Mathematical Methods and its Applications
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Lecture – 19
Solution of first order non – linear equation – I

Hello friends, welcome to my lecture on solution of first order non-linear equation. So, this is our first lecture on the how to find the solution of non-linear equations that are of first degree, that are of first order, but not of first degree.

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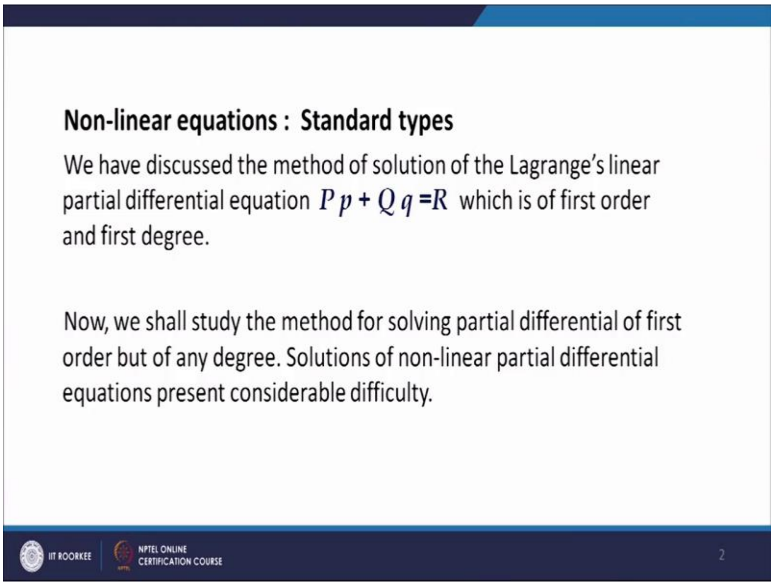

So, let us see we have so far discussed how to determine the solution of a Lagrange's linear partial differential equation which is of the form P into p plus Q into q equal to R, where P, Q and R as you know they are functions of x, y, z. So, this is a first order and first degree linear partial differential equation I mean partial differential equation which is linear in P and Q. Now, we shall be considering first order partial differential equations, which are not of first degree. So, if there are not of first degree then they will be non-linear partial differential equations. So, let us see how we solve first order non-linear partial differential equations and such solution of non-linear partial differential equations present considerable difficulties.

So, first we will focus on before giving the general method that is the Charpit method of solving such equations, we shall present some special types of equations whose solutions can be easily determined. So, many equations we shall see can be reduced to few standard forms and therefore can be solved generally which by methods which are shorter than Charpit method. So, let us first discuss the equations of the form f p. q equal to 0, such partial differential equations where the partial derivatives p and q that is delta z by delta x and delta z by delta y occur. And they do not involve the variables x, y and z they consider they come under this category of the equations of the form f p, q equal to 0.

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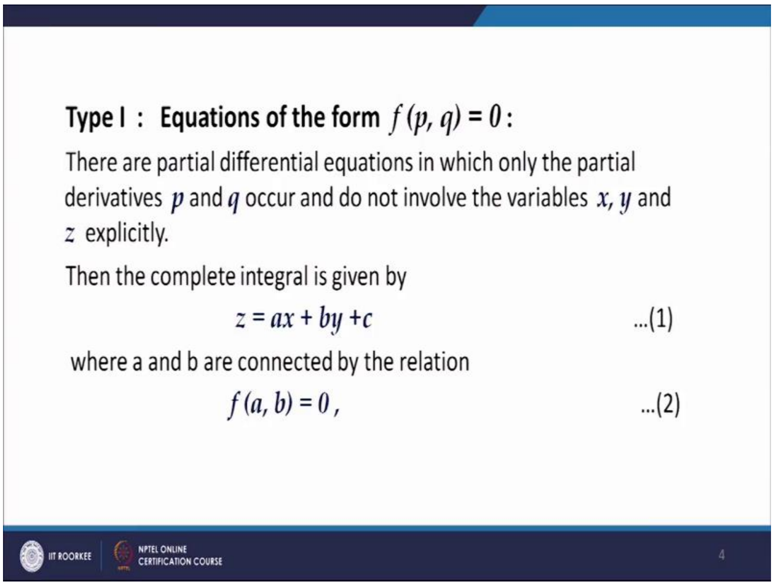

So, in the case of these equations, f p, q equal to 0, we see that the complete integral is given by z equal to a x plus b y plus c. Now, I want to remind you that the complete integral is defined as the solution of the differential equation, which contains as many arbitrary constants as there are independent variables in the differential equations. So, there are two independent variables here x and y. So, this is a solution of the partial differential equation f p, q equal to 0, because here a and b are connected by the relation f a, b equal to 0. If you find the partial derivatives of this equation z equal to a x plus b y plus c with respect to x and y, then what we get is the derivative of z with respect to x which is p, we have denoted by p comes out to be a.

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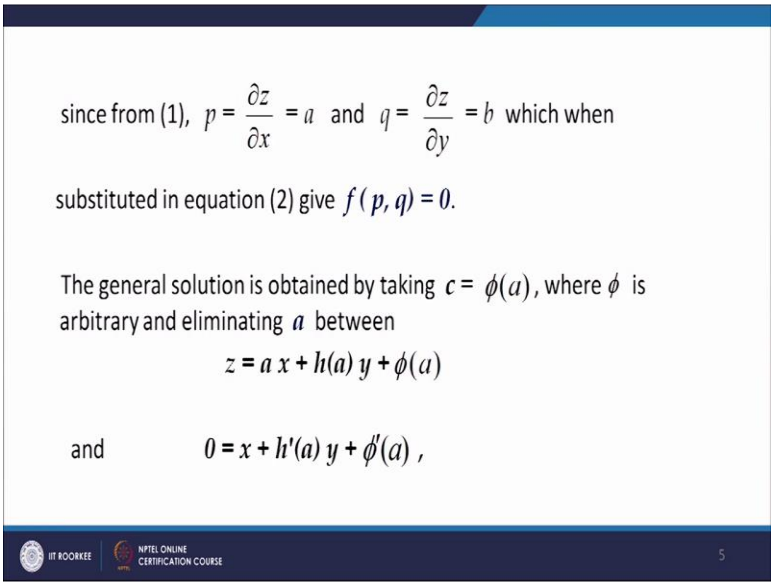

And the derivative of z with respect to y which we have denoted by q comes out to be b, which when substituted in equation 2 that is f a, b equal to 0 give us f p, q equal to 0. So, z equal to a x plus b y plus c is a solution of f p q equal to 0, and moreover it involves two arbitrary constants they are a and c; b is not an arbitrary constant, b is connected to a by the relation f a, b equal to 0. So, b can be said as the function of a. So, there are two arbitrary constants a and c. And because of these two arbitrary constants and this is a solution of the equation f p, q equal to 0 which contains two independent variables, it can be called as these complete integral of f p, q equal to 0.

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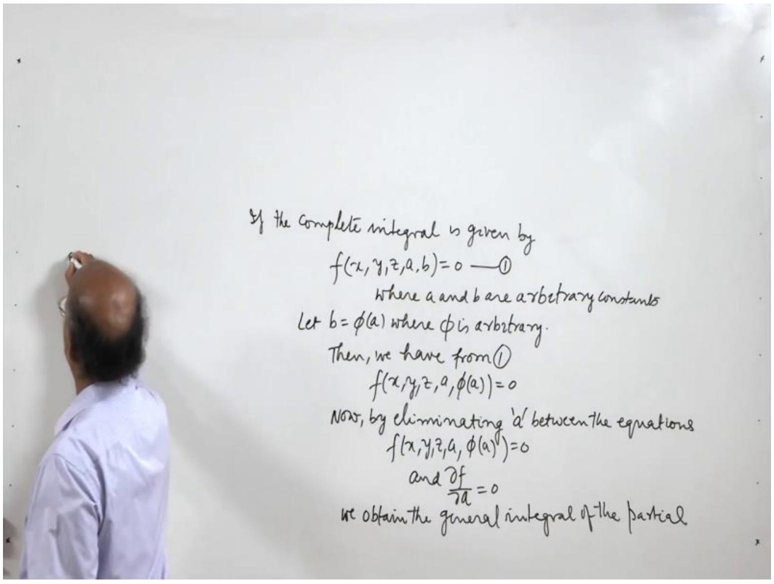

Now, if you remember two from the complete integral one can determine the general solution. If the complete integral is given by f x, y, z, a, b equal to 0 where a and b are arbitrary constants, then to determine the general integral what we do is let us say let b be an arbitrary function of a where phi is arbitrary. So, then we have the equation one then we have from 1 f x, y, z, a, phi a equal to 0. Now, by eliminating a the arbitrary constant a between the equations f x, y, z, a, phi a equal to 0; and the partial derivative of f with respect to a equal to 0. Between these two equations, we obtained the general integral of the partial differential equation of first order differential equation of f of f x, y, z, p, q equal to 0. So, this we have known already.

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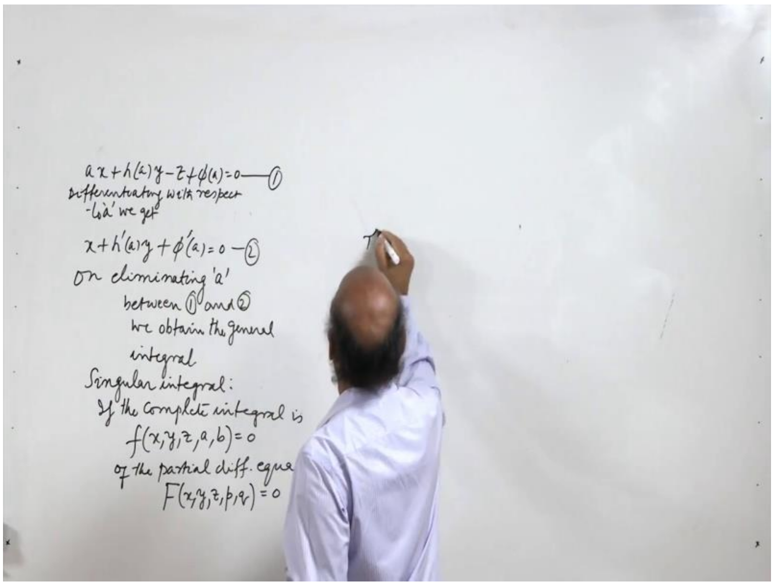

So, in order to find the general solution from the complete integral, the complete integral is z equal to a x plus b y plus c; f a, b equal to 0 gives you since f a, b is equal to 0, we can say we may take b as a function of a. And the arbitrary constant c and c is equal to phi a, where phi is an arbitrary function. So, then z will be equal to a x plus h by a h a into y plus phi a, so that we have the equation a x plus h a into y minus z plus phi a equal to 0 thus we have this equation. So, this equation is same as the equation f x, y, z, a, phi a equal to 0, we differentiate it with respect to a. So, differentiating it, we get x plus h dash a into y plus phi dash a equal to 0. So, we then on eliminating a between this equation let me call it as 1 and this as 2. So, when we eliminate a, on eliminating a between 1 and 2, we get the general solution, the general integral.

So, this how we will obtain the general integral from the complete integral we have already said that a partial differential equation is said to be completely solved if we get the complete integral, general integral and the similar solution of the partial differential equation. So, we have obtained the complete integral, general integral we can obtain from here. And let us now discuss how we can obtain the complete integral. Now, if you how to obtain the singular integral, to obtain the singular integral from the complete integral let us recall that if the complete integral is f x, y, z, a, b equal to 0. If the complete integral is this of the partial differential equation f x, y, z p q equal to 0 then the complete integral is obtained. Then the similar integral is obtained by eliminating a and b between the equations f x, y, z, a, b equal to 0 delta f over delta a equal to 0 and delta f over delta b equal to 0.

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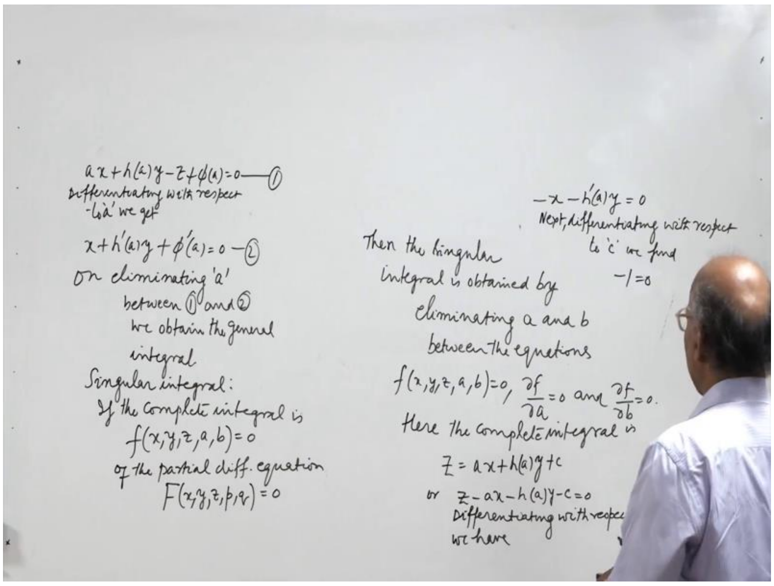

So, by eliminating the two arbitrary constants a and b from these three equations we arrive at the singular integral. Now, in this case what happens is let us look at the complete integral the complete integral is z equal to, so here the complete integral is z equal to a x plus b y, b is equal to h a, so h into a into y plus c. So, there are two arbitrary constants a and c, we can write it in the form f x b y z is equal to 0. So, are z minus a x minus h a into y minus c equal to 0. So, this equation is of the form f x, y, z, a, b equal to 0, here the arbitrary constants are a and c.

Now, let us differentiate these with respect to a. So, differentiating with respect to a, we have so minus x minus h dash a into y equal to 0. And when we differentiate this with respect to c, next differentiating with respect to the other arbitrary constant c, we find minus 1 equal to 0. So, we have to know eliminate the two arbitrary constants a and c between this equation this equation and this equation, now minus 1 equal to 0 is not possible. So, therefore, there is no singular integral in this case. So, in this case of the partial differential equation, we have only complete integral and general integral.

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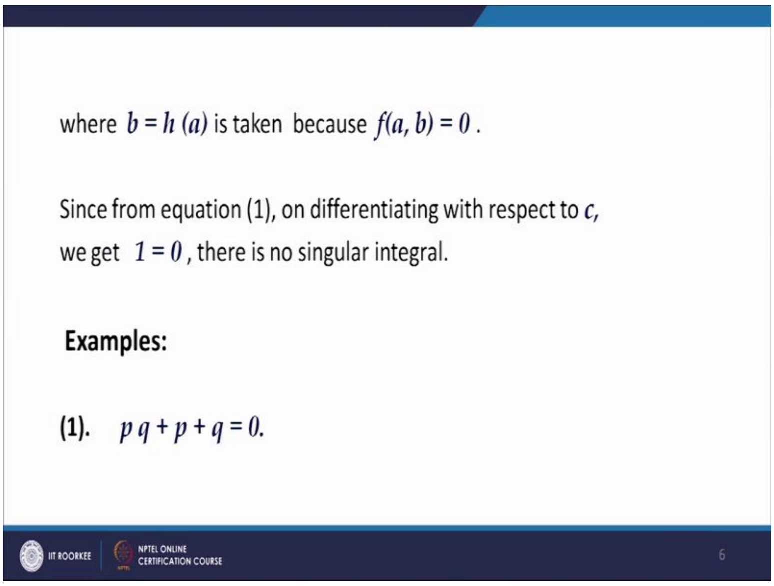

Now, let us go the example 1, p q plus p plus q equal to 0.

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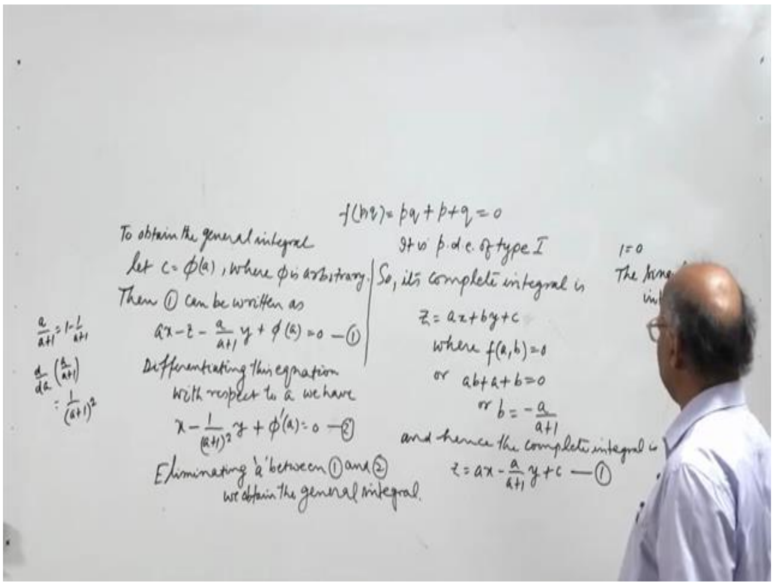

So, let us consider p q plus p plus q equal to 0. Now, we can see that this differential equation is a non-linear differential equation because the partial derivatives p and q they occur as a product in the first term. So, it is a first order differential equation, but it is not linear in p and q. So, it end more over there only here p and q occur, x, y, z do not occur. So, it is of the type one it is a partial differential equation of type one. So, it is complete integral could be written as z equal to a x plus b y plus c, where if I write it is as f p, q equal to p q plus p plus q then here f a, b equal to 0. So, f a b equal to 0 or you can say a b plus a plus b equal to 0. So, from here I can get the value of the b equal to minus a upon a plus 1. And hence the complete integral is z equal to a x minus a over a plus 1 into y plus c. From the complete integral we can obtain the general integral.

So, to obtain the general integral, let us take c to b some arbitrary function of a, where phi is arbitrary. Now, then we can say then one can be written as a x minus z minus a over a plus 1 into y plus phi a equal to 0. And differentiating this with respect to a, we have x minus now a over a plus 1 is nothing but 1 minus 1 upon a plus 1. So, when you differentiate a over a plus 1 with respect to a, what you get is d over d a of a over a plus 1, this gives you 1 over a plus 1 whole square. So, here we shall have minus 1 upon a plus 1 whole square into y plus phi dash a equal to 0. So, let me call it as equation number 1, and this is equation number 2, eliminating a between 1 and 2, we obtain the general integral.

Now, to obtain the singular integral as we did said earlier a and c are two arbitrary constants in this complete integral. So, we shall differentiate it partially with respect to a and with respect to c; and when we differentiate this equation with respect to c what we will have is 1 equal to 0. If we write it as a x minus a over a plus 1 by plus c minus z equal to 0, and then differentiate it with respect to c, we obtain 1 equal to 0 and 1 equal to 0 is not possible, so this is the singular solution are singular integral does not exist.

Again I repeat we have the complete integral z equal to a x minus a over a plus 1 y plus c. So, you can write it as a x minus a over a plus 1 y plus c minus z equal to 0. Then we have to differentiate partially with respect to a and with respect to c, when we differentiated with respect to c, we get 1 equal to 0, which is not possible. So, this singular solution does not exist in this case.

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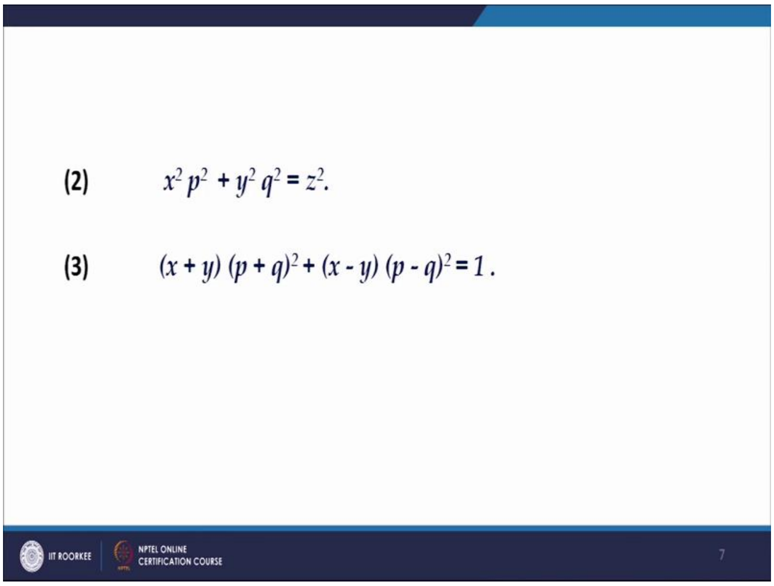

Now, let us go over to next question, where we will see that by suitable substitutions, we shall be able to bring this differential equation partial differential equation of first order, which is non-linear to the form f p, q equal to 0.

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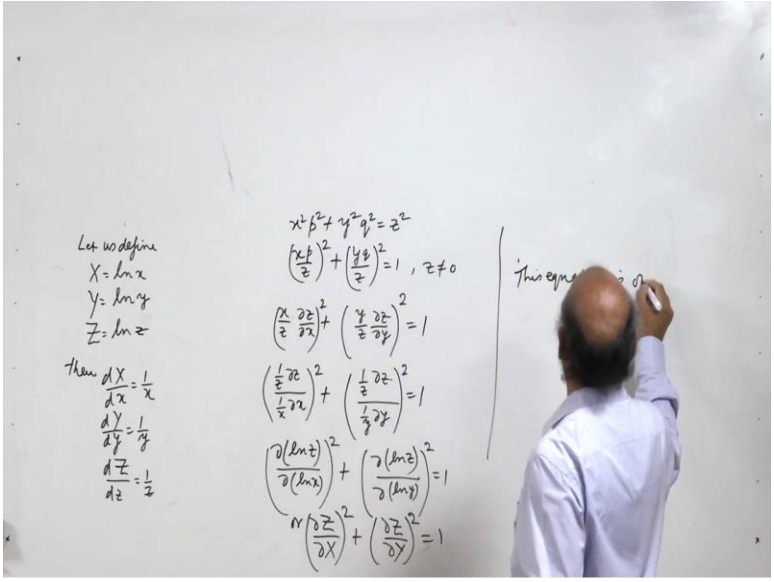

So, let us analyze this equation closely. We have x square p square plus y square q square equal to z square. Let us divide this equation by z square assuming that z is not equal to 0. So, we have x p over z whole square plus y q divided by z whole square equal to 1, if z is not 0. Now, this is what x over z p is delta z by delta x plus this is y over z, we can write like this. Now, this can also expressed as or 1 over z delta z divided by 1 by x delta x whole square plus 1 by z. So, let us define capital X equal to l n small x capital Y equal to l n small y and capital Z equal to l n small z this small z .

So, then d X over dx will be 1 over x, d Y over dy will be 1 by y, and dZ this capital Z this is big Z equal to 1 by z. So, this can be written as delta l n z, I can write like this delta l n x whole square plus delta l n z over delta l n y whole square equal to 1 or we can say delta z over delta x whole square plus delta z over delta y whole square equal to 1. Now, this is of the type this equation is of the type one. So, it is complete integral is z equal to a x plus b y plus c.

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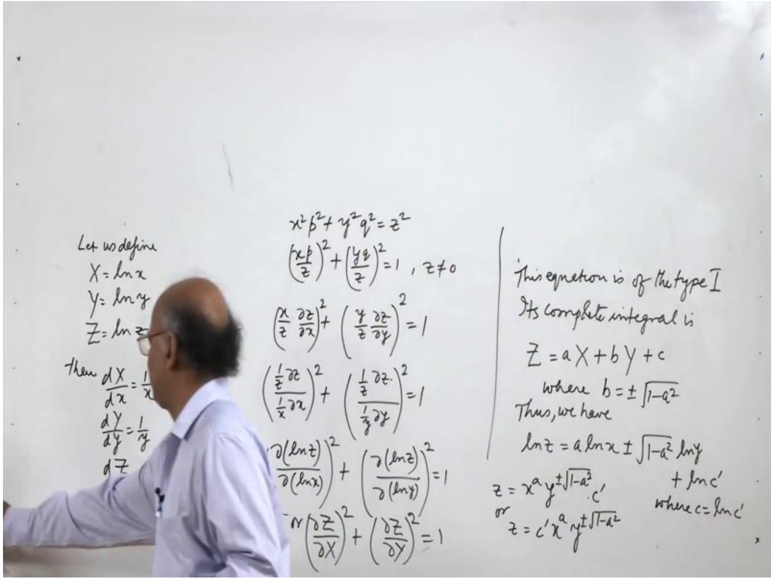

Let me write small c a x plus b y plus c, where we have, so where if delta z over delta x, this is a. So, then b square is equal to 1 minus a square, so where b is equal to plus minus under root 1 minus a square. So, thus we have this is l n z and then a times l n x plus minus under root 1 minus a square into l n by, and this c we can write as l n c dash, let us write c as where c is equal to l n c dash. So, then we can write z equal to x to the power a y to the power plus minus under root 1 minus a square into c dash. Let me write it as z equal to c dash times x to the power a y to the power plus minus under root 1 minus a square, where a and c dash are two arbitrary constants. So, this is complete integral of the given partial differential equation. Now, what we will do is to determine the general integral from this equation, we have to take c dash equal to phi into a.

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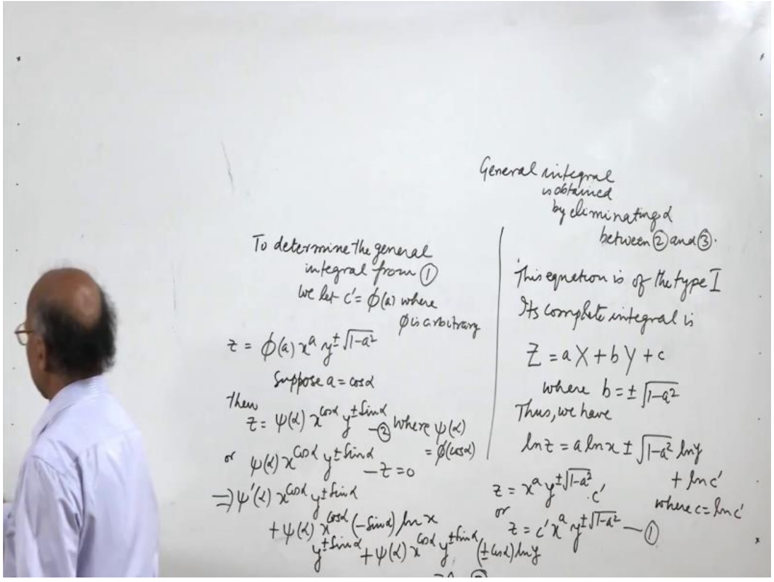

So, to determine the general integral from the complement integral from let me call it equation 1, from 1, we let c dash equal to some arbitrary function of a, where phi is arbitrary. So, we have phi a into x to the power a y to the power plus minus under root 1 minus a square, this is z, this is equal to z. Now, we can make it simpler by taking say a equal to suppose a equal to cos alpha, then we shall have z equal to phi of cos alpha. So, I can write it as some psi alpha, where psi alpha is phi of cos alpha x to the power cos alpha and then y to the power plus minus psi alpha, where psi alpha is phi of cos alpha.

Now, we differentiate this equation with respect to alpha. So, we shall have or we can say psi alpha minus x to the power cos alpha sorry psi alpha into x to the power cos alpha y to the power plus minus sign alpha minus z equal to 0. So, what we will do is let me call it as equation number 2, and then we differentiate this with respect to alpha. So, the psi dash alpha into x to the power cos alpha y to the power plus minus sign alpha plus psi alpha x to the power cos alpha when we differentiate x to the power cos alpha with respect to alpha, what we get is x to the power cos alpha. And then x to the power a, if we differentiate with respect to a to the power x, if we differentiate with respect to x, we get a to power x log a.

So, x to the power this, we are differentiate a minus sign alpha and then we get x to the power alpha, we are differentiating with respect to alpha. So, x to the power cos alpha into minus sign alpha into l n x we shall have y to the power plus minus sign alpha, and then psi alpha x to the power cos alpha y to the power plus minus sign alpha into plus minus cos alpha l n y minus equal to 0. So, this our equation number let me call it equation number 3. So, by eliminating alpha between equation number 2 or this equation and 3, we get the general integral. So, general integral is obtained. This singular integral is obtained by eliminating a and c dash or you can say by eliminating a and c dash between equation one and then its partial differentiation with respect to a and c dash will give you two more equations by eliminating a and c dash.

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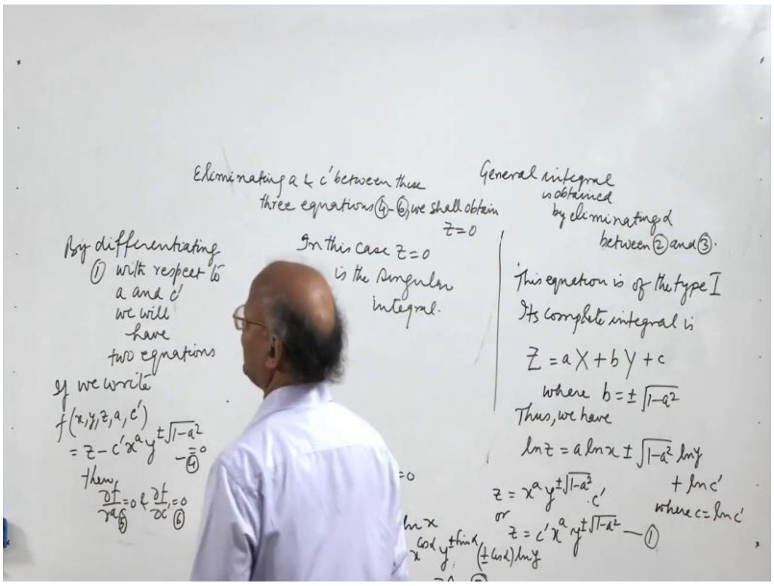

So, by differentiating equation 1 with respect to a and c dash, we will have two more equations, we have we will two more equations, we will have two equations. If we take it as, if we write f x, y, z a c dash equal to z minus c dash times x to the power a y to the power plus minus under root 1 minus a square then equal to 0. Then delta f by delta a equal to 0, and delta f by delta c equal to 0. So, if we write f x, y, z c dash to be equal to this then by differentiating this with respect to a and with respect to c dash, we shall have these two equations. So, eliminating a and c dash between these three equations, we shall see that three equations, I can call them as 4, 5, 6, 4 to 6, we shall have we shall obtain z equal to 0, therefore the singular integral.

So, in this case, z is equal to 0 is the singular integral. We can see that z equal to 0 satisfies this equation x square p square plus y square q square equal to z square, because z equal to 0 means p and q both are 0s. So, both sides are 0 0.

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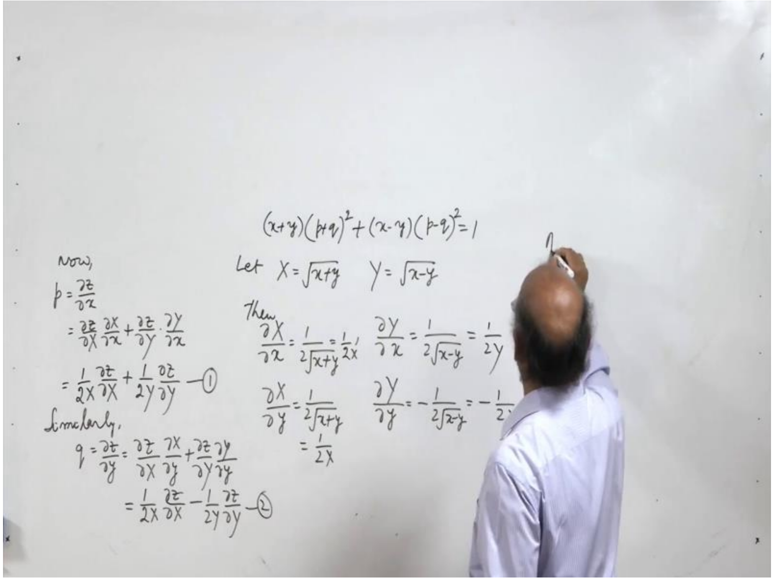

So, let us take third question where we have x plus y times p plus q whole square and then x minus y p minus q whole square. So, we shall make some suitable substitution and will reduce it to the form which we have considered in type one. So, let us define X equal to root x plus y and Y equal to root x minus y. Let us define capital X equal to this and capital Y equal to this. Then delta X over delta x is 1 over 2 root x plus y delta X over delta y is 1 over 2 root x plus y; and here when we differentiate y with respect to x partially we get 1 over 2 root x minus y. And here we get delta Y over delta y as minus 1 over 2 root x minus y.

Now, p is delta z over delta x. We can write it as delta z over delta X into delta x by delta x plus delta z over delta Y into delta y over delta x, because capital X capital Y both are functions of x. So, we have here delta x over delta x is 1 over 2 root x plus y is 1 over 2 x. So, 1 over 2 x and then y over delta x delta y delta x 1 over 2 root x minus y and 1 over 2 that x minus y is 1 over 2 y, this is 1 over 2 x. So, we have 1 over 2 y. And similarly, q which is delta z by delta y it can be written as yes and this is equal to delta x over delta y this is one over two x and delta y over do delta y is minus 1 over 2 y. So, we will have one over.

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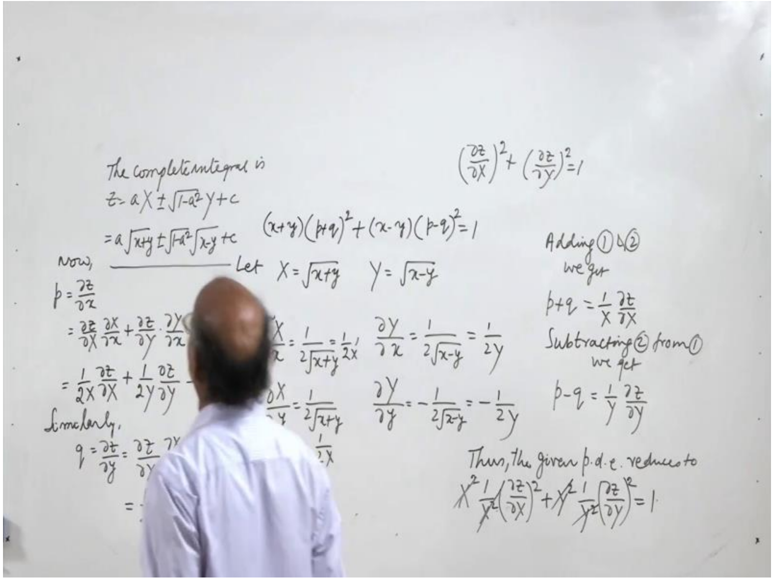

Now, let us add the equations 1 and 2. So, adding 1 and 2, we get p plus q equal to 1 over 2 x delta z by delta x we get, 1 over x. And p minus q similarly and subtracting 2 from 1, we get p minus q equal to 1 over y delta z over delta y. So, we will put the values here. So, x plus y, x plus y is x square. So, thus the given equation becomes partial differential equation reduces to we have x square into p plus q whole square. So, 1 over x square delta z by delta x square plus x minus y x minus y is y square into p minus q whole square. So, 1 over y square this equal to 1.

So, here we have x square. So, this cancel with this and this cancels with this. And we have, which is a partial differential equation of the form one where z is the dependent variable capital X, capital Y are the independent variables. So, it is of type one. So, the complete integral is z equal to a x plus minus under root 1 minus a square, this is capital X here, capital Y plus some constant let us say c. Let us put the value of capital X. So, a times square root x plus y plus minus under root 1 minus a square then y is a square root x minus y plus c. So, z equal to a times X square root x plus y plus minus square root 1 minus a square root x minus y plus c, this is a square plus b square equal to 1. So, b is plus minus under root 1 minus a square which we have put here. So, this is the complete integral.

Now, from the complete integral as we have seen on the earlier examples, for the general integral we will write c as phi a and then differentiate the equation with respect to a. And then eliminate a between this equation where c is replaced by phi a, and it is derived equation where we differentiate this equation with respect to a partially. And then in order to find and then we eliminate a, so to get the general integral. And when we want to find the singular integral we eliminate the two arbitrary constants a and c between this equation and it is derived equations which are the partial derivatives of this equation with respect to a and c.

So, by eliminating a and c between this equation, and those equations, we can obtain the singular integrals that the usual that is means by the usual method of obtaining the singular integral we can do this. So, once we have the complete integral, we can obtain the general integral and the singular integral for the given partial differential equation. So, this how we solve the problems which occur in the case f p, q equal to 0 that is of type one, where the differential equation only contains the partial derivatives p and q, the variables x, y, z do not occur. With that, I would like to conclude my lecture.

Thank you very much for your attention.