Mathematical methods and its applications
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Lecture – 16
Formulation of partial differential equations

Hello friends. Welcome to my lecture on Formulation of Partial Differential Equations.

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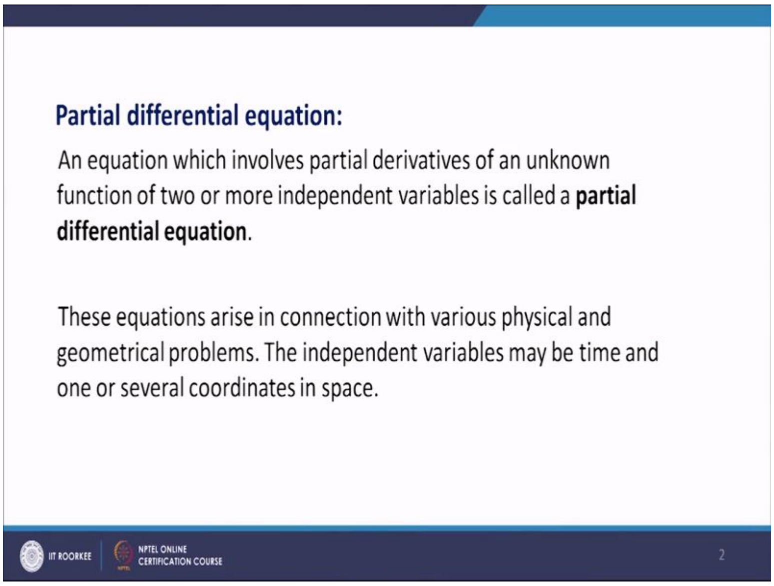

Let us define what do we mean by a partial differential equation first, and equation which involves partial derivatives of an unknown function of 2 or more independent variables is called a partial differential equation. These equations arise in connection with various physical and geometric problems. The independent variables may be time and one or several coordinates in space. We shall see later on an example of a physical problem.

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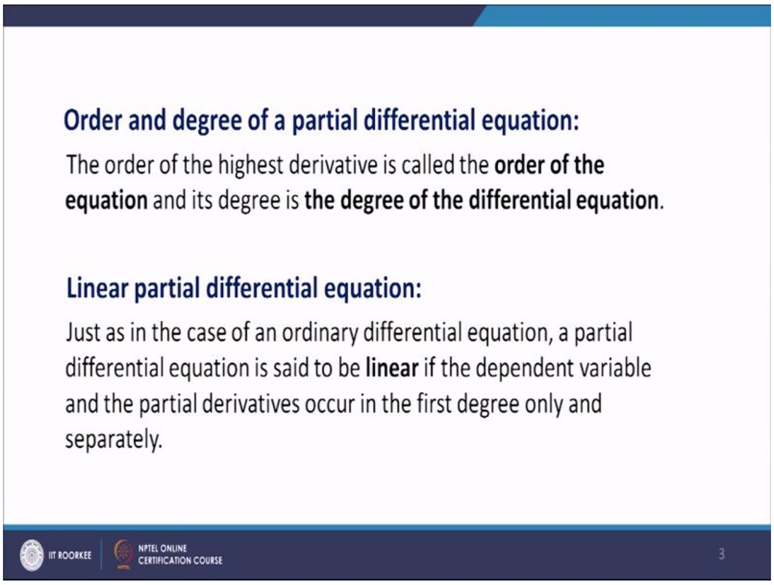

And see how the partial differential equation arises there. The order and degree of a partial differential equation are defined as the order of the highest derivative that occurs in the differential equation is called the order of the partial differential equation. And the degree of the highest order derivative is defined as the degree of the partial differential equation.

A partial differential equation is defined as linear, just as in the case of an ordinary differential equation, if the dependent variable. And the partial derivatives they occur in the first degree and separately. So, I mean in the partial differential equation the independent variable and partial derivatives should be of first degree and they should not occur as products. If such a thing is not true, then the partial differential equation will be called as non-linear.

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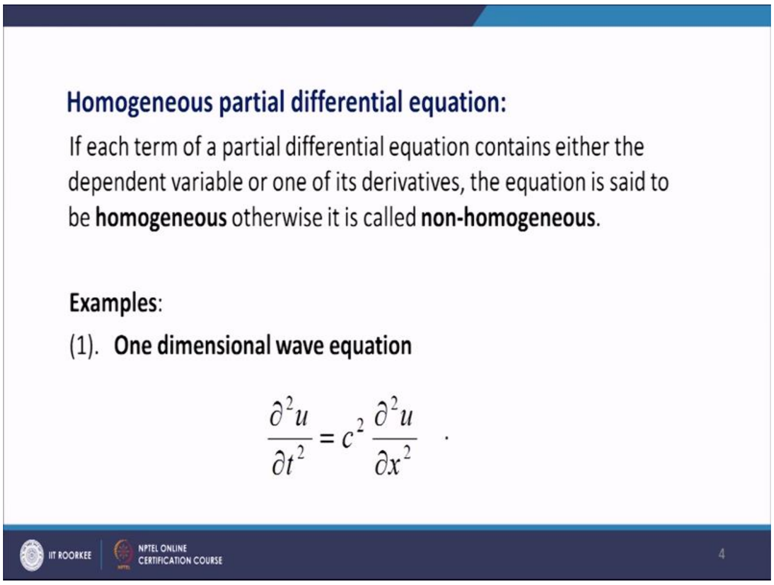

Now, we define homogenous partial differential equation, if each term of a partial differential equation contains either the dependent variables or one of it is derivatives the equation is called as a homogenous differential equation otherwise we shall call it as a non-homogenous equation.

For example, let us look at the examination example of a one dimensional wave equation. This is delta square u by delta t square c square delta square u by delta x square. This is one dimensional wave equation. We can see that the there is second order differential equation because second order derivative is the highest order derivative. Here moreover the second order derivatives are of power 1, their degree is 1. So, it is first second order with first degree and moreover each term of the differential equation contains the derivative and therefore, it is a a homogenous differential equation. So, this second order first degree linear differential equation which is homogeneous.

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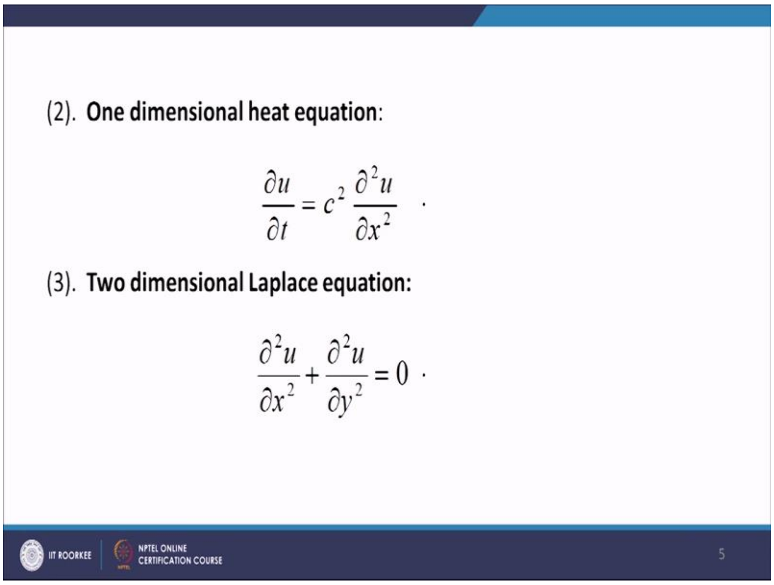

Here, let us take another example one dimensional heat equation. Here this is again second order differential equation because we have second order derivative here, which is the highest order derivative and it is power is 1. So, it is second order first degree moreover each term is containing the derivative. So, it is homogenous equation, and also it is linear because the dependent variable u and it is derivatives. They occur in first degree and they are occurring separately.

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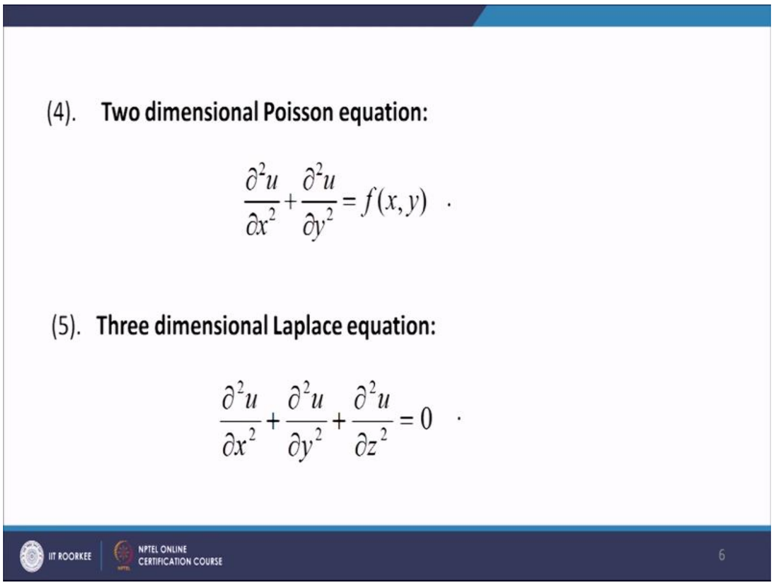

So, it is second order first degree linear homogenous equation. And this is second order 2 dimensional Laplace equation. This is again second order linear differential equation which is a first degree it is homogenous and it is linear.

Let us consider 2 dimensional Poisson equations. So, this is again second order linear second order first degree linear non homogenous equation if f x y is not equal to 0. If f x y is equal to 0 for all x y in some region, then it will also become a homogenous equation and this is 3 dimensional Laplace equation. So, here this is second order first degree homogenous linear differential equation.

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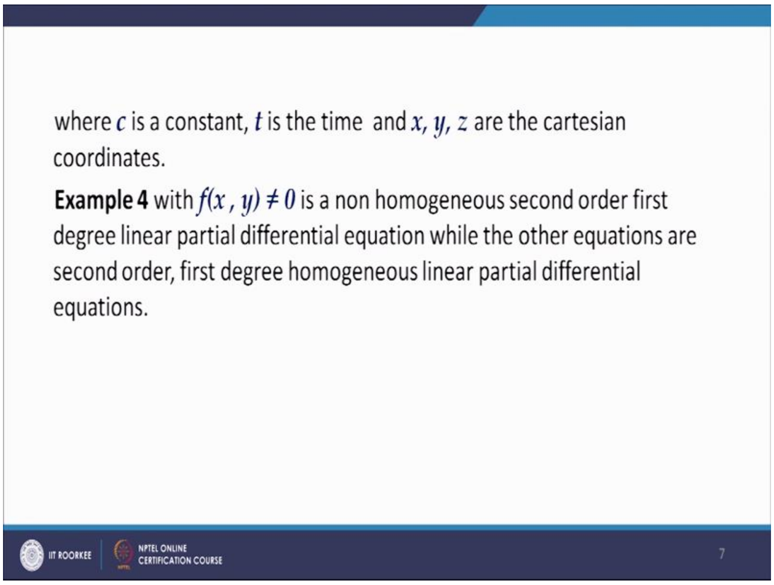

Now, so t in these examples t is the time and x y z are Cartesian coordinates. And as I said f x y I it is non 0 then in the example 4 then we have a non-homogenous second order first degree linear partial differential equation, while the other equations are second order first degree homogenous linear partial differential equations.

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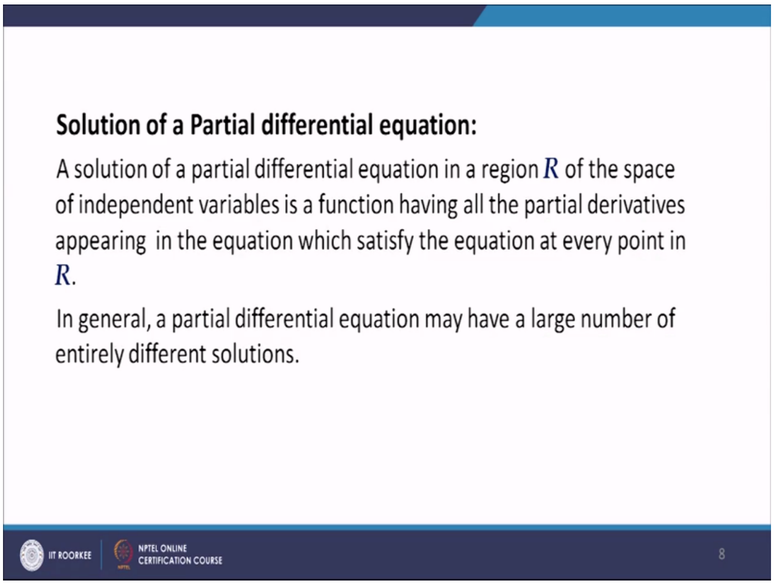

Now, solution of a partial differential equation:a solution of a partial differential equation in a region r of these space of independent variables is a which has all the partial derivatives that occur in the partial differential equation, and satisfies the differential equation at every point in r. In general, partial differential equation may have a large number of entirely different solutions. For example, if you look at the example 3; this one delta square u by delta x square plus delta square u by delta y square. 2 dimensional Laplace equation, then we can verify that x square minus y square plus x.

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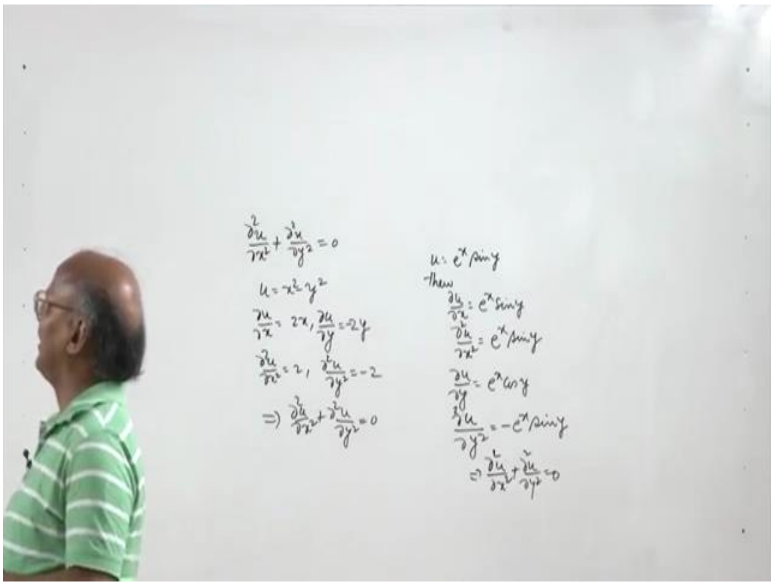

Suppose delta square u by delta x square, plus delta square u by delta y square. Let us consider the case of this Laplace equation. Then if you take u equal to x square minus y square we can verify that u equal to x square minus y square is a solution of this Laplace equation. Because if you take the first find the first order partial derivatives here, then they are 2 x and minus 2 y, and when you find the second order partial derivatives they come out to be 2 and minus 2 and so which implies that this is equal to 0.

So, u equal to x square minus y square is a solution of this. Similarly, we can see that if u is equal to e to the power x sin y. Then u x or you can say delta u y delta x is e to the power x sin by u x x while u y, So, delta square u by delta y square is minus e x sin by, So, delta square u by delta x square plus delta square u by delta y square is equal to 0 and so u equal to e to the power x sin y is a solution of this Laplace equation.

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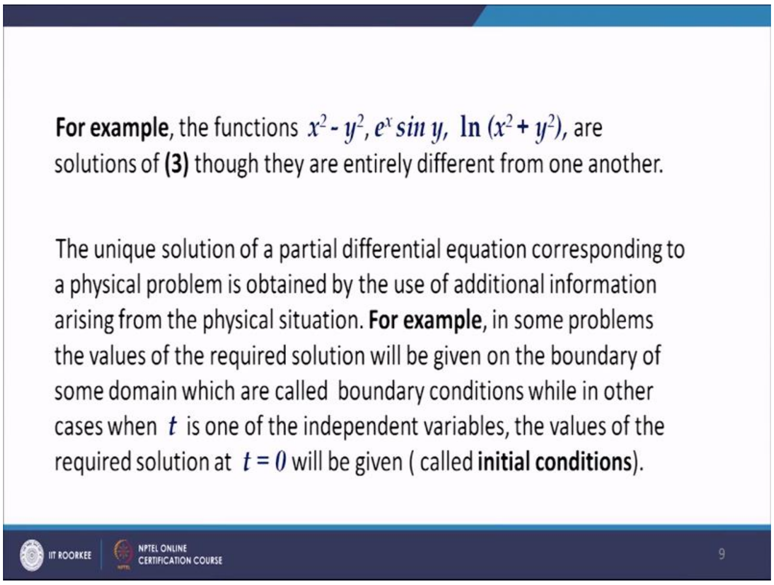

Similarly, you can verify that u equal to l n x square plus y square also satisfies the Laplace equation. So, and these solutions are entirely different in the sense that there is a polynomial in x and y while it involves exponential and trigonometry functions. And this involves a logarithmic function. So, they are entirely different from one another.

So, we have a large number of entirely different solutions of this differential equation now. So, one can then see how we can obtain a unique solution of a partial differential equation. The unique solution of a partial differential equation corresponding to physical problem is obtained by using the initial and boundary conditions, which arise from the physical situation in some problems.

You will see that the values of the required solution, they are only given on the boundary of some domain which are called boundary conditions. While in some cases when t is one of the independent variables the values of the required solution at t equal to 0 will be given. And they are called as the initial condition. So, using those initial and boundary conditions, we arrive at a unique solution of the partial differential equation corresponding to a physical problem.

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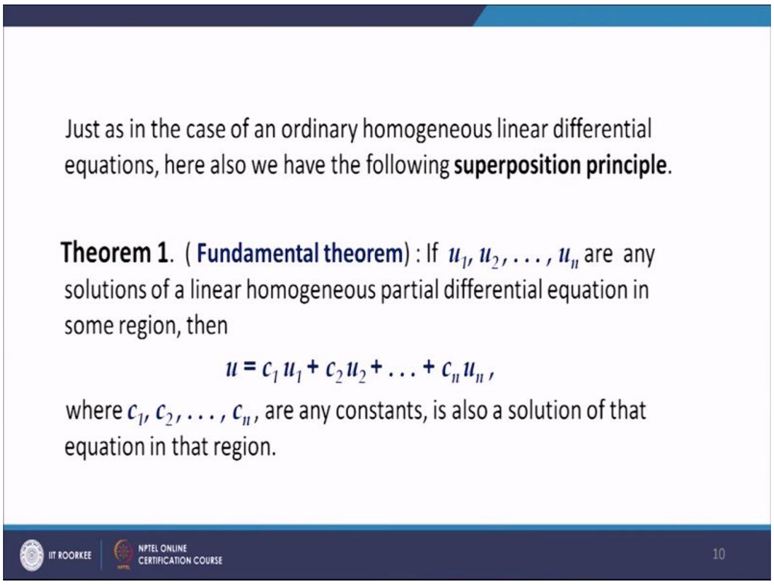

Now, let we have seen that in the case of ordinary differential equations, they have the super position principle. That is if u 1 if y 1 y 2 y n are n independent solutions of the homogenous linear differential equation. Then c 1 y 1 plus c 2 y 2 plus c n y n is also a solution of that. So, here also we have the super position principle, this theorem 1. If u 1 u 2 u n are any solution of a linear homogenous partial differential equation in some region.

Then c 1 u 1 plus c 2 u 2 plus c n u n where c 1 c 2 c n are any constants is also a solution of that equation in that region. So, this you can easily verify by picking up any homogenous linear partial differential equation, it can be easily seen. Now we shall be discussing in our lectures solutions of some special types of partial differential equations.

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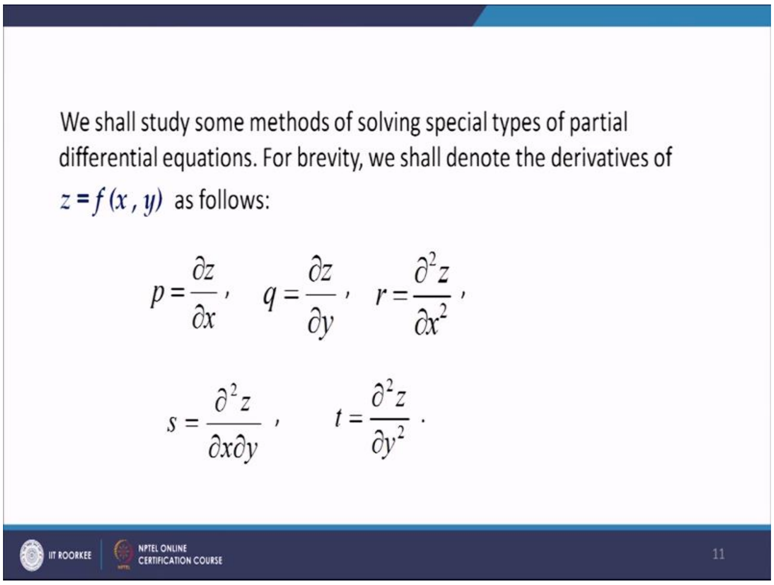

And so for gravity we shall be assuming that the partial derivative of z with respect to x is denoted by p partial derivative of z with respect to y is denoted by q. So, first order partial derivatives with respect to x and y are denoted by p and q. And second order partial derivatives that is z x x is r z x y is s and z y y is denoted by t.

Now, we are going to see how the partial differential equations arise as a result of eliminating arbitrary constants and arbitrary functions. So, suppose we have relation in x y z and a b we have a b are arbitrary constants.

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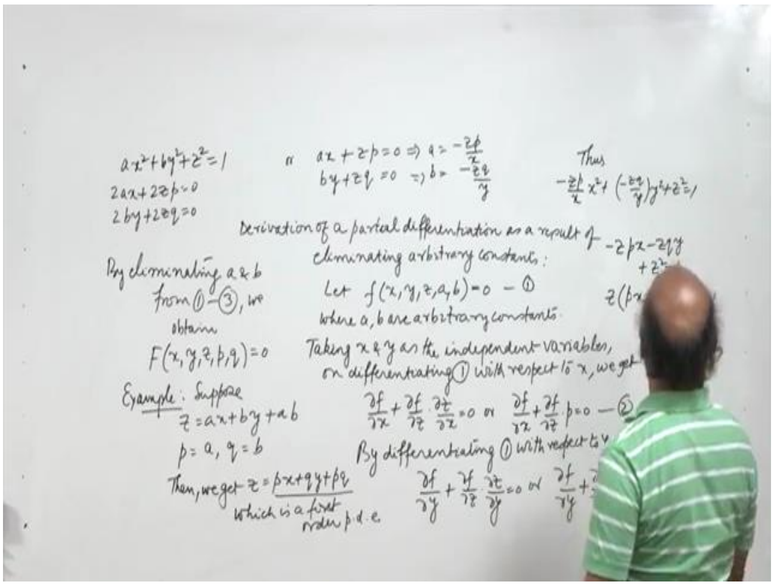

So, let us see derivation of a partial differential equation as result of eliminating arbitrary constants. So, let us say we are given a relation f x y z a b equal to 0. Where a and b are arbitrary constants. So, then taking x and y as the independent variables, we shall be getting the following, taking x and y as the independent variables.

On differentiating this equation 1 on differentiating 1 with respect to x, we get delta f by delta x plus delta f by delta z into delta z by delta x equal to 0 r by the notation p equal to delta z by delta x, we have this equation is equal to 0 and by differentiating 1 with respect to y we get r. So, we get 2 more equations. So, in order it does we have 3 equations, 1 2 3 from these 3 equations in general the 2 constants a and b can be eliminated and then we get. So, by eliminating a and b from 1 2 3 we obtain a relation between f x y z p q equal to 0 which is a first order linear first order differential equation that is linear in p and q.

So, which is a first order linear equation? So, whenever we eliminate 2 arbitrary constants occurring in the relation, it leads us to a first order equation. Say for example, suppose z is equal to a x plus b y plus a b. So, this will give you p equal to a and q equal to b. When you differentiate this equation with respect to x we get delta z by delta x equal to o which is p equal to a and delta z by delta y equal to v is which gives us q equal to b. So, we get 2 more equations p equal to a q equal to b now eliminate a and b from these 3 equations. So, then we get z equal to p x plus q y plus p q which is a first order differential equation, first order partial differential equation.

Now, if you take say for example, another example a x square plus b y square plus z square equal to 1, here also there are 2 arbitrary constants a and b. So, let us find the differential equation from here. So, 2 we differentiated with respect to x. So, 2 a x plus 2 z into p equal to 0, 2 b y plus 2 z into q equal to 0 when we differentiate this with respect to x, they arrive at 2 a x plus 2 z p equal to 0, and when we differentiate it with respect to y. We get 2 b y plus 2 z q equal to 0. So, we get r a x plus z into p equal to 0 and b y plus z into q equal to 0. So, we get here the value of a, a equal to minus z p divided by x and this will give you the value of b, b equal to minus z q divided by y. So, let us substitute these values of a and b in the given equation.

So, we have thus minus z p upon x into x square plus minus z q upon y into y square plus z square equal to 1. Or we shall have minus z p x minus z q y plus z square equal to 1, or we can write it as z p x z times p x plus q y equal to square minus 1. So, this is the first order differential equation that we get after eliminating the 2 arbitrary constants a and b. Now let us find let us derive a partial differential equation by eliminating arbitrary functions. So, we differentiate derivation of a partial differential equation as a result of eliminating arbitrary functions.

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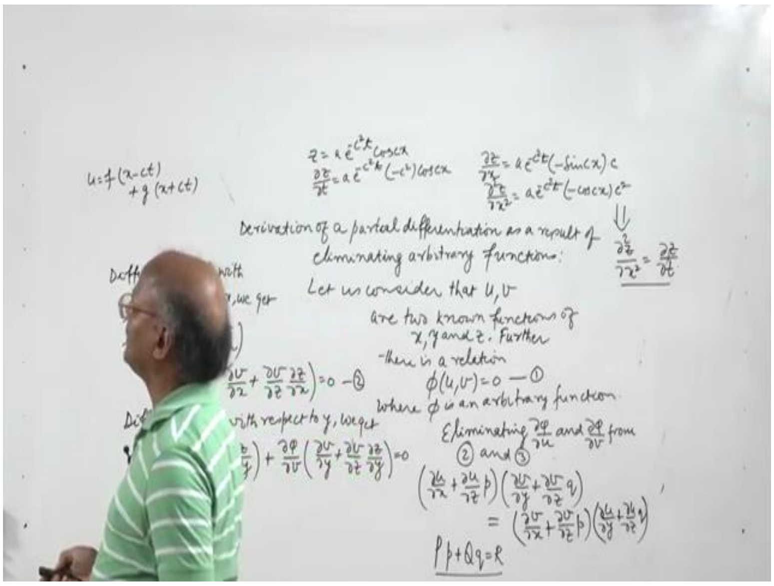

So, let us consider that u and v are 2 non functions of x and y and z. Let us consider that u comma v r 2 known functions of x y and z. And further there is a relation phi u v equal to 0 where phi is an arbitrary function.

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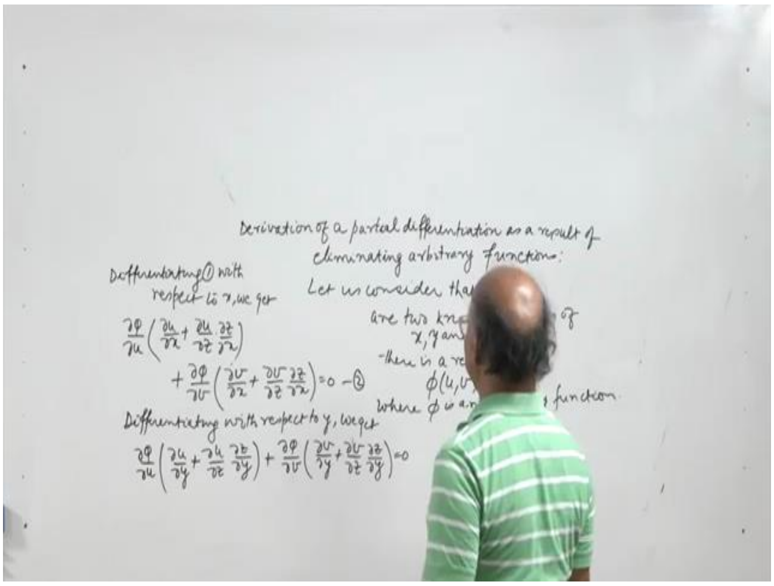

So, let us derive the partial differential equation from this relation phi u v equal to 0. So, differentiating equation 1 with respect to x, we get delta phi over delta u to delta u by delta x plus, similarly differentiating with respect to v we get with respect to y we get. So, differentiating with respect to y the equation 1, we shall arrive at, now we eliminate the known quantities delta phi delta u and delta phi over delta v from 2 and 3, eliminate we get this I can write as p into this I can write as q. So, this into this plus this into this equal to 0; so I can write it as equal to then I can divide.

So, I will get minus sign and minus sign we can also see equality here we will get equal to. So, this into that this into this; so v x plus v z p into u y plus u z into q which is a first order which is a first order partial differential equation and it can be expressed in the form p into p plus q into q equal to r, with suitable values of p q and r which can be obtained from here. So, this is a first order partial differential equation.

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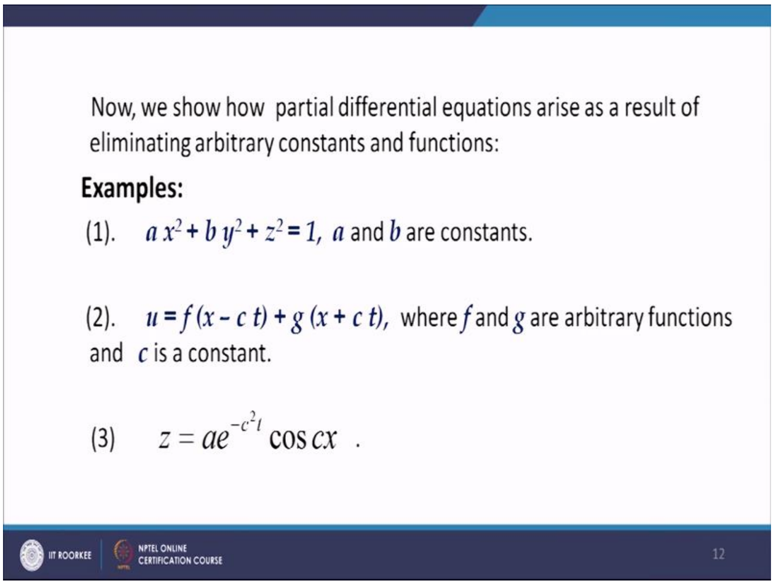

So, when you take say for example, u equal to f x minus c t plus g x plus c t, when you will eliminate the 2 arbitrary functions f and g here, you will arrive at a second order differential equation u equal to f x minus c t plus g x plus c t. This will lead you to second order differential equation. And similarly here while you take the third case equal to a e to the power minus c square t cos c x here there are 2 arbitrary constants a and c you will get a differential equation z equal to a times e to the power minus c square t cos c x.

So, this will give you partial derivative with respect to t as a e to the power minus c square t into minus c square cos c x, and when you differentiate that with respect to x what you get when you differentiate again what you get. So, if you compare this and this what you get is, we arrive at this equation a c e to the power minus c square t into c square cos c x that is delta z by delta t. So, we get this second order differential equation on eliminating the arbitrary constants which occur there.

Let us now go over to we are now going to see how the partial differential equation arise in a physical problem. So, let us consider the one dimensional heat equation. We shall see how we get the first order the second order differential equation delta u by delta t equal to c square delta square u by delta x square. So, let us consider the flow of heat by conduction in a uniform bar.

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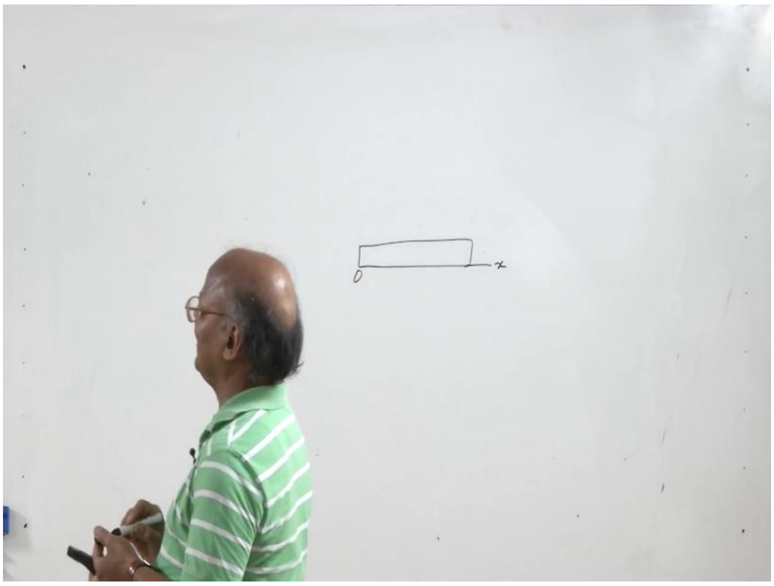

Let us consider a uniform bar and they are; so let us consider the flow of heat by conduction in a uniform bar whose sides are insulated.

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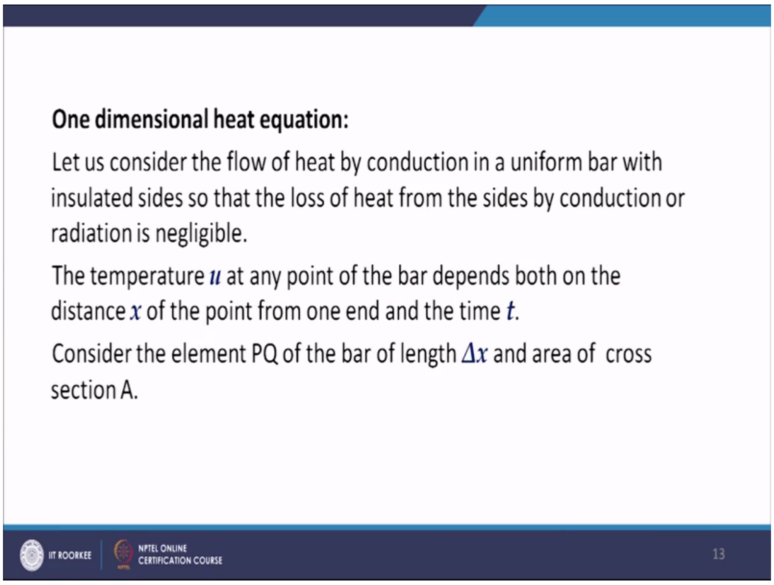

So, let us consider the sides of the bar to be insulated so that that there is no loss of heat by conduction r radiation from the sides. The temperature u, u is denoting the temperature here the temperature u will be function of x and t, x is the distance of the point under consideration from one end and t is the time.

So, consider the element p q of the bar, let us say o p is x. So, end o q is x plus delta x. So, let us consider the element p q of the bar of length delta x and area of cross section a, if K is the thermal conductivity of the material then the amount of heat that flows into the element across.

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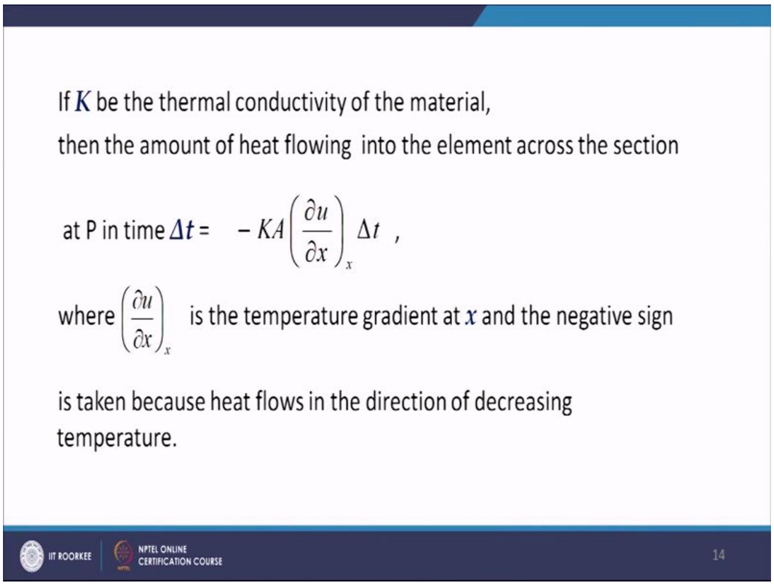

The cross section at p in time delta t is given by minus K a into delta u by delta x at into delta t where delta u and delta x is the temperature radiant at the point p and negative sign is taken because the heat is flowing in the direction of decreasing temperature. Now similarly the amount of heat that flows out of the element across the section at q in time delta t will be minus K a delta u by delta x at the point q which is x plus delta x into delta t.

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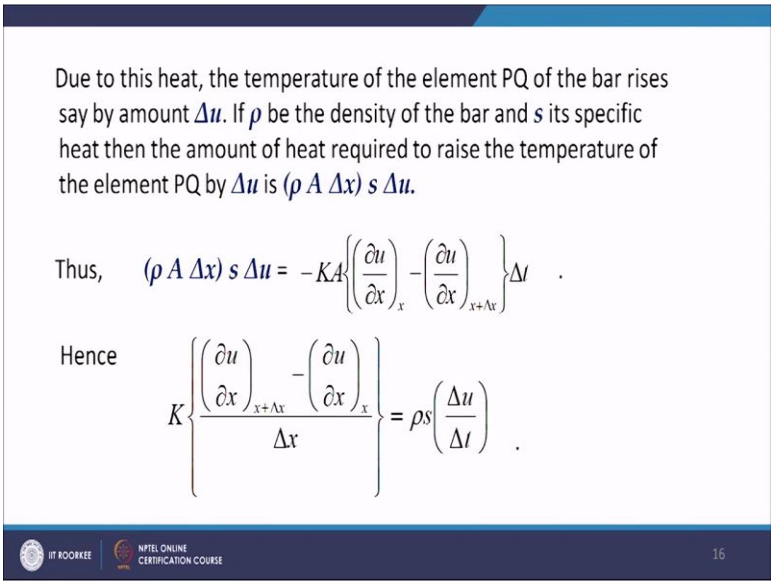

So, the amount of heat that is returned by the element p q in time delta t will be minus K a delta u by delta x at x minus delta u by delta x at x plus delta x into delta t. Now due to this heat the amount of temperature of the element p q will rise, say suppose it rises by an amount delta u then if role the density of the bar and s it is specific heat then the amount of heat required to raise the temperature of the element p q by delta u is rho into a into delta x, rho into a into delta x is the mass of the element p q because rho is the density a delta x is it is volume. So, rho into a into delta x is the mass into s into delta u. So, thus rho into a in to delta x into s delta u is equal to minus K delta u by delta x at x minus delta u by delta x at x plus delta x into delta t. Now dividing by delta and x and delta t both sides we arrive at K times this equal to rho s delta u by delta t.

Now, let us take delta x and delta t 2 go to 0. So, when we let delta x go to 0 and delta t goes to 0. Then this becomes delta square u by delta square, by the definition of limit. So, K times delta square u by delta x square becomes rho times s into delta u by delta t.

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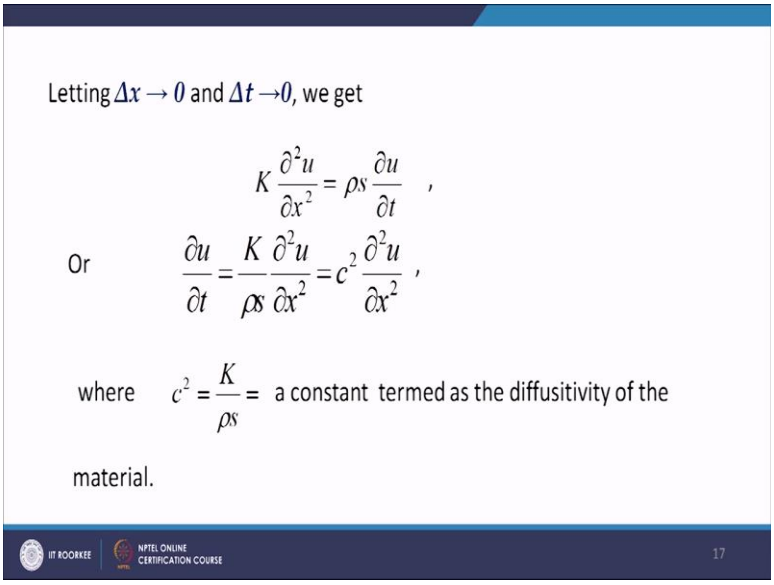

So, rho into s into delta u by delta t or we can say delta u by delta t or we can say delta u by delta t K by rho s into delta square u by delta x square. Since K rho and s are constants, you can denote them by c square. We can denote K by rho K by rho s by c square, so c square delta s square u by delta x square. So, this c square is a constant and it is termed as the diffusivity of the material.

So, this is how we have the differential equation of second order in in the physical problem that is the heat conduction the flow of heat in a bar of uniform length of uniform cross section. So, when we consider the flow of heat by conduction in uniform bar, it results in to this differential equation, delta u by delta t equal to c square delta x square u by delta x square. So, this is an example of physical example where the partial differential equation is occurring.

With that, I would like to conclude this lecture.

Thanks.