Mathematical Methods and its Applications
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Lecture – 18
Solution of Lagrange's equation – II

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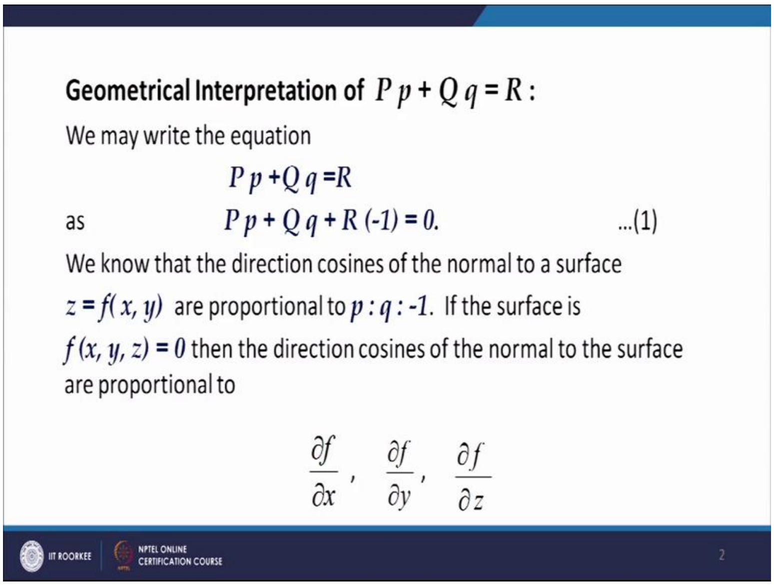

Hello friends, welcome to my second lecture on solution of Lagrange's equation. It is in continuation of my previous lecture on this topic. Here first we discuss the how geometrical interpretation of the Lagrange's linear equation, which is P into p plus Q into q equals R, where small p and small q represent the partial derivative subject with respect to x and y.

Now, we may write this equation P p plus Q q equal to R s, P p plus Q q plus R times minus 1 equal to 0. Now, we know that the direction cosines of the normal to a surface z is equal to f x y are given by delta z over delta x, delta z over delta y and minus 1. So, direction cosines of the normal to the surface z equal to f x y are proportional to p, q and minus 1. Now, if the surfaces f x, y, z equal to 0 then the direction cosines of the normal to the surface are proportional to its partial derivatives of f with respect to x by z. Because we know that del f is a vector normal to the surface f x, y, z equal to 0.

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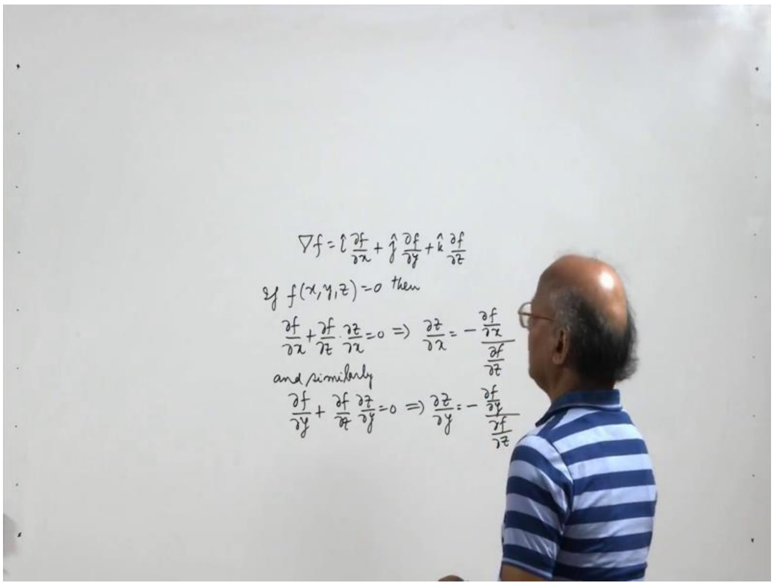

And del f is we know that del f is equal to i del f by del x plus j del f by del y plus k del f by del z. So, it is known that if the surface is given by f x by z equal to 0 then del f is a vector normal to the surface. So, the direction cosines of the normal to the surface are proportional to partial derivatives of f with respect to x, y, z that is del f over delta f over delta x delta f over delta y delta f over delta z.

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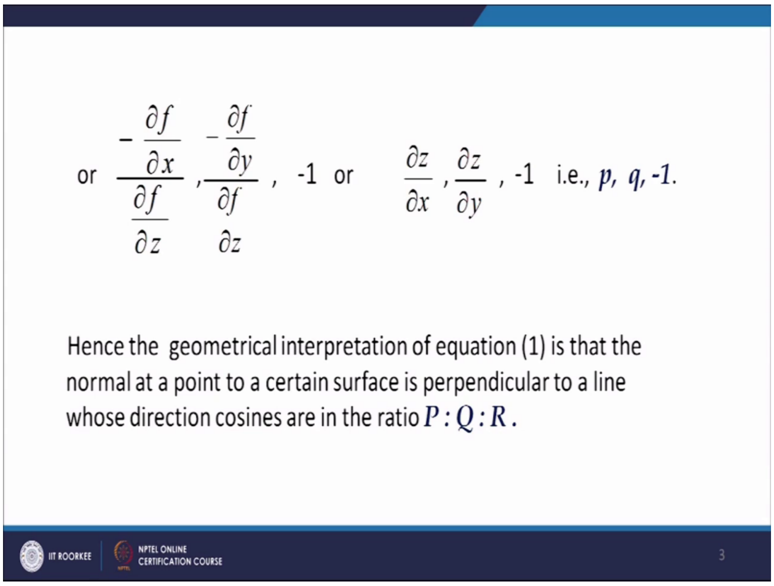

Now, or we can also say that this is there are proportional to minus delta f over delta x over delta f over delta z minus delta f over delta y over delta f over delta z and minus 1, or delta z over delta x delta z over delta y and minus 1. Because if f x, y, z equal to 0 then when we differentiate it with respect to x what we get is, so which implies that. And similarly, when we differentiate with respect to y, the equation f x, y, z equal to 0, we get. So, the ratio is minus delta f by delta x over delta f by delta z becomes delta z by delta x minus delta f over delta y over delta f by delta z becomes delta z over delta y. So, we can say that these ratios are same as the partial derivatives of z with respect to x and y. So, delta z over delta x delta z over delta y and minus 1 that is p q and minus 1.

Now, hence the geometrical interpretation of the equation 1 is that let us look at this P into p plus Q into q plus R into minus 1 equal to 0. Now, we are seeing that the direction cosines of the normal to the surface are proportional to p, q and minus 1. And therefore, we can geometrically interpret this equation 1 as the normal at a point to a certain surface is perpendicular to a line whose direction cosines are in the ratio P, Q and R, because we know that if the two lines are perpendicular to each other then the dot product of their direction ratio is equal to 0. So, we have normal to a point, at a point to a certain surface is perpendicular to join whose directional ratios are in the ratio P, Q and R.

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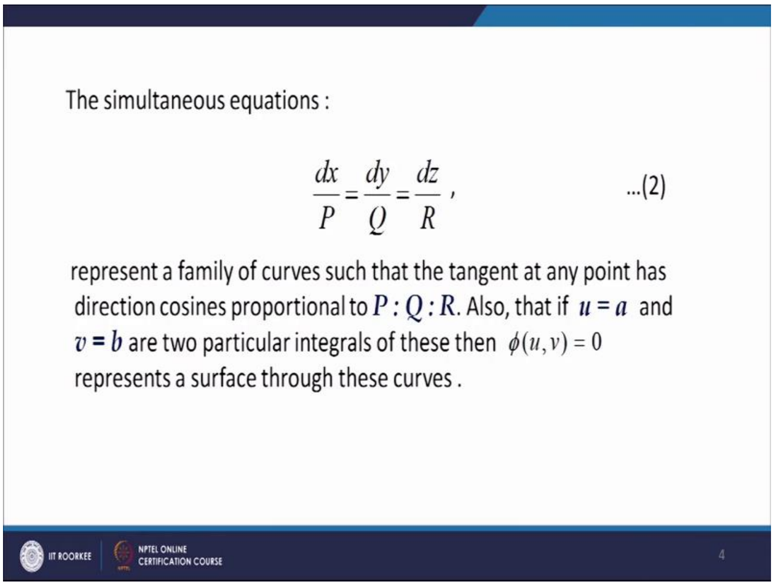

Now, let us look at the simultaneous equations dx by P equal to dy by Q equal to dz by R it represents the family of curves such that the tangent at any point has the direction ratios proportional to P, Q and R. Now, if u is equal to a, and v is equal to b are two particular integrals of these equations, then phi u, v equal to 0 represents a surface through these curves. Let us take a point P dash on the surface represented by phi u, v equal to 0 then through this point passes a curve of the family which lies entirely on the surface. The normal to the surface at the point P dash must therefore, be at right angles to the tangent at this point to the curve that is it is perpendicular to a line whose direction cosines are proportional to P, Q and R. And this is exactly what we require in the partial differential equation 1.

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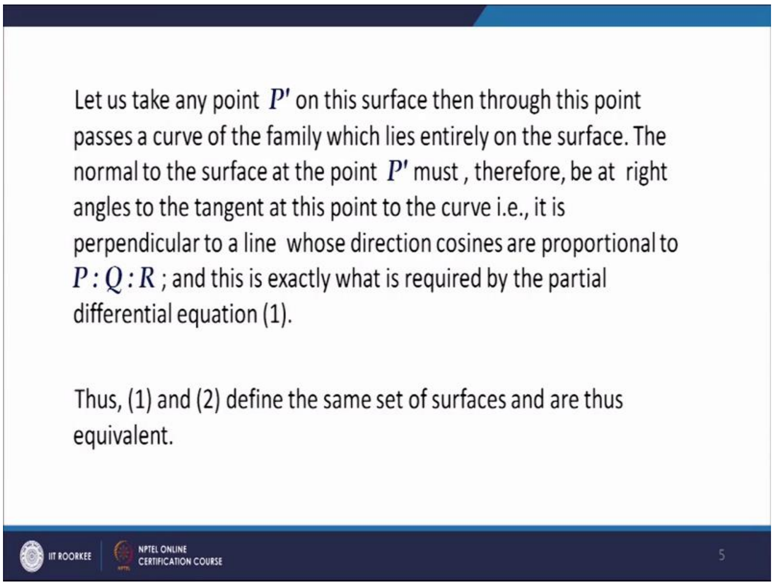

So, 1 and 2, this is 2, 1 and 2, define the same set of surfaces and are therefore, equivalent. So, if the solutions the solutions u and v of the equations subsidiary equations dx over P equal to dy over Q equal to dz over R can be obtained easily when the variables are separable.

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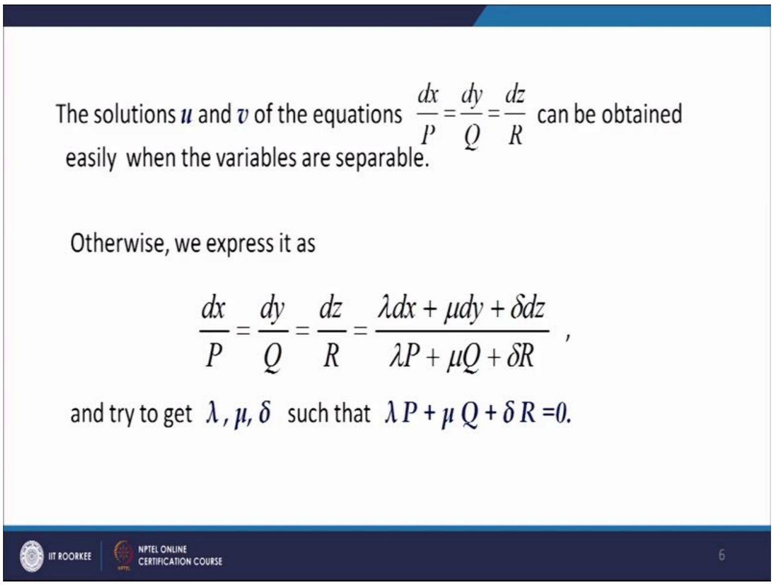

If the variables are not separable then we shall express it in the form dx over P equal to dy over q equal to dz over R equal to lambda dx plus mu dy plus delta dz over lambda P plus mu Q plus delta R, where lambda, mu, delta are certain functions of x, y, z such that lambda P plus mu Q plus delta R equal to 0. So, we search the functions lambda mu and delta in such a way that lambda into P plus mu Q plus delta R equal to 0. Now, if this equation one lambda P plus mu Q plus delta R equal to 0, we will have lambda dx plus mu dy plus dz equal to 0. Now, you on choosing lambda, mu, nu and lambda mu and delta in such a way that lambda P plus mu Q plus delta R equal to 0; from here it turns out that lambda dx plus mu dy plus delta dz equal to 0.

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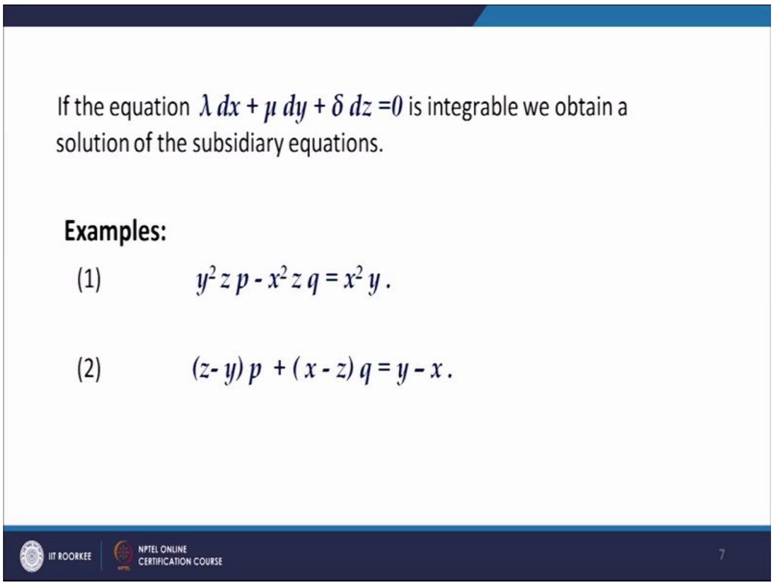

So, now if the equation lambda dx plus mu dy plus delta dz equal to 0 is integrable, we will obtain a solution of the subsidiary equations. Then we can make another choice of lambda mu delta and obtain another solution are from this solution using this solution we can obtain another independent solution of the subsidiary equations. So, how we will arrive at the general integral of a partial differential equation of first order and first degree using this Lagrange’s method let it is illustrated in the examples below. So, let us first take the example of y square z into p minus x square z into q equal to x square y.

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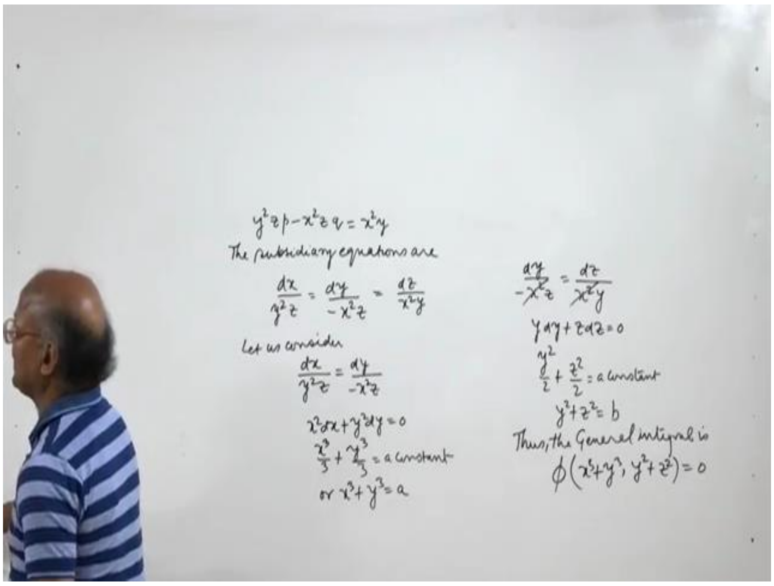

So, y square z into p minus x square z into q equal to x square y. So, the subsidiary equations are dx over p, p is y square z equal to dy over minus x square z that which is q and then R is x square y. So, dz upon x square y we will find two independent solutions u and v from the subsidy equations. So, let us first consider dx over y square z equal to dy over minus x square z. Now, here so this z and z can be cancelled out, and then we see that x square dx plus y square dy equal to 0. So, the variables x and y can are separated here. And we can integrate, this integrating this we get x cube by 3 plus y cube by 3 equal to a constant or we can say x cube and y cube equal to a, where a is an arbitrary constant.

Now, we can find another solution of this is u x, y, z equal to a, we can find v x, y, z equal to v from the other equations dy over minus x square z equal to dx upon x square y. Now, here we can cancel x square and we get y dy plus z dz equal to 0. So, again z dz equal to 0. So, again the variables are separable and therefore, here we can integrate and we will get y square by 2 plus z square by 2 equal to a constant. So, we can write y square plus z square as some constant, let us say b. So, this is b x, y, z equal to b. And thus the general integral is phi is an arbitrary function phi u, v equal to 0, so x cube plus y cube y square plus z square equal to 0. So, this is how we will find the general solution of the Lagrange's equation given in example 1. Here we have seen that it is easy to solve this subsidiary equation, because it is the case of case where the variables are separable, we can easily separate the variables x and y. So, it was easy to solve.

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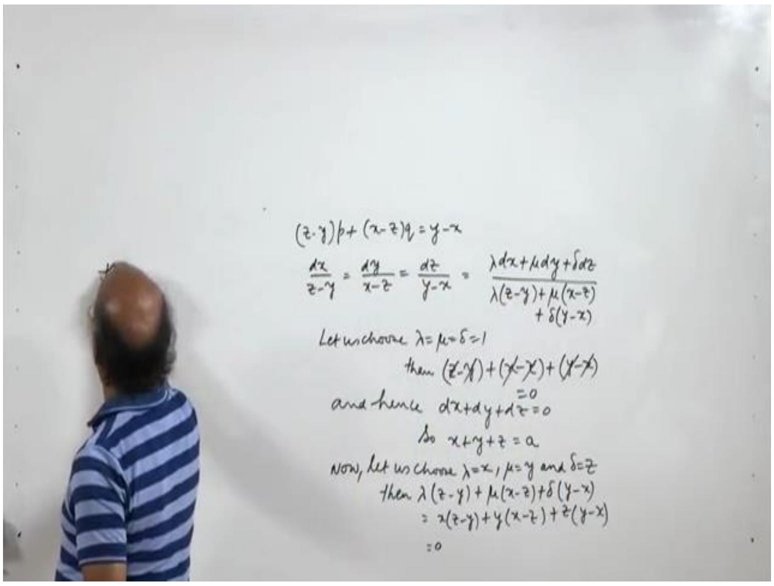

Now let us take the example 2. So, in example 2, z minus y into p plus x minus z into q equal to x y minus x, so we will have subsidiary equations as dx over z minus y equal to dy over x minus z equal to dz over y minus x. Now, we can see that if you can see that dx over z minus y equal to dy over x minus z, the variables x, y, z are not separable. So, it is a case where the variables are not separable. So, we shall consider this equal to lambda dx plus mu dy plus delta dz divided by lambda times x minus y plus mu times x minus z plus delta times y minus x.

Now, let us make a choice of lambda mu and delta, in such a way that the denominator here is 0. So, let us choose lambda mu and delta all to be equal to 1, let us choose lambda mu delta equal to 1, then we see that x minus y plus z minus y z minus y this is z minus y z minus y x minus z. This is z minus y plus x minus z plus y minus x, this cancels with this y cancels with y and z cancels with z, so this is equal to 0. And hence dx plus dy plus dz equal to 0 from here and so x integrating we get x plus y plus z equal to constant a. So, we have got one solution u x, y, z equal to a, where u x, y, z is x plus y plus z.

Now, let us make another choice of lambda mu delta. Now, let us choose lambda equal to x, mu equal to y, and delta equal to z. Then we notice that lambda times z minus y plus mu times x minus z plus delta times y minus x is again 0, because this is x times z minus y plus y times x minus z plus z times y minus x. So, we have x y which cancels with x y here we have x z which cancels with x z here then we have y z y z cancels with this is equal to 0.

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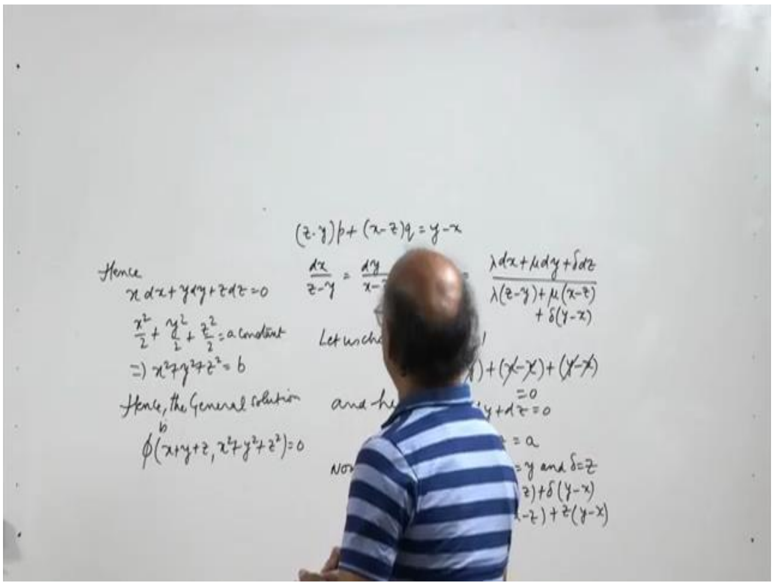

So, hence we have another equation. Hence, x dx plus y dy plus z dz equal to 0. And on integration, we get x square by 2 plus y square by 2 plus z square by 2 equal to a constant which can be written as x square plus y square plus z square equal to a constant b. So, this is v x, y, z equal to b, and the general solution is phi x plus y plus z x square plus y square plus z square equal to 0.

So, now we have seen an example of a Lagrange's linear equation, where the variables are not separable and we have to make a choice of lambda mu and delta such that lambda P plus mu Q plus delta R equal to 0. Now, by choosing two by making two different choices of lambda mu and delta, we arrive at lambda P plus mu Q plus delta R equal to 0. And then integrating the two for cases on integration we get the two independent integrals u and b, u equal to a, and b equal to the b of the subsidiary equations. So, we obtain the general solution. Now, let us take one more case another case where we will see that we obtain one solution of the auxiliary equation; and using that solution, we shall be getting the second solution.

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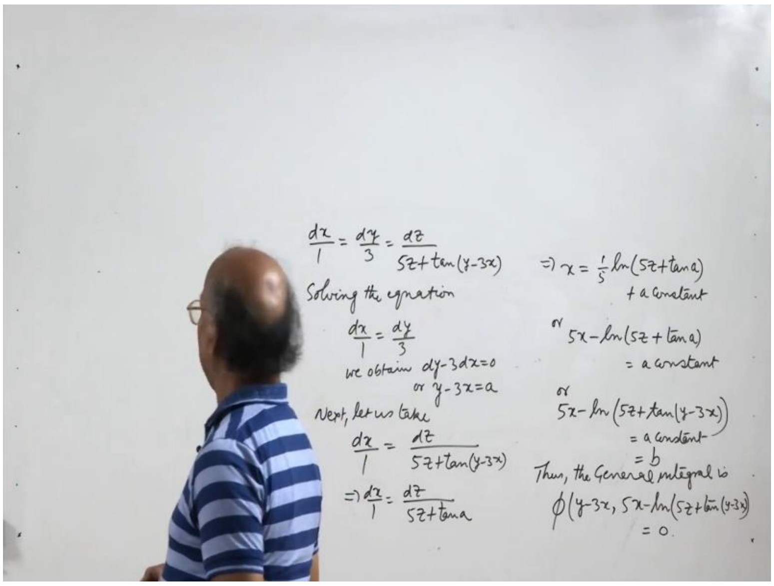

So, in example 3, it is a different type of example here we have dx by 1 dy by 3 equal to dz upon 10 by 5 z plus 10 by minus 3 x. So, these are subsidiary equations. Now, solving the equation dx over 1 equal to dy over 3, we obtain dy minus 3 dx equal to 0 or y minus 3 x equal to a, where a is some constant. Now, in order to find the second solution, what we do is let us consider, next let us take dy over 3 or we can take dx over 1 equal to dz over 5 z plus 10 y minus 3 x.

Now, we shall in order to solve this question, we shall be using the solution y minus 3 x equal to a. So, y minus 3 x equal to a we can put here, so which implies that dx by 1 equal to dz upon 5 z plus tan a. Now, we can this is the case of separation of variables, on one side, we have x; on the other side, we have z. So, we can integrate, so we get x equal to 1 by 5 l n 5 z plus tan a plus some constant, a constant, or we can say 5 x minus l n 5 z plus tan a equal to some constant. Now, let us replace back the value of a. So, 5 x minus l n 5 z plus 10 by minus 3 x equal to a constant let us say b. So, thus we have then thus the general solution is phi y minus 3 x and then 5 x minus l n 5 z plus tan y minus 3 x equal to 0. So, the general integral in this case, here we have seen that by using one solution, we can obtain the second independent integral of the subsidiary equations.

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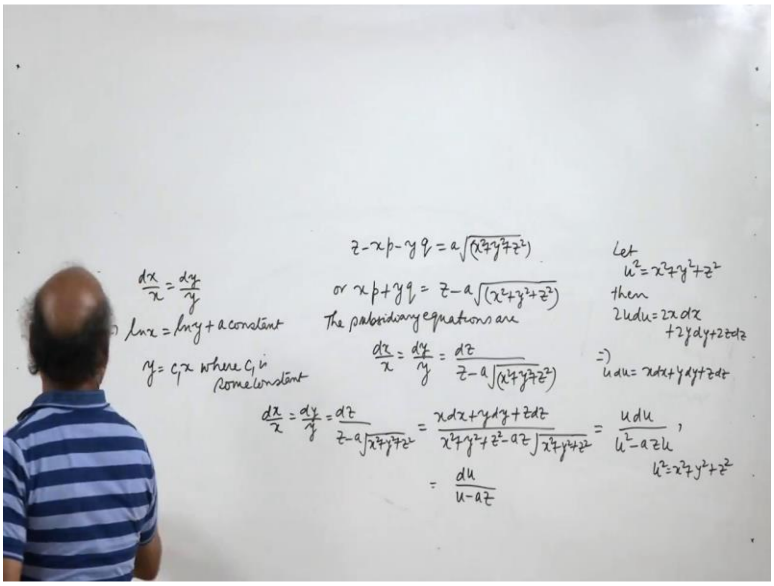

Now, let us take one more example of a similar type. So, z minus x p minus y q equal to a under root x square plus y square plus z square where a is some constant it is given. Now, we can put it in this standard form or x p plus y q equal to z minus a under root x square plus y square plus z square. Now, this is in the form of the Lagrange's equation P p plus Q q equal to R. So, the subsidiary equations are dx over x equal to dy over y equal to dz over z minus a under root x square plus y square plus z square, now dx over x equal to dy over y gives us the solution very easily. So, this is l n x equal to l n y plus a constant that this can be written as y equal to c into x, I can get the constant as minus l n c. So, then we will get l n y equal to l n x plus l n c are by equal to c x where c 1 x let us say, where c 1 is some constant.

Now, let us try to find the second solution. So, we have, so dx over x equal to dy over y equal to dz upon z minus a under root x square plus y square plus z square is also equal to x dx plus y dy, let us multiply y x y and z plus z dz divided by x square plus y square plus z square minus a z under root x square plus y square plus z square. So, let us now take let u square b equal to x square plus y square plus z square, then 2 u du is equal to 2 x plus 2 y 2 x dx plus 2 y dy plus 2 z dz or u du equal to x dx plus y dy plus z d z. So, this is equal to u du divided u square minus a z into u on putting where u square is equal to x square plus y square plus z square. And this is same as du over u minus a z, du over u minus a z we get.

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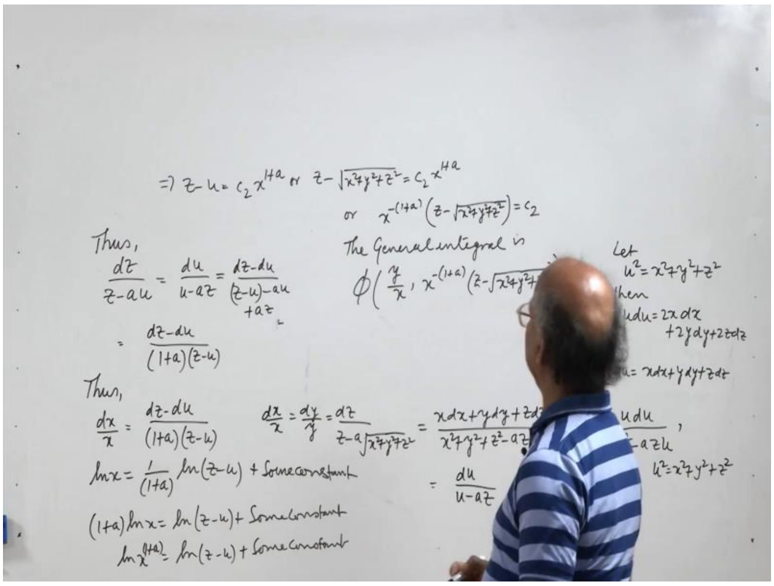

Now, what we do is. So, we have dz upon z minus a u thus dz upon z minus a u is equal to this is z minus a u, z minus a u equal to du over u minus a z which is equal to dz minus d u divided by z minus u minus z z minus a u plus a z. So, this is equal to dz by du divided by 1 plus a times z minus u. So, dz upon z minus a u is equal to du over u minus a z, which is equal to dz minus du divided by z minus u minus a u plus a z. So, we get z minus u times 1 plus a. Now, we can easily integrate this. So, what we will do is. So, this is equal to this. So, thus dx by x dx by x equal to dy by y equal to dz by z minus a u equal to du by u minus a z equal to dz by du minus upon 1 plus a into z minus u. So, we get this.

And this will give you l n x equal to 1 plus a 1 over 1 plus a l n z minus u can put z minus u s t then do z minus du equal to dt. So, dt over 1 plus a into t. And when we integrate, we get one over one plus a ln z minus u. So, this is we can put it as plus some constant. We can simplify this now you can multiply by 1 plus a. So, 1 plus a l n x equal to l n z minus u plus some constant, and this can then be written as l n x to the power 1 plus a. So, we can write it as z minus u equal to some constant c 2 times x to the power 1 plus a this can be written as minus l n c 2. So, we will get z minus u equal to c 2 x to the power of 1 plus a. Now, or z minus under root x square plus y square plus z square equal to c 2 times x to the power 1 plus a. So, thus we get the following solutions general. So, integral we get as follows.

So, one solution was y equal to c 1 x, the other solution is I can say x to the power minus 1 plus a into z minus under root x square plus y square plus z square equal to c 2. So, the general integral is phi y over x and x to the power minus 1 plus a z minus under root x square plus y square plus z square equal to 0. So, this is the general integral in this case. So, we have discussed four different types of examples. In the first case, we have the case of a suppression of variables. And the second case, we had the example where we had to make two different choices of lambda mu and delta, so that lambda P plus mu Q plus delta R equal to 0.

In the third case, we had an example where we use the one solution y minus 3 x equal to a to find the second solution independent solution. In the fourth case, it was a typical example where one the solution very easy to find which was y equal to c minus x, but for the second solution, we had to do I mean several this thing substitutions and all. So, the second solution was not so easy to obtain, but that is how we get the second solution. And the general solution is therefore given by phi u, v equal to 0. So, with that I would like to conclude my lecture.

And I thank you all for attention.