some constant k.

Therefore, they are both linearly independent on the interval minus infinity to infinity. Now we take the case of say 4 functions y 1 x equal to cos square x, y 2 x equal to sin square x, y 3 x equal to sec square x, and y 4 equal to tan square x, we can see that they are linearly independent on the interval minus pi by 2 to pi by 2.

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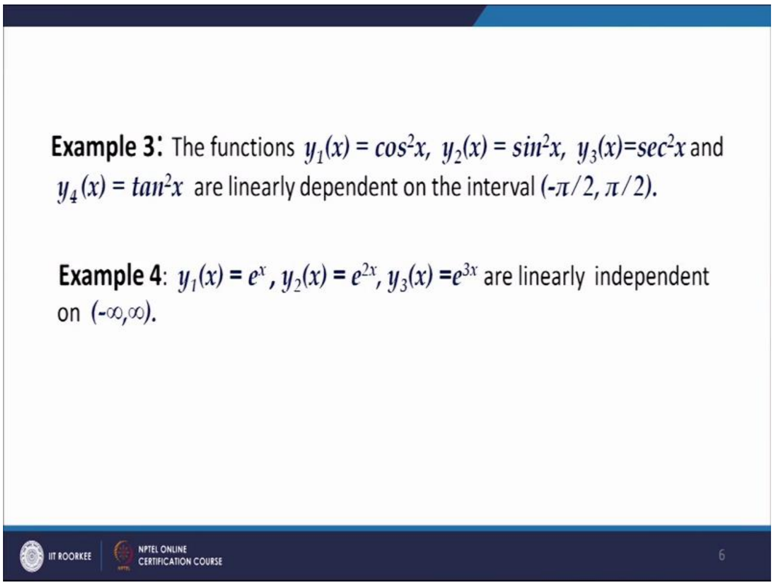

So, this interval is very important; here because you can see that this is sec square x is 1 over cos square x. So, if you go beyond minus pi by 2 to pi by 2, then what will happen is that sec square x is 1 over cos square x, it will be 0 at pi by 2 and minus pi by 2. So, Sec Square will not be defined. So, we can find ourselves and more over in tan square x will not be defined at pi by 2 n minus pi by 2. So, we will consider the interval minus pi by 2 pi by 2.

Now, here let us see we have y 1 equal to cos square x, y 2 x equal sin square x, y 3 x equal to sec square x, and y 4 x equal to tan square x. We have to show that they are linearly dependent on the interval minus pi by 2 to pi by 2. So, what we do is we have to find our constant c 1, c 2, c 3, c 4 such that which are not all zeros such that c 1, y 1 plus c 2, y 2 plus c 3, y 3 and c 4, y 4 is equal to 0. So, we have to find constants c 1, c 2, c 3 and c 4 not all 0 such that c 1, y 1 plus c 2, y 2 plus c 3, y 3 plus c 4, y 4 is equal to 0.

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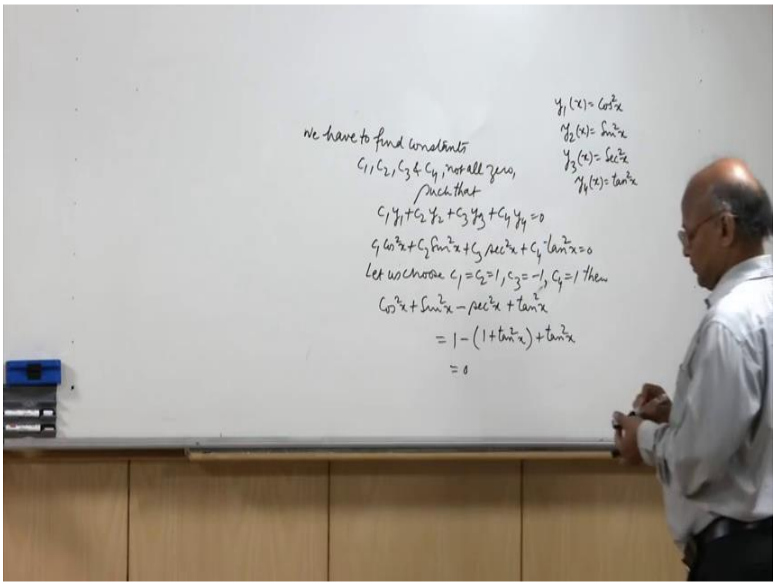

So, what we do is c 1, y 1; y 1 is cos square x. So, c 1 times cos square x then c 2 times in square x, then c 3 times sec square x, and then c 4 times tan square x. We know that sec square x is 1 plus tan square x. So, and we moreover we know that sin square x plus cos square x is 1. So, let us choose c 1, c 2 to be equal to 1. So, that we have here cos square x plus sin square x that is 1, and here we will have c 3 into 1 plus tan square x.

So, let us take c 3 to be minus 1. So, let us take c1, c 2 be equal to 1 c 3 equal to minus 1 and c 4 equal to 1; then we can see that cos square x, plus sin square x, minus sec square x plus tan square x; will be equal to this is 1 minus 1 plus tan square x, plus tan square x. So, this will be equal to 0. So, we are able to find the constants c1, c 2 c 3, c 4 not all zeroes such that c 1, y 1 plus c 2, y 2 plus c 3, y 3 plus c 4, y 4 is equal to 0. So, they are linearly dependent on the interval minus pi by 2 to pi by 2.

And the example 4: we consider y 1 x equal to e to the power x, y 2 x equal to e to the power 2, x y 3 x equal to e to the power 3 x. Let us show that they are linearly independent on the interval minus infinity. So, in order to prove that they are linearly independent, we have to show that whenever we write c 1, y 1 plus c 2, y 2 plus c 3, y 3 equal to 0 it will always imply that c1, c 2, c 3 all are zeroes.

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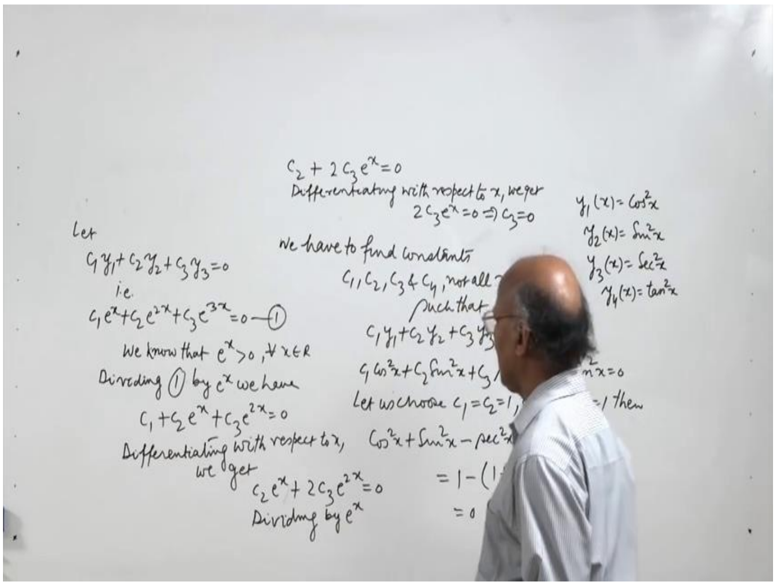

So, let us say let c 1, y 1 plus c 2, y 2 plus c 3, y 3 is equal to 0; that is c 1 e to the power x, plus c 2 e to the power 2 x, plus c 3 e to the power 3 x equal to 0. Our aim is to prove that c1, c 2, c 3 all are zeros. Now we know that e to the power x is always positive for every real x, we know that e to the power x is greater than 0 for all x belonging to r.

So, what we do is we divide this equation by e to the power x. So, dividing the equation 1 by e to the power x, we have c 1 plus c 2, e to the power x, plus c 3 e to the power 2 x equal to 0; again divide this equation by e to the power x. So, we have c 1 no, now what we will do is let us differentiate this equation, this equation holds for every x belonging to r. So, differentiating this equation with respect to x we have we get this is 0, c 2, e to the power x, plus c 3 into 2 times e to the power 2 x.

So, we get this. Now what we do is we again divide by e to the power x. So, dividing by e to the power x, we will get c 2 plus 2 c 3, e to the power x equal to 0, again this equation is valid for every x belonging to r. So, differentiating with respect to x, we get 2 c 3 e to the power x, equal to 0. Now e to the power x is always positive; so this means that c 3 equal to 0, when c 3 is 0 from here we get c 2 equal to 0 and then c 3 and c 2 equal to 0 give us c 1 equal to 0. So, we get c1, c2 c 3 all are zeros. So, e to the power x, e to the power 2 x, and e to the power 3 x, they are linearly independent functions on minus infinity to infinity interval.

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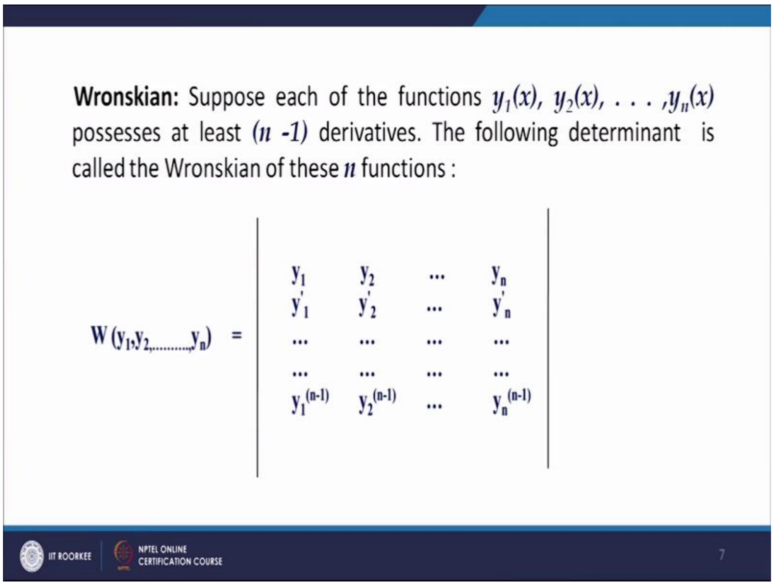

Now, let us consider the n functions. So, suppose we have n function y 1 x, y 2 x, y n x we have seen while doing these examples that it is not easy to check the linear dependence independence by the by finding the constant c1, c 2, c 3, c 4 and so on to. So, it is, but. So, there is another process by which we can check the linear dependence independence of n functions. So, if suppose we have n functions y 1, y 2, y n which possess say at least n minus 1 derivative with respect to x, then we will check the value of this determinant.

This is nth order determinant which we denote by W y 1, y 2, y n and call is at the Wronskian of these call is at the Wronskian of these n functions. So, in the first row we have y 1, y 2, y n second row has first order derivatives of y 1, y 2, y n and the last row has n minus oneth order derivatives of y 1, y 2, y n. We will see that this nth order determinant in the case of in other in the case of n function y 1, y 2, y n which are the solutions of the homogenous linear nth order differential equation, they will be linearly independent if and only if the Wronskian of this n functions is not equal to 0 on the interval i.

So, it is easy to check the value of the Wronskian of the n solution of the nth order linear differential equation to decide whether they are linearly dependent or independent; let us see how we prove this result.

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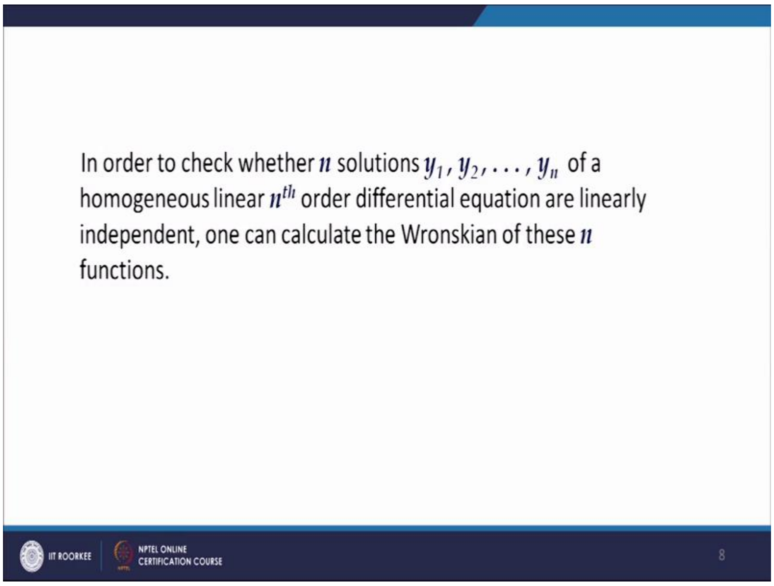

So, let us consider an initial value problem; here this is called initial value problem because we have a second order differential equation with 2 conditions y at x naught equal to y naught, y dash at x naught equal to y 1which are called the initial conditions.

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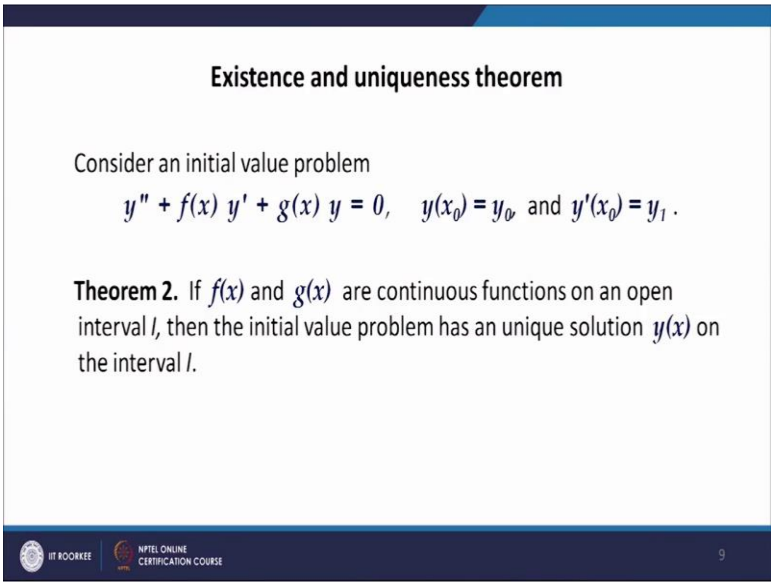

So, second order differential equation of with 2 initial conditions is called as a initial value problem. So, let us consider this initial value problem, if f x and g x are continuous functions here on an open interval I, then this initial value problem has in unique solution by x on the interval I, this is extended result. So, we shall make use of this result.

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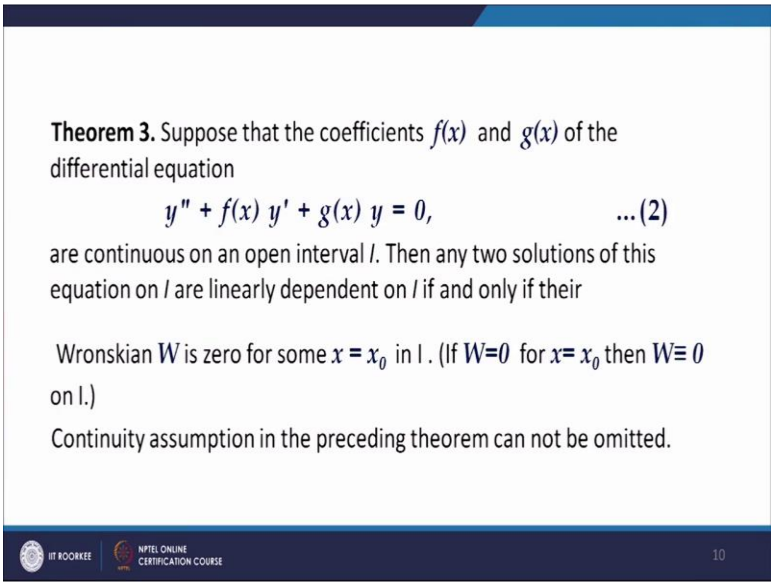

So, let us suppose that the coefficients effects in g x of this differential equation or continuous in an open interval I, then any 2 solutions of this equation or linearly dependent on the interval I if and only if. The Wronskian is 0 for some x equal to x naught; if W 0 for some x equal to x naught then w is identically 0 on an I. So, here this is a homogenous a linear differential equation, the 2 functions y 1 and y 2 let us say the 2 solutions y 1 and y 2 of this differential equation will be linearly dependent if and only if their Wronskian is 0 identically 0. So, this result can be extended to n functions we are proving it for n equal to 2.

So, in the case of nth order differential equation, the n solutions will be linearly dependent if and only if their Wronskian is 0. So, we will do it for simple for simplicity we will do it for n equal to 2. So, let us see how we prove this result. So, let us assume that.

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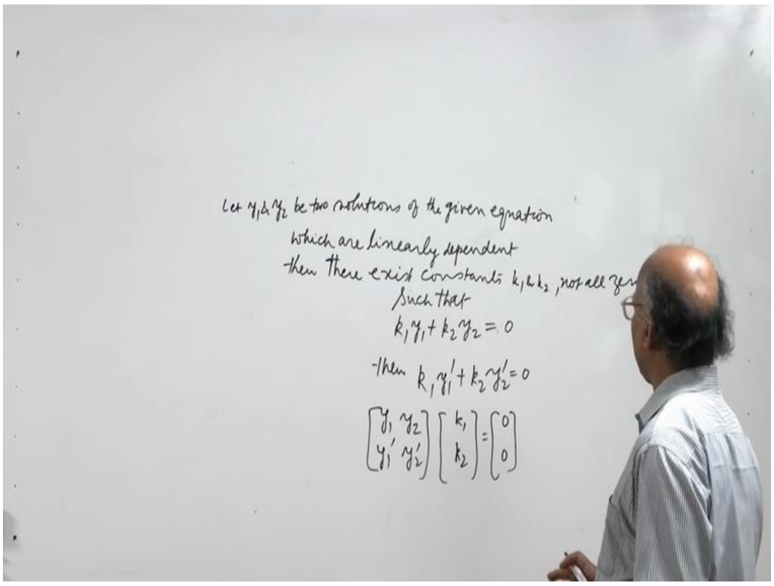

Let us assume that y 1 and y 2 be 2 solutions of the equation 2, which are linearly dependent. Let y 1 and y 2 be 2 dependent be 2 solutions of the given equation which are linearly dependent. Then why the definition of linear dependence then there exists constants k 1 and k 2 say not all 0, such that such that k 1, y 1 plus k 2, y 2 is equal to 0. So, by the definition of linear dependence we arrive at 2 constants k 1 and k 2, which are not all which are not both 0. So, that k 1, y 1 plus k 2, y 2 is the equal to 0. We have to prove that the Wronskian of the functions y 1 and y 2 is equal to 0, for some x equal to x naught in I. So, we can see from here that then k 1, y 1 dash, plus k 2, y 2 dash is equal to 0, differentiating this equation with respect to x we have k 1, y 1 dash, plus k 2, y 2 dash is equal to 0.

So, from these 2 equations what we have y 1, y 2, y 1 dash, y 2 dash, this coefficient matrix into k 1, k 2. So, this is a homogenous system of linear equations where k 1, k 2 are not both 0 therefore, determinant of this matrix must be 0.

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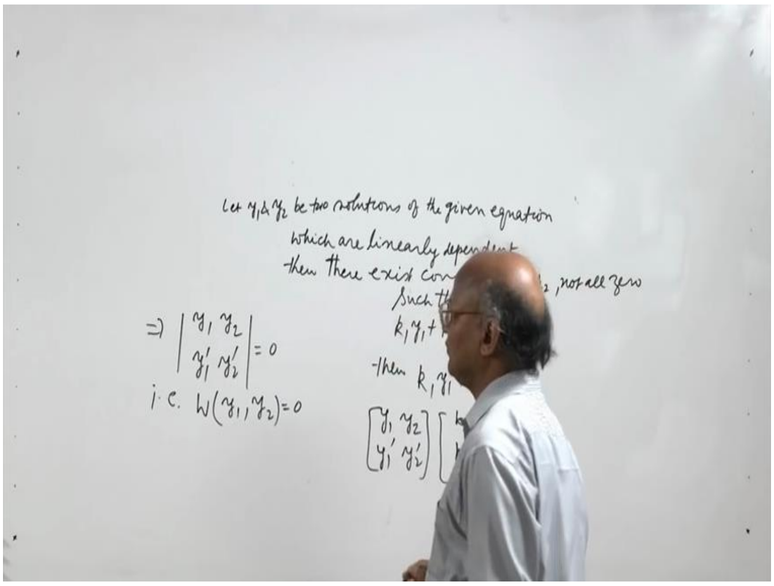

So, these imply that modulus over this y 1, y 2, y 1 dash, y 2 dash must be equal to 0. So, the determinant of the functions y 1, y 2, y 1 dash, y 2 dash is equal to 0, that is the Wronskian of the function y 1, y 2 is equal to 0. So, this proves that whenever the 2 functions y d 2 solutions y 1, y 2 of the equation 2 are linearly dependent, then their Wronskian is 0. So, it is 0 for some x naught.

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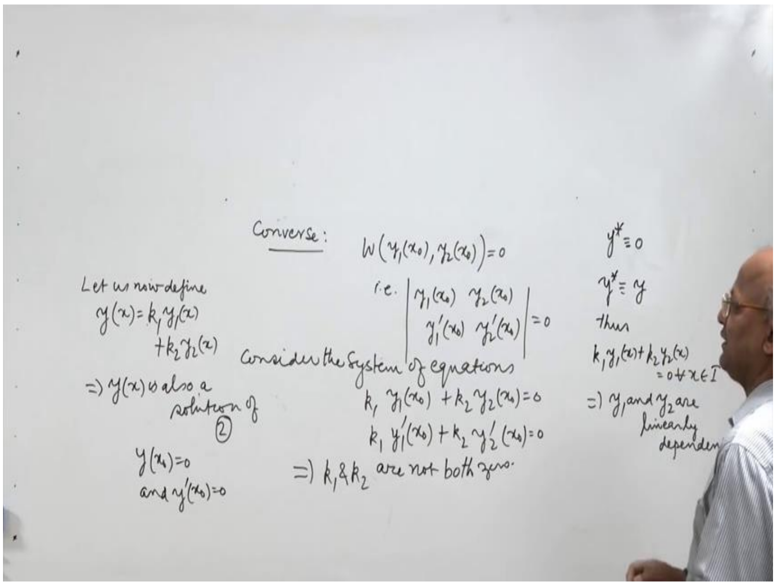

Now, let us see we can prove the converse. So, for the converse let us assume that. So, let us assume that the Wronskian W is 0 for some x equal to x naught in i. So, W x naught W y 1 x naught, y 2 x naught is equal to 0, that is y 1 x naught, y 2 x naught, y 1 dash x naught, y 2 dash x naught is equal to 0. We are given that the 2 solutions y 1 and y 2 of the equation 2 are such that the Wronskian w is 0 for some x equal to x naught. So, from that we have this determinant equal to 0; now let us consider the system of equations, k times y 1 x naught, plus k 2 times y 2 x naught, let us consider this system of equations. So, here we see since the determinant of k 1 x naught, since the determinant of y 1 x naught, y 2 x naught, y 1 dash x naught, y 2 dash x naught is equal to 0; this homogenous system of linear equations implies that k 1 and k 2 are not both 0. So, this implies that k 1 and k 2 are not both 0.

And now then let us define y x equal to k times k 1 times y 1 x, plus k 2 times y 2 x. Since y 1 and y 2 are solution of the equation 2, and equation 2 is homogenous equation by the super cheap position principle, it follows that k 1, y 1 plus k 2, y 2 is also a solution of equation 2. So, y x is also a solution of equation 2. Now from these equations it follows that y at x naught is equal to 0, if you put x naught here k 1, y 1 x naught, plus k 2, y 2 x naught is 0. So, y at x naught is equal to 0 and y dash at x naught is also 0 y data y data y dash at x is equal to k 1, y 1 dash x, plus k 2, y 2 dash x, when you put x equal to x naught, we fee and use this equation we get y dash at x naught equal to 0. So, now, what do we see that y x is a solution of equation 2, and also y at x naught equal to 0 y dash at x naught is equal to 0.

Now, another solution of equation 2 is y star equal to 0 solution; that is 0 solution satisfies the equation 2, and also y dash is equal to 0 satisfies y dash at x naught equal to 0, y dash y star dash at x naught equal to 0. So, this solution satisfies equation 2 along with the 2, conditions that at x naught it is 0 and its derivative is also that are x naught. Now let us apply the theorem uniqueness theorem adjutancy uniqueness theorem; this theorem tells us that in when we have this homogenous equation or we have this initial value problem where y at x naught is y naught, y dash at x naught is y 1, then the solution is unique, the initial value problem has unique solution. So, the both the solutions y x and this solution must be identical and therefore, y star must be equal to y. So, we can say that y must be identically 0. So, k 1, y 1 x, plus k 2, y 2 x is equal to 0 for all x belonging to I. So, by the uniqueness theorem it follows that k 1, y 1 plus k 2, y 2 is equal to 0 for all x belonging to I, and k 1, k 2 are not both 0 this implies that y 1 and y 2 are linearly dependent y 1 and y 2 are linearly dependent.

Now, y 1 and y 2; so y 1 and y 2 are linearly dependent, so whenever w Wronskian w is 0 for some x equal to x naught, it we have proved that y 1 and y 2 are linearly dependent. Now let us apply the proof of the first part, in the proof of the first part we have shown that whenever 2 solutions are linearly dependent then W is identically 0. So, here y 1 and y 2 are linearly dependent, now use the proof of a first part to say that W is 0, W is identically 0 on I.

So now, use now using the proof of the necessity part we have W is identically 0. So, this theorem tells us that whenever 2 solutions of equation 2 are linearly dependent, then W is identically 0 on I. Here the continuity assumption in this theorem cannot be omitted, let us see why the continuity junction cannot be dropped here this is evident from this example. Just take y 1 x equal to x cube for all real x and y 2 x equal to x cube when x is greater than or equal to 0 and minus x cube when x is less than 0.

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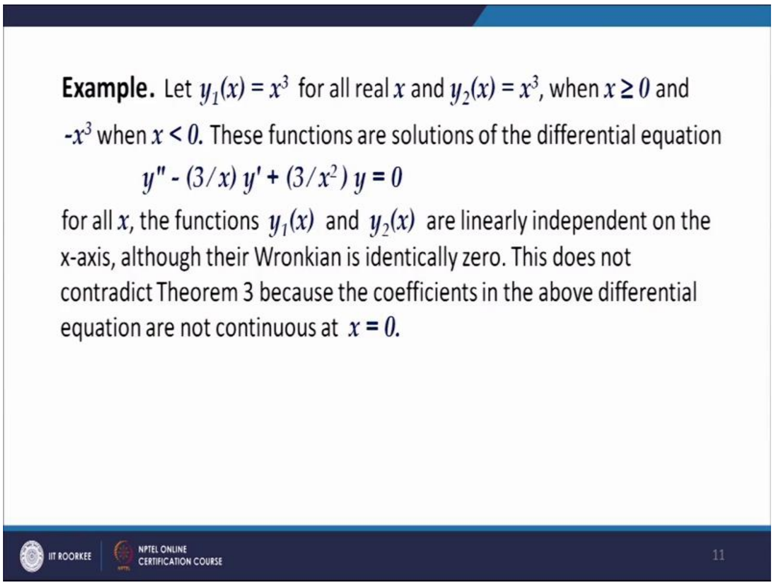

You can easily check that y 1 x equal to x cube satisfies this equation for all real x, and y 2 x equal to x cube when x greater than or equal to 0, and minus x cube when x less than 0; they also satisfy this second order differential equation for all x. So, both y 1 and y 2 are now moreover we can see that y 1 x is x cube, for ordial x while y 2 x is x cube when x is greater than or equal to 0, minus x cube when x is less than 0.

So, they are both linearly independent on the x axis one is not a scalar multiple of the other. Although their Wronskian identically 0, we can easily check that Wronskian of y 1 and y 2 is equal to is identically 0. So, this does not contradict with theorem 3; here we are seeing that although the function y 1, y 2 are linearly independent their Wronskian turns out to be identically 0. This step this example does not contradict theorem 3 because the coefficients in this equation f x is minus 3 y x, and g x is 3 y x square they are not continuous at x equal to 0. So, because of the fact that they are not continuous at x equal to 0, it does not contradict theorem 3.

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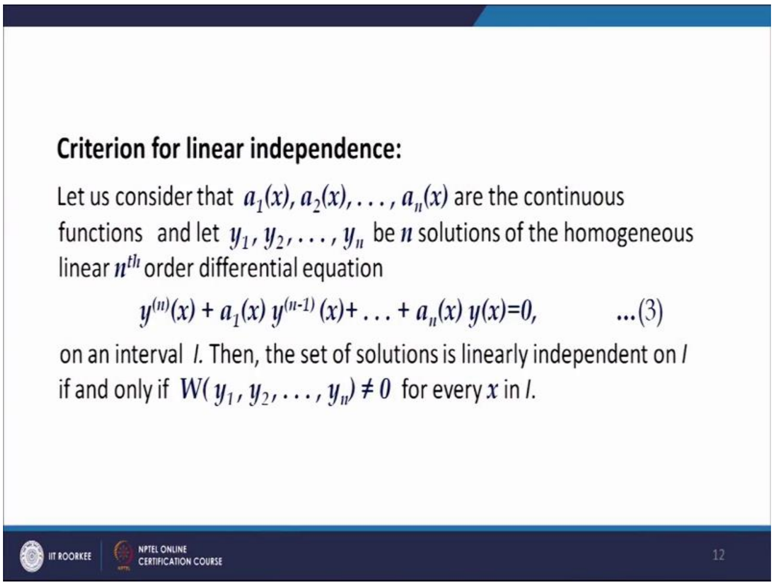

Now, when suppose we consider the nth order linear homogenous linear differential equation, we are a 1 x, a n x are continuous functions then the then the result which we proved for n equal to 2; this result can be generalized to n functions. So, we have the set of solutions n solutions y 1, y 2, y n is linearly independent I and I if and only if W y 1, y 2, y n is not equal to 0 for every x in I.

So, that can be generalized to this nth order homogenous linear differential equation. Now a set y 1, y 2, y n of n linearly independent solutions of the homogenous nth order differential equation on an interval I is called a fundamental set of solutions, and it can be shown that there exists a fundamental set of solutions for the homogenous linear nth order differential equation on an interval I .

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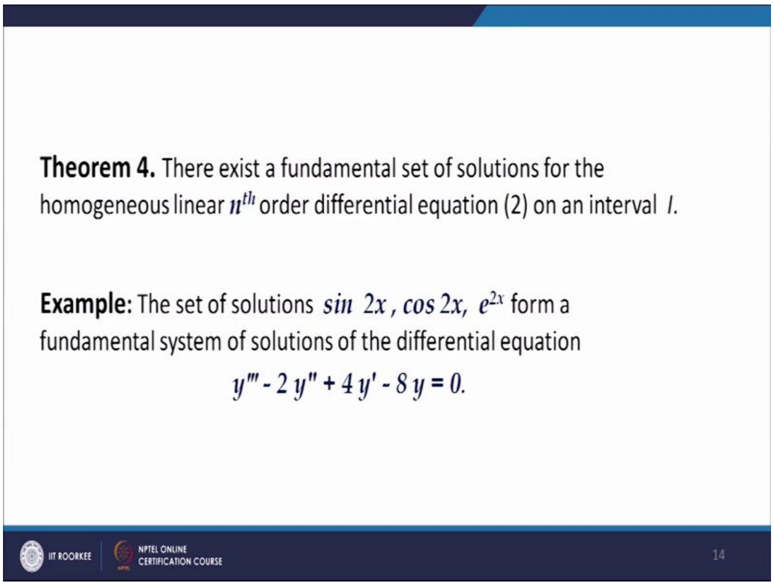

For as an example we can consider the set of solutions sin 2 x, cos 2, x e to the power 2 x these 3 functions we can easily check that, they are solutions of this third order homogenous linear differential equation, and they form a fundamental set. Fundamental set means we have to show that the 3 functions are linearly independent, which we can easily show by using the Wronskian.

So, let us consider the Wronskian of these 3 functions, and show that these 3 functions are linearly independent and so form a fundamental set.

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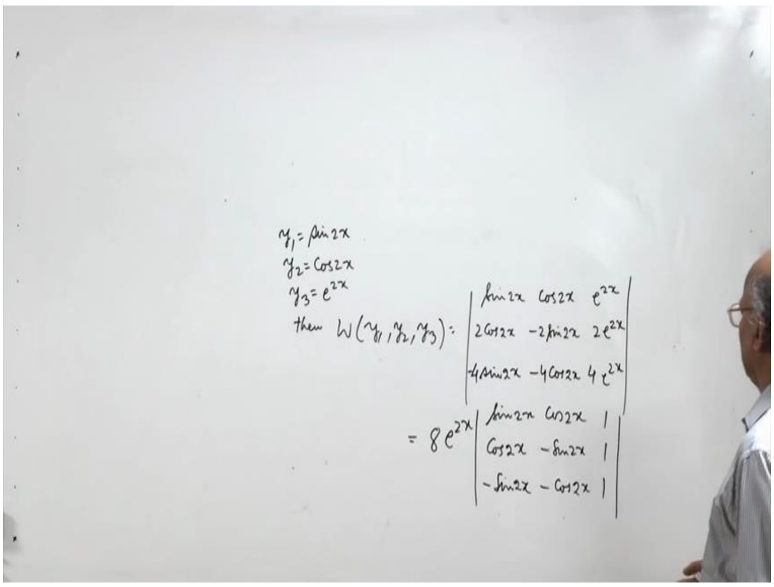

So, let us consider the Wronskian of these 3 functions. So, Wronskian so that the function y 1 be equal to sin 2 x, y 2 be equal to cos 2 x, y 3 x be equal to e to the power 2 x, then W y 1, y 2, y 3 this is equal to e to the sin 2 x, cos 2 x, e to the power 2 x. Next row contains the derivatives. So, 2 cos 2 x, this will be minus 2 sin 2 x, then here we will have 2 e to the power 2 x; and here second order derivatives of each of these functions. So, we have 4 minus 4 sin 2 x, and then here we have cos 2 x into minus 4 cos 2 x, here we will have 4 e to the power 2 x.

Now e to the power 2 x is strictly positive. So, we can take it for 2 x out of the last column. So, e to the power 2 x and then we can take common from e last column and then we take 2 and 4 from the first and second and third row. So, I will have here 8 times, what I have done is I have taken e to the power 2 x common from the third column, and 2 and 4 from the second and third rows. So, 8 times e to the power 2 x.

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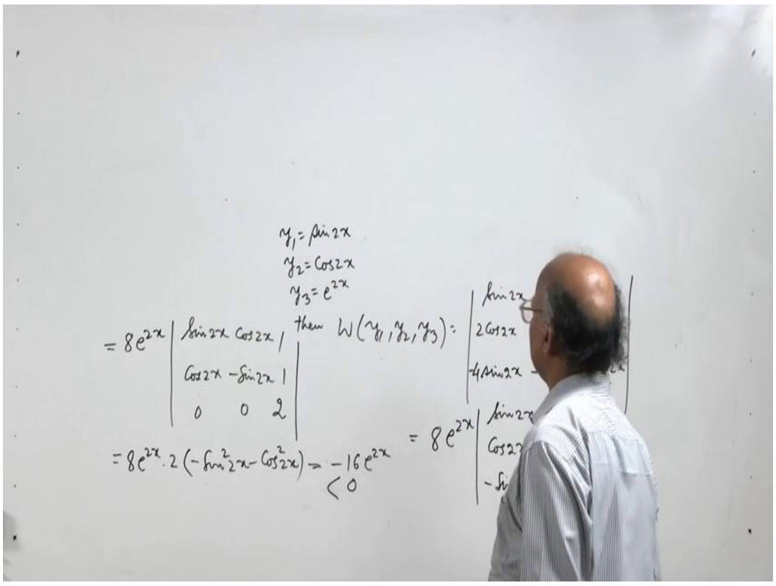

Now, let us add first row to the third row. So, we will get here 0, 0 and we will get here 2. Now let us expand it by the last row. So, we will have here eight e to the power 2 x into 2 times, minus sin square 2 x minus cos square 2 x, which is minus 1. So, we get minus 16 e to the power 2 x, which is in fact less than 0. So, the Jacobean is not 0 and therefore, the 3 functions are linearly independent and so form a fundamental set of solutions of this third order differential equation.

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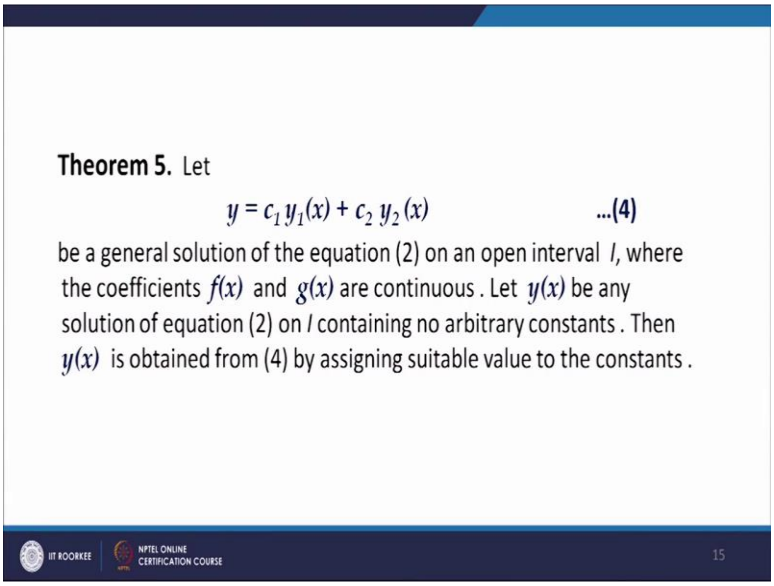

Now, this last theorem tells us that if you take y equal to c 1, y 1 x, plus c 2, y 2 x and it be a general solution of the homogenous, equation 2 this then it includes all solutions of equation 2. So, let us see how we get this. Let y equal to c 1, y 1 plus c 2, y 2 be a general solution of equation 2 on an interval I where the coefficients f x and g x are continuous, and y be a any solution of equation 2 containing no arbitrary constants, then this y is obtained from 4 by assigning a suitable value to the constants. So, this general solution includes all solutions of the equation 2. So, this is what is conveyed by this theorem with this I would like to conclude my lecture.

Thank you very much for your attention.