Mathematical methods and its applications
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Lecture – 17
Solution of Lagrange's equation –I

Hello friends, welcome to my first lecture on solution of Lagrange's equation. We know that an ordinary differential equation has solutions which involving a few arbitrary constants.

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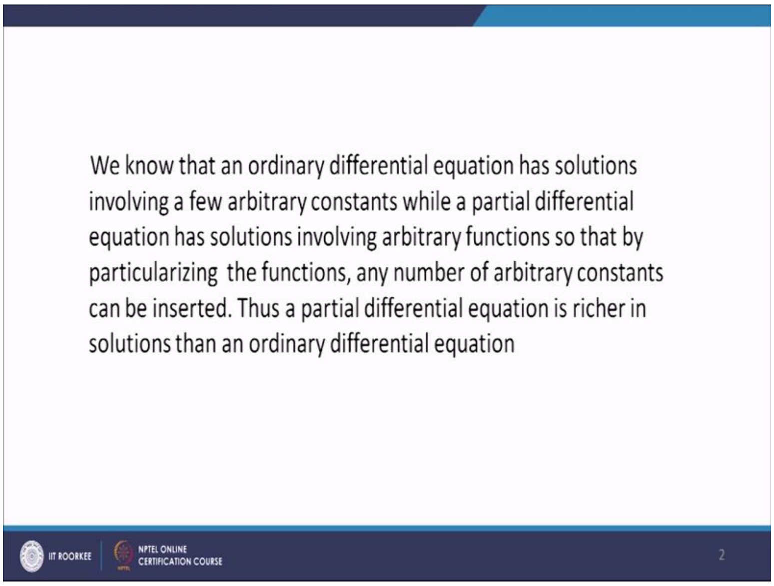

If you take nth ordinary differential equation then it solution involves, general solution involves n arbitrary constants. Now in the case of a partial differential equation, we have solutions which involve arbitrary functions and so by particularizing the arbitrary function any number of arbitrary constants can be inserted and therefore, we can say that a partial differential equation is richer in solutions than an ordinary differential equation.

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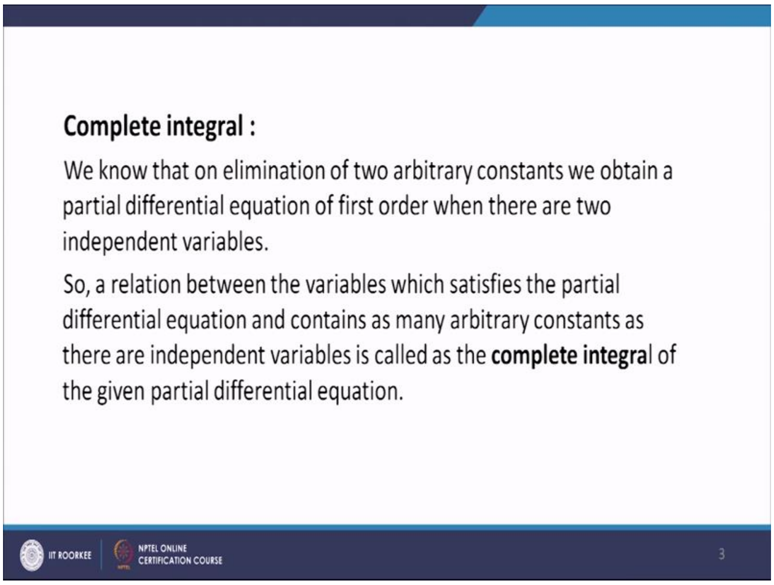

Now, let us define what do we mean by a complete integral? We know that when you eliminate two arbitrary constants we obtain a partial differential of first order, differential equation of first order when there are independent variables. In my last lecture we have seen that when the when we eliminate the two arbitrary constants we obtain a partial differential equation of first order, in case of two independent variables. So, a relation between the variables which satisfies the partial differential equations, equation and contains as many arbitrary constants as there are independent variables is called as the complete integral of the given partial differential equation.

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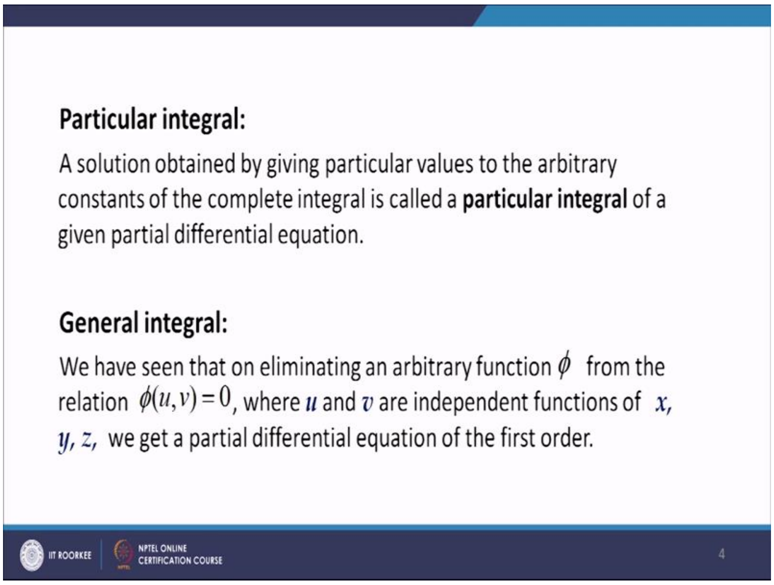

Now, solution obtained by giving particular values to the constants occurring in the complete integral is called as a particular integral of the given partial differential equation. Now we also in the previous lecture we have also seen that when we eliminate an arbitrary function phi from the relation phi u v equal to 0 where, u and v are independent functions of x, y and z; we get a partial differential equation of the first order.

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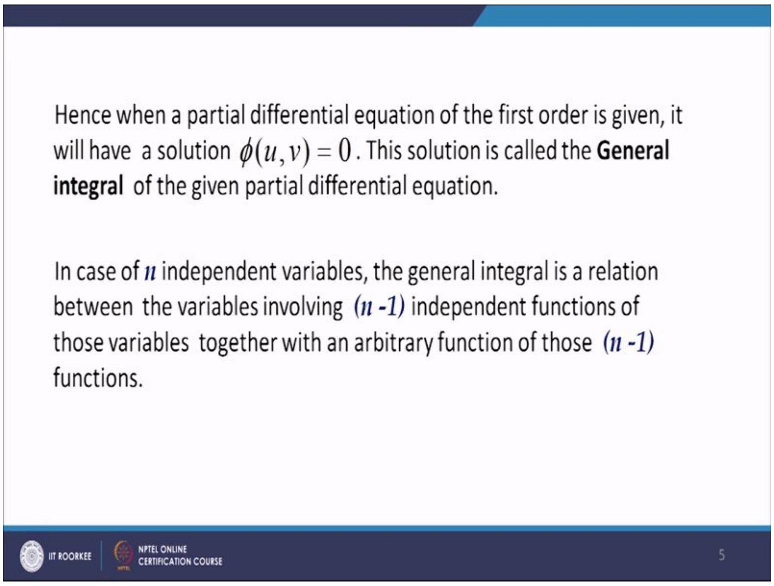

And therefore, when a partial differential equation of the first order is given we will have a solution phi u, v equal to 0. This solution is called as the general integral of the given partial differential equation. Now we can generalize this to the case of n independent variables. When we have independent variables the general integral is a relation between the variables involving n minus 1, independent functions of those variables together with an arbitrary function of those n minus 1 functions.

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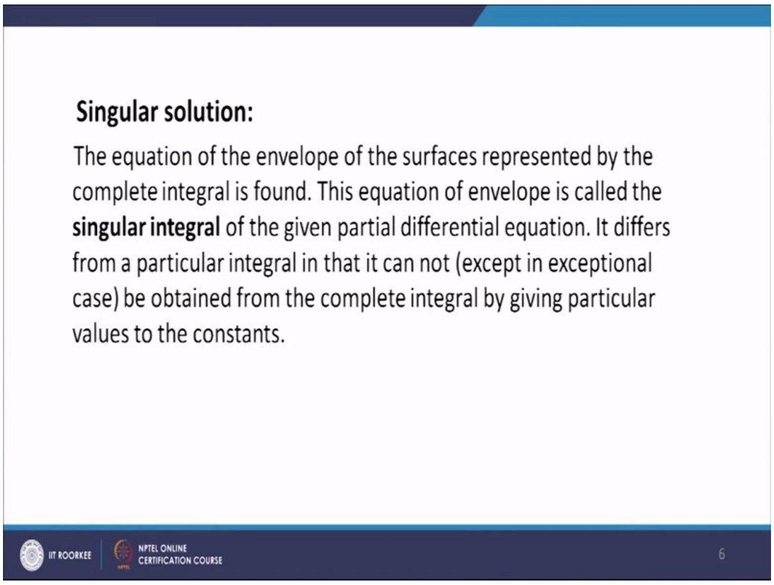

Now, let us define a singular solution. The equation of the envelope of the surfaces represented by the complete integral is found. This equation of envelop is called the singular integral of the given partial differential equation. Now this singular integral differs from the particular integral in that, it cannot be obtained from the complete integral by giving particular values to the constants. Let us say, we have the partial differential equation f x, y, z, p, q equal to 0 which, has been derived from the relation F x, y, z, a, b equal to 0 where, a and b are two arbitrary constants.

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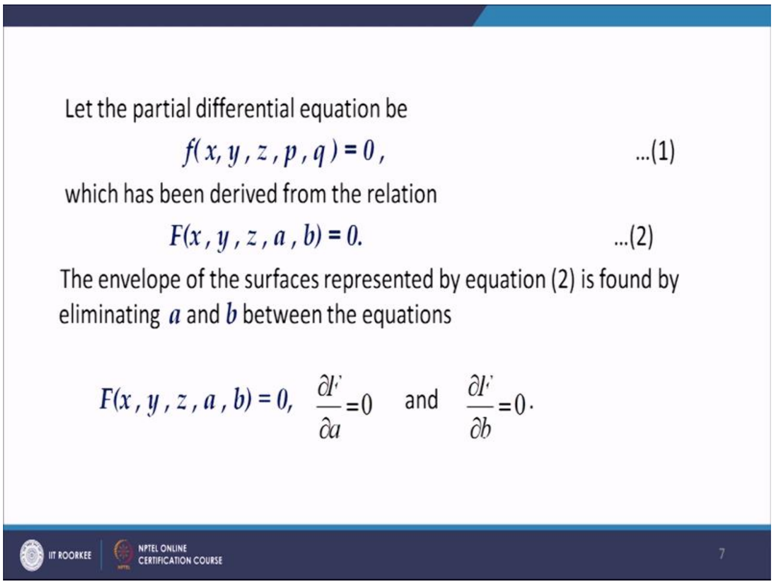

We know that when you eliminate the two arbitrary constants occurring in the relation F x, y, z, a, b equal to 0, but you get is a first order partial differential equation.

So, f x, y, z, p, q partial differential equation is derived from the relation F x, y, z, a, b equal to 0. The envelope of the surfaces represented by the equation 2 is found by eliminating a and b; the arbitrary constants a and b between the equations F x, y, z, a, b equal to 0 delta F over delta a equal to 0 and delta F over delta v equal to 0. Now singular integral is a relation between the variables involving no arbitrary constants. In general singular integral, if it exists it is distinct from a complete integral, but in the exceptional cases it occurs as a particular case of the complete integral obtained by giving a, special values to the arbitrary constant. This is also in the exceptional case.

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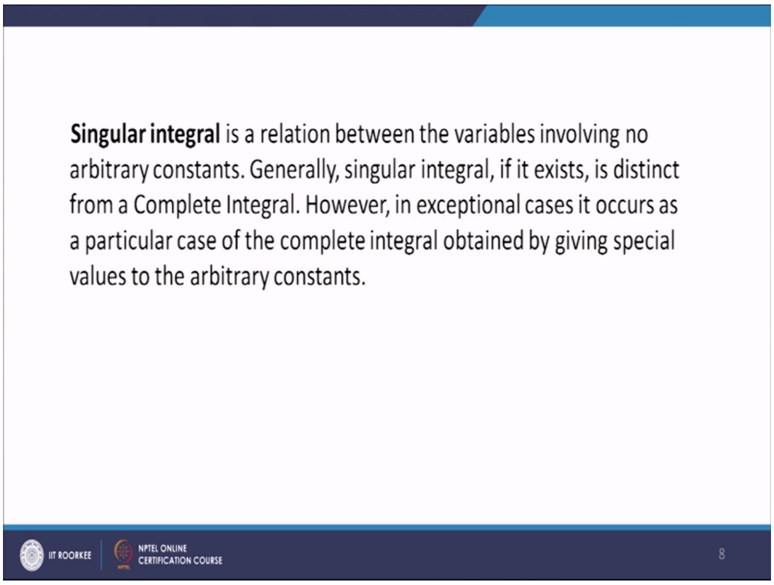

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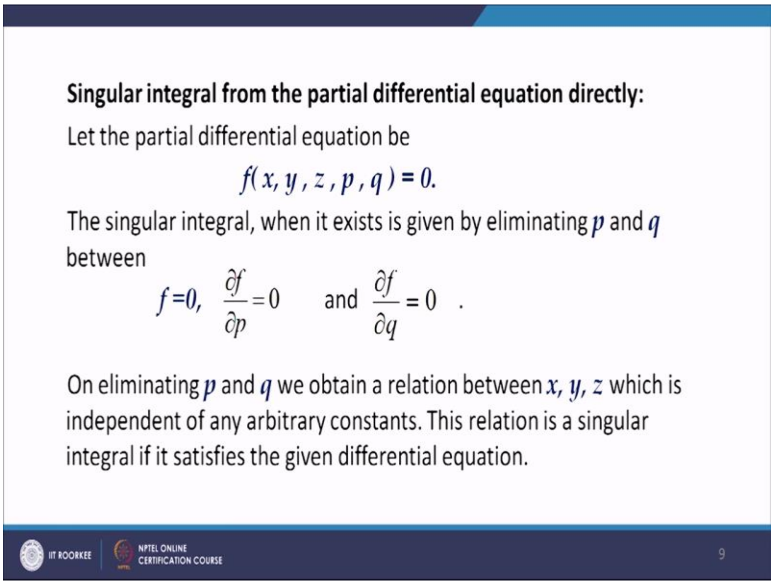

Now, let us see how we get the singular integral from the partial differential equation directly. Suppose we have the partial differential equation of first order f x, y, z, p, q equal to 0 then, the singular integral where it exists is obtained by eliminating p and q between the equation f equal to 0 the its partial derivative with respect to p and its partial derivative with respect to q equal to 0. So, I am eliminating p and q, we shall obtain the relation between x, y, z which will be independent of any arbitrary constants and this relation is a singular integral provided it satisfies the given differential equation.

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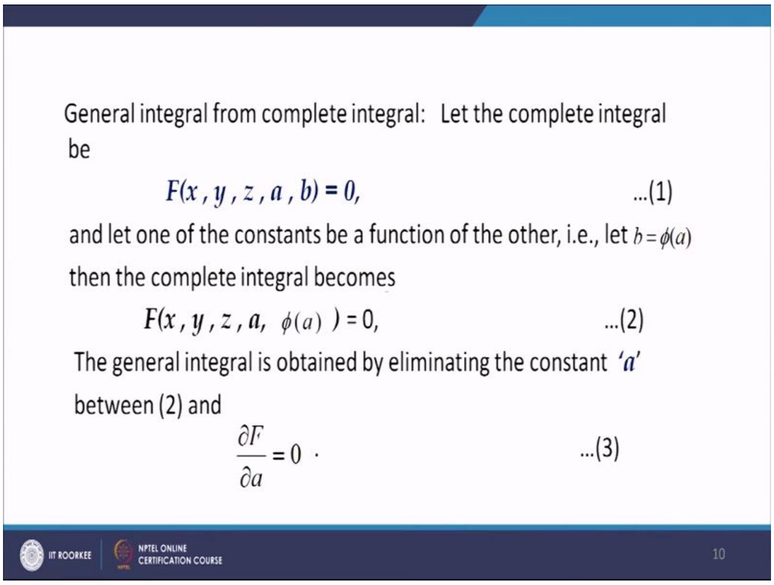

So, now let us see how we can obtain general integral from the complete integral. Let us say the complete integral of the differential equation b, F x, y, z, a, b equal to 0. So, here x and y are taken two independent variables and we have two arbitrary constants. So, we have defined the complete integral as the solution involving as many arbitrary constant as there are independent variables, so in the relation. So, here this can be called as a complete integral. So, F x, y, z, a, b equal to 0, let this be this complete integral and one of the constant suppose a can be written as a function of, b can be written as a function of a. So, b equal to phi a then, we can write the complete integral as F x, y, z, a phi a equal to 0.

The general integral is then obtained by eliminating the constant a between equation 2 and equation 3, which is the equation delta F over delta a equal to 0.

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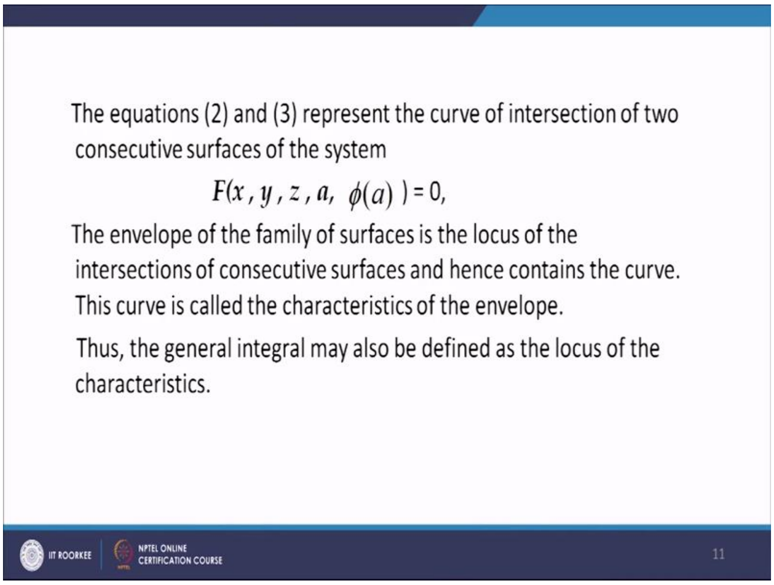

The equations 2 and 3 represent the curve of intersection of two conjugated surfaces of the system F x, y, z, a phi a equal to 0. The envelope of the family of surfaces is the locus of the intersections of consecutive surfaces and hence contains the curve. So, this curve is called as the characteristics of the envelope that, the general integral may also be defined as the locus of the characteristics. Now let us see how we find the solution of a partial differential equation.

When while solving a partial differential equation we must not only find the complete integral, but should also indicate the singular and particular general integrals. The partial differential equation is not said to be completely solved, if we just find the complete integral.

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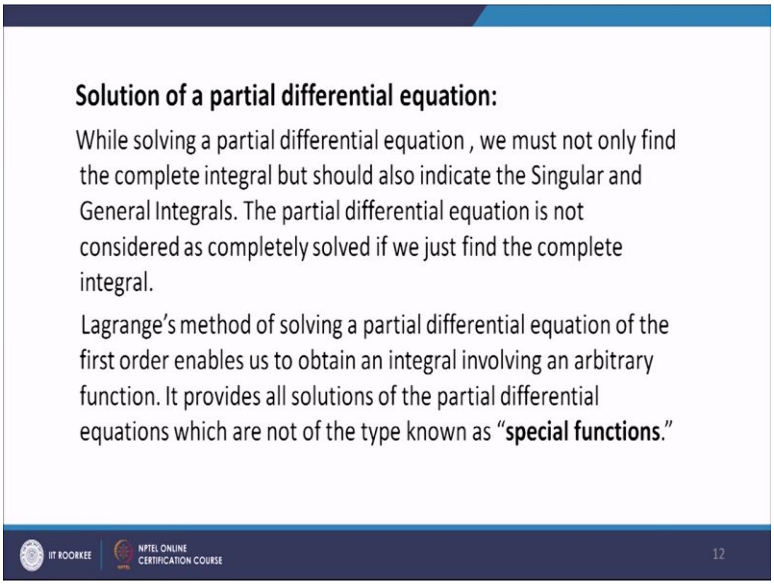

We shall now discuss how to solve a partial differential equation of first order and first degree. So, Lagrange's method of solving a first order and first degree partial differential equation provides such an integral involving an arbitrary function. And this solution actually this integral which involves an arbitrary function, it provides all solutions of the partial differential equation which are not of the type known as a special functions.

So, this we shall see later on. This case where solution of the partial differential equation is not actually contained in the general integral. So, this we shall see later on, it just an exceptional case where all solutions of the partial differential equation are not obtained from the general integral.

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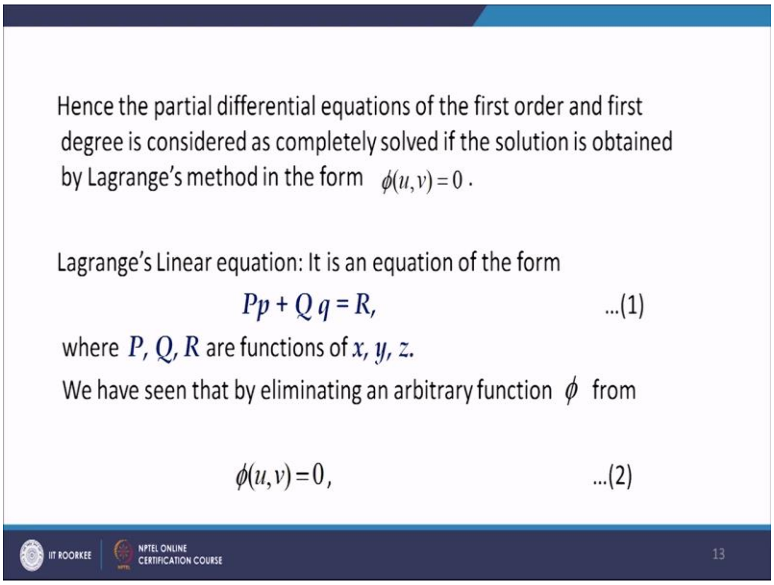

Now, hence the partial differential equations of the first order and first degree is considered as completely solved. If the solution of the differential equation is obtained by the Lagrange's method in the form phi u, v equal to 0. Let us see the most general form of the Lagrange's first order and first degree differential equation.

So, Lagrange's linear equation is an equation of the form P p plus Q q equal to R, let us remember that this small p represents the partial derivative of z with respect to x and this small q represent the partial derivative of z with respect to y, so where this P, Q and R are functions of x, y, z. So, thus we have a first order and first degree differential equation and we have seen that by eliminating the arbitrary function phi from phi u, v equal to 0.

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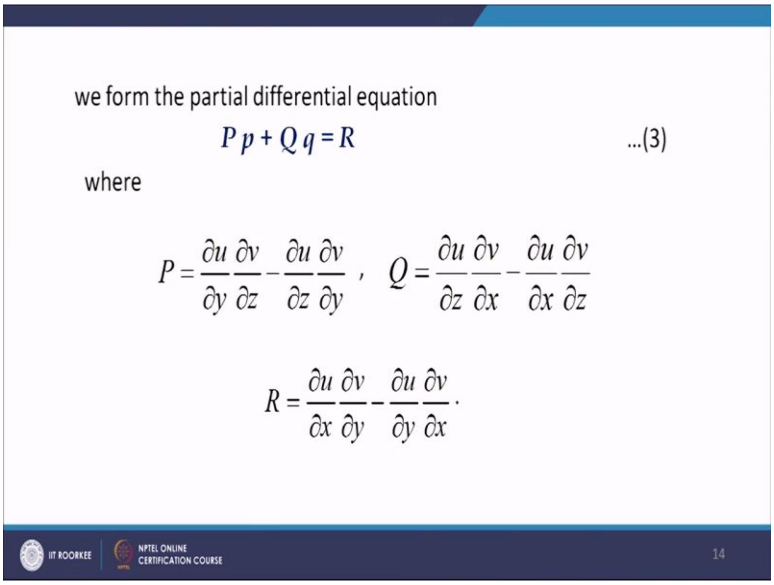

We find the partial differential equation P p plus Q q equal R where, P comes out to be equal to this and Q comes out to be equal to this and R comes out to be equal to this.

Hence when we have an equation given by 3, let us say when we have an equation given by 3, we have an integral given by equation 2; this integral. So, when we consider a partial differential equation of the first order and first degree its solution will be given by phi u, v equal to 0 which will be called its general integral.

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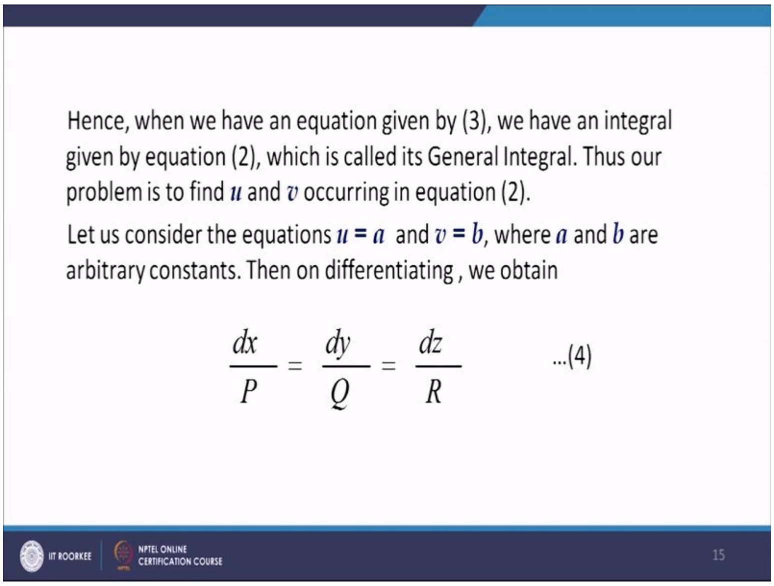

Thus our in problem is now reduces to finding the independent functions u and v that occurring in equation 2. If we know the independent functions u and v then we shall be able to write phi u, v equal to 0; the general integral of the given partial differential equation of first order first degree.

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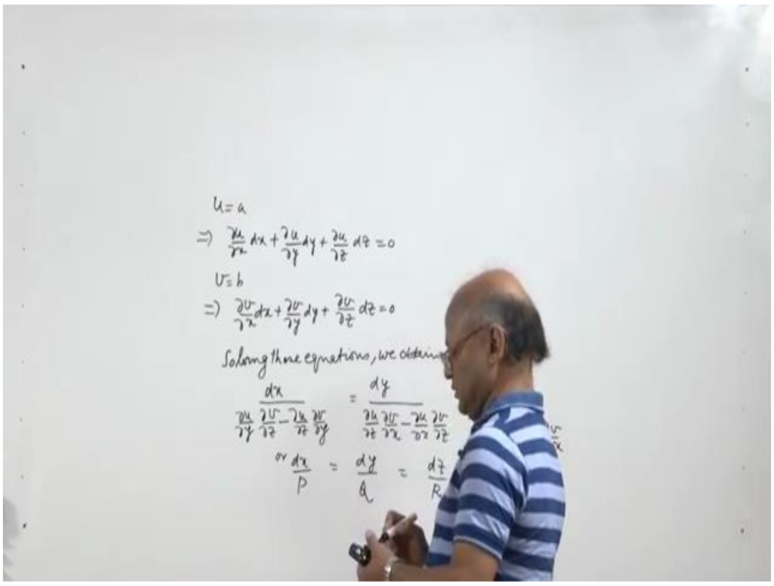

So, let us consider, let us now consider the equations u equal to a and v equal b where, a and b are arbitrary constants. So, then on differentiating what do we get, u equal to a gives you delta u by delta x because u is a function of x, y, z. So, delta u by delta x d x delta u by delta y d y delta u by delta z d z equal to 0 because a is a constant and similarly b equal to v gives you delta b over delta x d x, delta b over delta by d y plus delta b over delta z d z equal to 0. Now let us solve these two equations. So, solving these two equations we get, we obtain d x over u y minus u y v z minus u z v y and then d y over u z v x minus u x v z.

And then we have u x v y minus u y v x. Now this is equal to this P. So, we have d x over P equal to d y over Q equal to d z over R. So, then on solving, then on differentiating the equations u equal a and b equal to v, we arrive at the subsidiary equations d x by P equal to d y by Q equal to d z by R. The equation 4, these equations there are two independent equations. So, there are the equation 4 are differential equations, whose solutions are u equal a and v equal to b.

Hence to obtain a solution of the partial differential equation P p plus Q q equal to R, we write down the subsidiary equations d x over P equal to d y over Q to d z over R.

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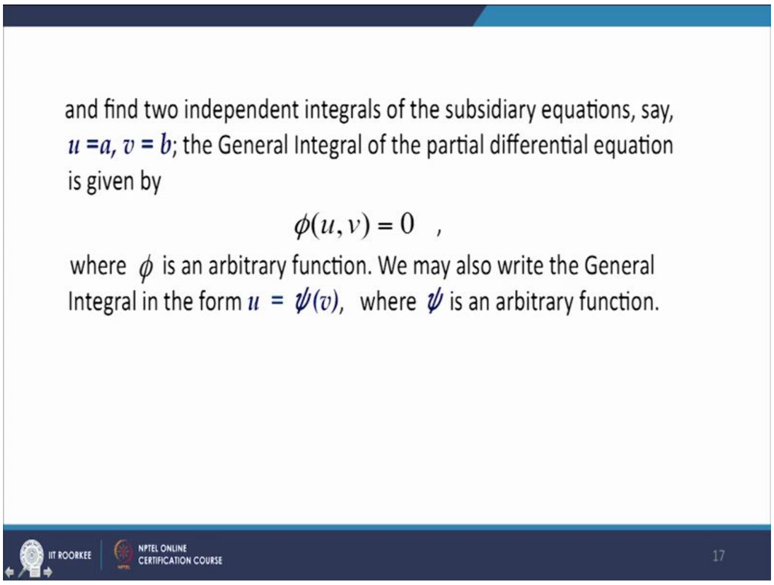

And find two independent integrals of the subsidiary equations say u equal to a, v equal to b then the general integral of the partial differential equation will be given by phi u, v equal to 0 where, phi is an arbitrary function. Now since we are only interested in an arbitrary function relating u and v, we may also write the general integral in the form u equal to psi v where, psi is an arbitrary function.

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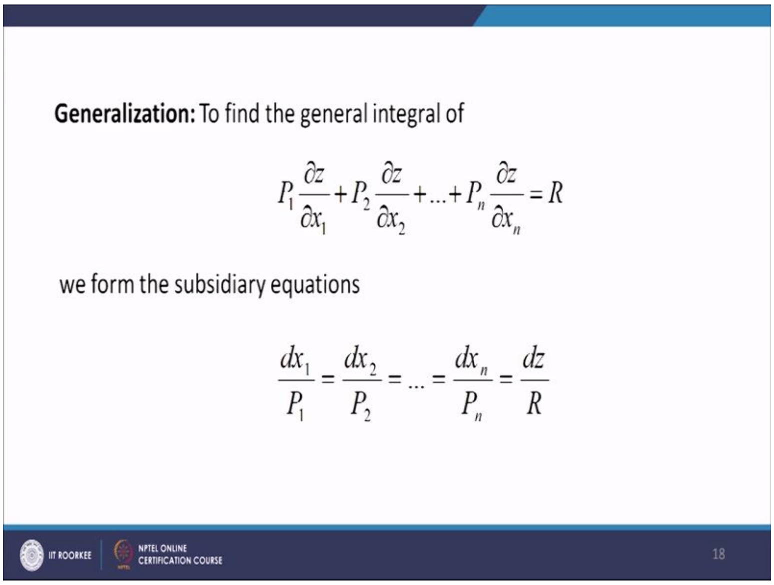

Now, this can be generalized to the case of n independent variable. Suppose z is a function of n independent variables x 1, x 2, x n and we are given the partial differential equation P 1 delta z by delta x 1 plus P 2 delta z by delta x to P n delta z by delta x n equal to R where, P 1, P 2, P n and R are functions of the variables x 1, x 2, x n and z then we form the subsidiary equation d x 1 over P 1 equal to d x 2 over P 2 equal to d x n over P n equal to d z by R and obtain the n independent solutions u 1 equal to a 1, u 2 equal to a 2, u n equal to a n of the subsidiary equations.

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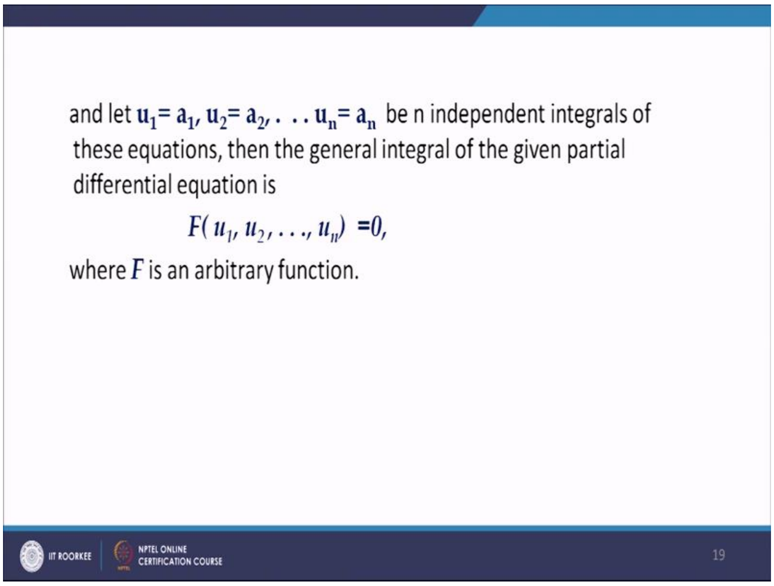

Then, the general integral of the given partial differential equation is written as F u 1, u 2 u n equal to 0 where, F is an arbitrary function.

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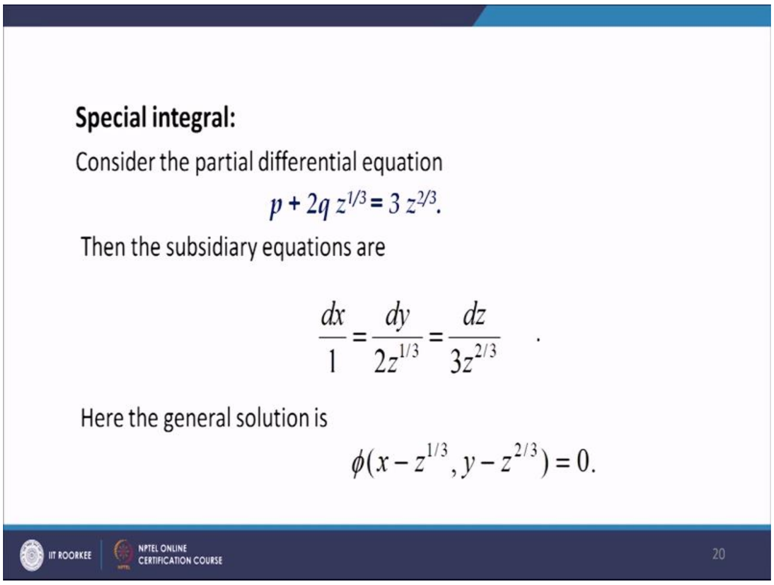

Now, we are considering a special type of Lagrange's linear equation where we shall see that the some solutions are not actually obtained from the general integral, such solutions are called as special integrals. For example, let us consider the partial differential equation p plus 2 q z to the power 1 by 3 equal to 3 z to the power 2 by 3. So, this is a Lagrange's linear equation where we shall see that capitol P is equal to 1, capital Q is equal to 2 z to the power 1 by 3 and R is equal to 3 z to the power 3 by 3 when we compare this equation with the Lagrange's linear equation general form.

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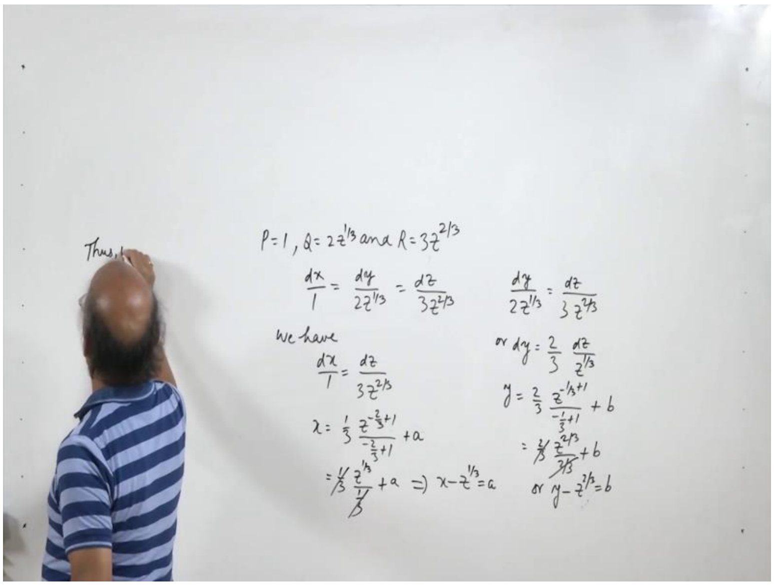

So, the subsidiary equations are d x over 1 equal to d y over 2 z to the power 1 by 3 equal to d z over 3 z to the power 2 by 3. Now, our aim is to determine, two independent integrals u equal to some constant and v equal to some constant, from the subsidiary equations. So, let us see this is very simple because it we can apply the method of separation of variables. So, we have let us consider the equations d x over 1 equal to d z over 3 times z to the power 2 by 3. So, when we integrate this equation, we get x equal to 1 by 3 z to the power minus 2 by 3 plus 1 divided by minus 2 by 3 plus 1. So, this will be plus some constant let us say, a. So, this will be equal to 1 by 3 z to the power 1 by 3 divided by 1 by 3 plus a.

So, this can be cancelled and we have x minus z to the power 1 by 3 equal to a. So, we have got the function u, x, y, z as x minus z to the power 1 by 3. Now similarly let us consider d y another equation d y over 2 z to the power 1 by 3 equal to d z upon 3 times z to the power 2 by 3. So, then we shall have d y equal to 2 over 3 d z upon z to the power 1 by 3. Now again let us integrate, integrating we get y equal to 2 by 3 z to the power minus 1 by 3 plus 1 divided by minus 1 by 3 plus 1 plus some constant let us say b. So, this will give you, 2 by 3 z to the power 2 by 3 divided by 2 by 3 plus b.

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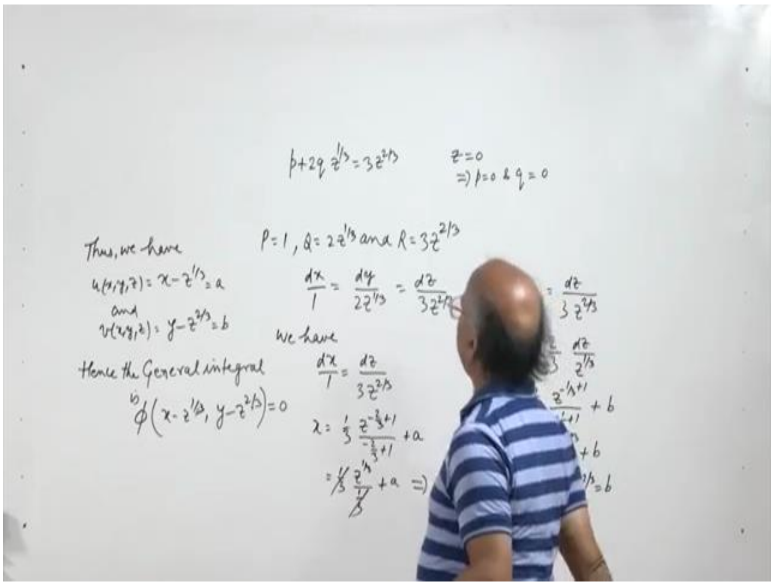

So, this will cancel and we get another integral y minus z to the power 2 by 3 equal to b. So, thus we have two independent integrals, u, x, y, z equal to x minus z to the power 1 by 3 equal to a constant, a and v, x, y, z equal to y minus z to the power 2 by 3 equal to

b. Now we can see that these two functions of x, y, z are independent of each other. So, hence, the general integral is given by some arbitrary function phi u, v. So, that means u means x minus z to the power 1 by 3 and y minus z to the power 2 by 3 equal to 0.

Now, we note that. So, we have got the general integral by applying the method given by Lagrange's. Now let us notice that the differential equation p plus 2 q z to the power 1 by 3 equal to 3 times z to the power 2 by 3. It is also satisfied when we take z equal to 0 because when you take z equal to 0 then, this will give you p equal to 0, its derivative with respect x equal to 0 and q equal to 0. So, p equal to 0 and q equal to 0 when you substitute and z equal to 0, the given partial differential equation is satisfied and the z equal to 0 cannot be obtained from this general integral, it is not an, it cannot be expressed z equal to 0 cannot be expressed as a function of x minus z to the power 1 by 3 and y minus z to the power 2 by 3.

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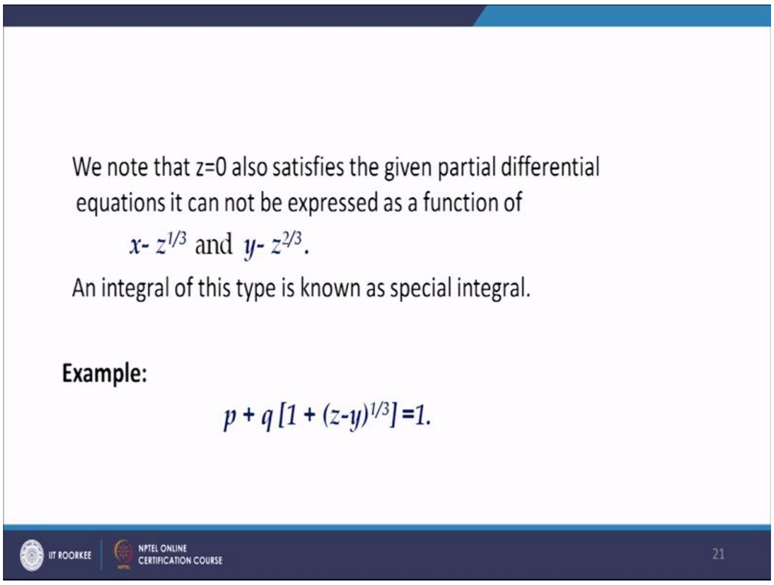

So, such a function is known as the special integral. So, there are some exceptional cases where we notice that solution of the partial differential equation is not expressible as a function of the independent integrals u and v, it cannot be obtained from the general integral. Now let us take one more case.

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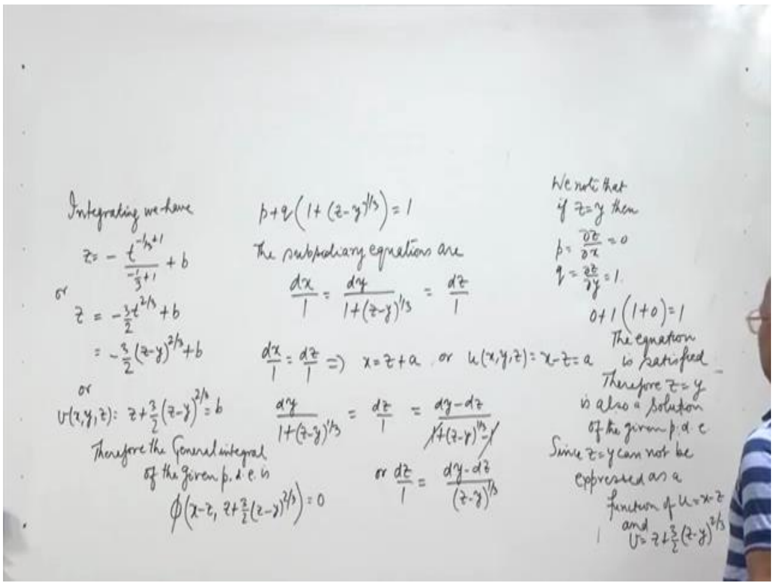

So, let us consider one more partial differential equation of this kind p plus q times 1 plus z minus y to the power 1 by 3 raise to the power 1 by 3 equal to 1. So, again we see that it is a Lagrange's linear equation. So, we can write the subsidiary equations, the subsidiary equations are d x by P. Now capital P here is 1. So, we have d x by 1 capital Q is 1 plus z minus y to the power 1 by 3.

So, we get d y upon 1 plus z minus y raise to the power 1 by 3 and d z upon R, R is equal to 1. Now we have to find two independent integrals, u equal to constant and v equal to constant from this subsidiary equations. So, let us first solve d x over 1 equal to d z over 1. So, this solving this, we get x equal to z plus some constant a or u x, y, z equal to x minus z equal to a. Now let us take, let us find the other solution. So, we can consider d y over 1 plus z minus y raise to the power 1 by 3 equal to d z upon 1. Now this is also equal to d y minus d z divided by 1 plus z minus y raise to the power 1 by 3 minus 1. So, this cancels with this. So, thus we have or d z upon 1 is equal to d y minus d z over z minus y to the power 1 by 3.

Now, let us define z minus y equal to some variable t then, d y minus d z or d z minus d y is equal to d t. So, we shall have d y minus d z upon z minus y to the power 1 by 3, we can write as minus d t upon t to the power 1 by 3. Thus, d z upon 1 is equal to minus d t over t to the power 1 by 3. Now let us integrate. So, integrating we have z equal to minus t to the power minus 1 by 3 plus 1 divided by minus 1 by 3 plus 1 plus some constant b.

This is equal to or z equal to minus t to the power 2 by 3 into 3 by 2 plus some constant b. So, this is also can be written as minus 3 by 2 z minus y to the power 2 by 3 plus v or we get u v, x, y, z equal to z plus 3 by 2 z minus y raise to the power 2 by 3 equal to b.

Thus, we have got two independent integrals u x, y, z equal to x minus z equal to a and v x, y. z equal to z plus 3 by 2 z minus y to the power 2 by 3 equal to b. And therefore, the general integral of the given partial differential equation which we can write in short as PDE is phi u, v equal to 0; that means, x minus z and z plus 3 by 2 z minus y raise to the power 2 by 3 equal to 0. Now this is the general integral which we have solved by which we have obtained by solving the subsidiary equations. Let us notice that if you take z equal to y.

So, we note that, if z is equal to y then due to the independence of the variables x and y when we differentiate z with respect to x we get 0 and when we differentiate z with respect to y we get 1. So, if you look at the given partial differential equation p is equal to 0, q is 1 and then z is equal to y. So, we will get 0 plus 1 into 1 plus 0 equal to 1 and therefore, z equal to 1. So, the equation is satisfied and therefore, z equal to y is also a solution of the given partial differential equation, but we observe that z equal to y cannot be expressed as a function of the functions independent integrals u and v.

So, since z equal to y equal to or z equal to y cannot be expressed as a function of u equal to x minus z and v equal to z plus 3 by 2 z minus y to the power 2 by 3, it is called as a special integral. It is a special integral of, so thus in some particular cases of partial differential equations we observe that there exists a solution which cannot be expressed as a function of the independent functions of x, y, z, u and v. Otherwise in almost all cases the general solution integral which is obtained from the subsidiary equation includes all the solutions of the given partial differential equation. So, this is what we have about how we solve a Lagrange's partial differential equation first order and first degree.

In our next lecture we shall discuss in detail this method of solving linear partial differential equation of first order and first degree and we shall also give the geometrical interpretation of the Lagrange's first order and first degree linear differential equation that is P p plus Q q equal to R. With that I would like to conclude my lecture.

Thank you very much for your attention.