Mathematical methods and its applications
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Lecture – 15
Methods for finding particular integral for higher-order linear differential equations with constant coefficients

Hello friends. Welcome to my lecture on Methods for Finding Particular Integral for Higher-order Linear Differential Equations with Constant Coefficients.

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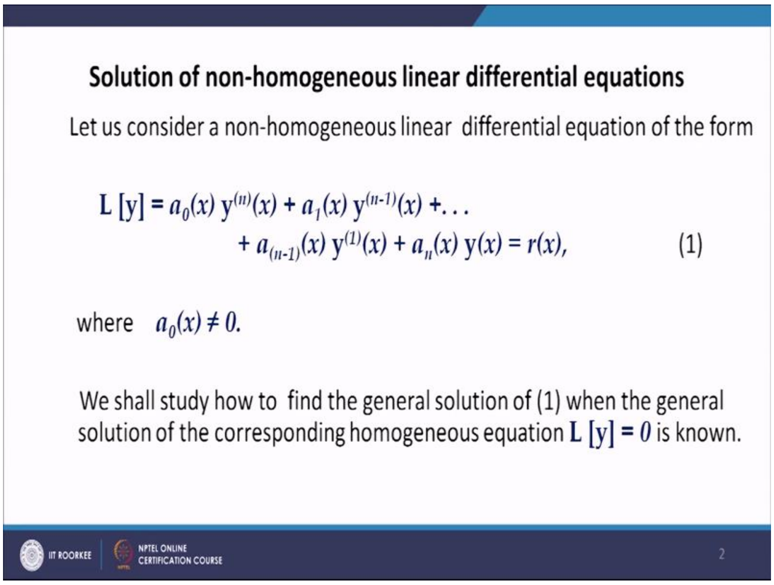

We shall begin with a non-homogenous linear differential equation which can be expressed as a naught x into nth derivative of y with respect to x plus a 1 x and minus 1 of the derivative of y with respect to x. Then n minus 1 x then d y by d x by 1 x means d y by d x plus a and x by x equal to r x where a naught x is not equal to 0. So, this is nth order linear differential equation which is non homogenous. We shall be finding general solution of this, assuming that the coefficients of the derivatives a naught x a 1 x a n minus 1 x and then a n x they are constants. So, we shall see how to find the general solution of equation 1, when the general solution of the corresponding homogenous equation is known.

So, we have in our previous lecture we have seen how to find the general solution of the homogenous nth order linear differential equation, So, so we shall see that, once we know the solution general solution of the nth order homogenous linear equation, then how to find the general solution of the nth order arbitrary non homogenous linear differential equation, So, let us see what we do is we have the following result.

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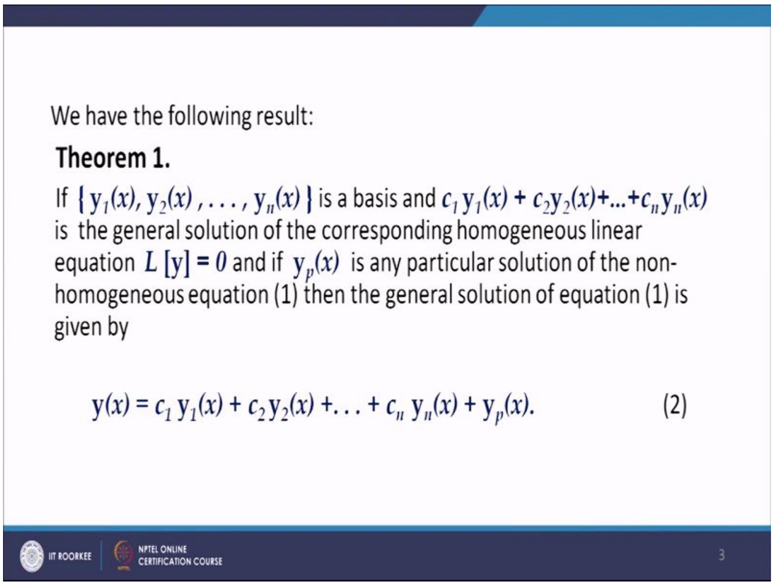

We actually due to this result for the case of n equal to 2, second order linear differential equation with constant coefficients. Now we are doing it for nth order differential equation, it is a that that result can easily be generalized. So, we have a this theorem is actually generalization of that result. So, if y 1 x y 2 x y n x is a basis means it is a fundamental set of solutions of equation of the homogenous equation associated with equation 1, if this is equation 1. So, y 1 x y 2 x and so on, y n x are actually n independent solutions of the homogenous equation here.

So, they form a fundamental set and fundamental set is also called as a basis of solutions. So, let y 1 x y 2 x and so on, y n x be a basis and c 1 y 1 plus c 2 y 2 and c n y n x is the general solution of the corresponding homogenous linear equation l y equal to 0. Then if y p x is any particular solution of the non-homogenous equation 1. The general solution of equation 1 can be expressed as sum of the general solution of the associated homogenous linear equation that is c 1 y 1 plus c 2 y 2 and so on, c n y n plus y p x. So, this is the result.

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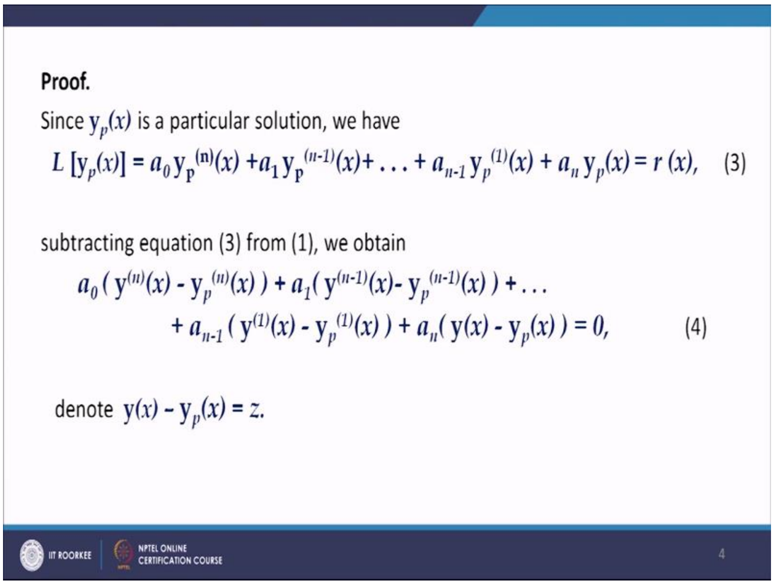

Let us see how we prove it the proof is very simple. So, since y p x is a particular solution of equation 1 it will satisfy equation 1. So, l y p x will be equal to a naught y p n x a 1 y p n minus 1 x and so on, a n minus 1 y p x plus a n y p x equal to r x.

Now, let us subtract this equation 3 from equation 1. So, when you subtract this equation 3 from equation 1, what we will get is the following. We will have a naught. So, these a naught a 1 a 2 a n minus 1 a n they are all now constants. They are no longer functions of x. So, a naught y n x minus y n minus y y p n x, plus a 1 y n minus 1 x, minus y p n minus x n minus 1 y n x, a p y n x, a n y n x, y x minus y p x, equal to 0. Denote let us denote y x minus y p x y z.

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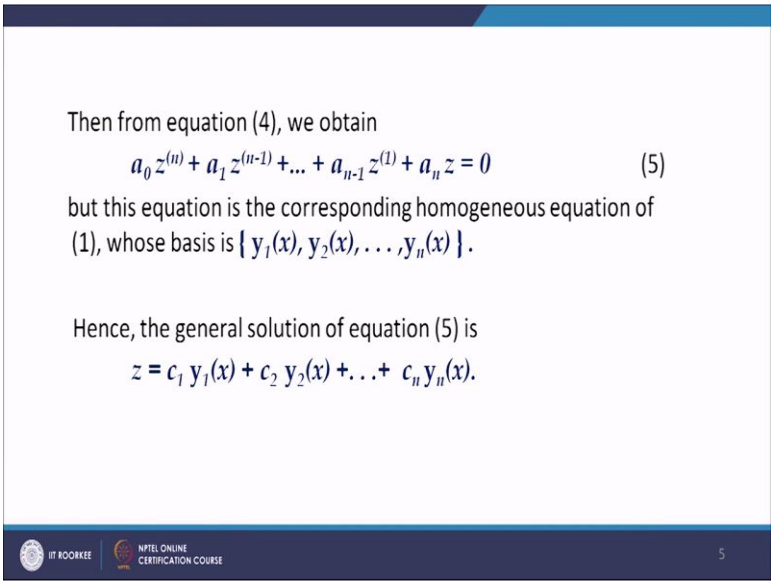

So, if we do that what we will get is a naught z n plus a 1, z n minus 1, a n minus 1 z 1, plus a n z equal to 0. But this equation is the corresponding homogenous equation of one, whose basis is y n x y 2 x and so on, y n x. So, the general solution of equation 5 is z equal to c 1 y 1 plus c 2 y 2 and so on, c n y n now z is equal to y x minus y p x. So, we shall have y x minus y p x equal to this quantity c 1 y 1 plus c 2 y 2 and so on, be n c n y n, and therefore, y x equal to c 1 y 1 plus c 2 y 2 c n y n plus y p x.

Now, this solution contains n arbitrary constants. So, it must be the general solution of equation 1. So, this is the proof. Let us now look do some examples on this.

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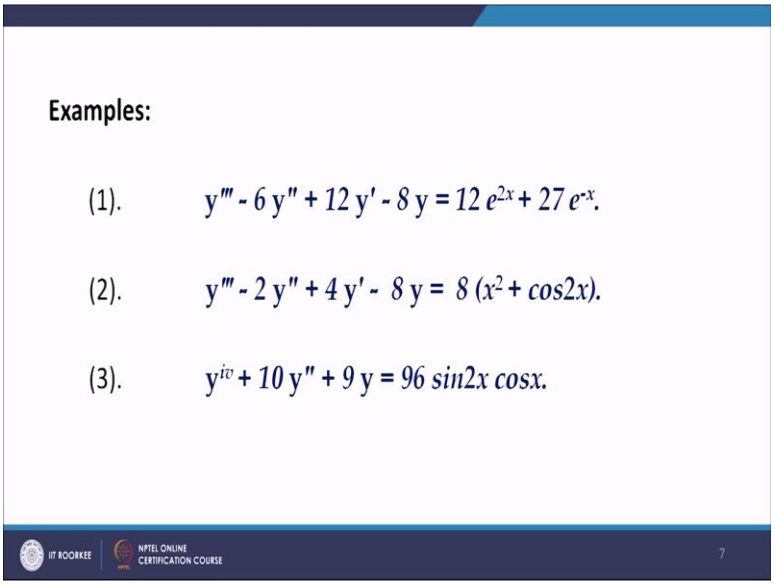

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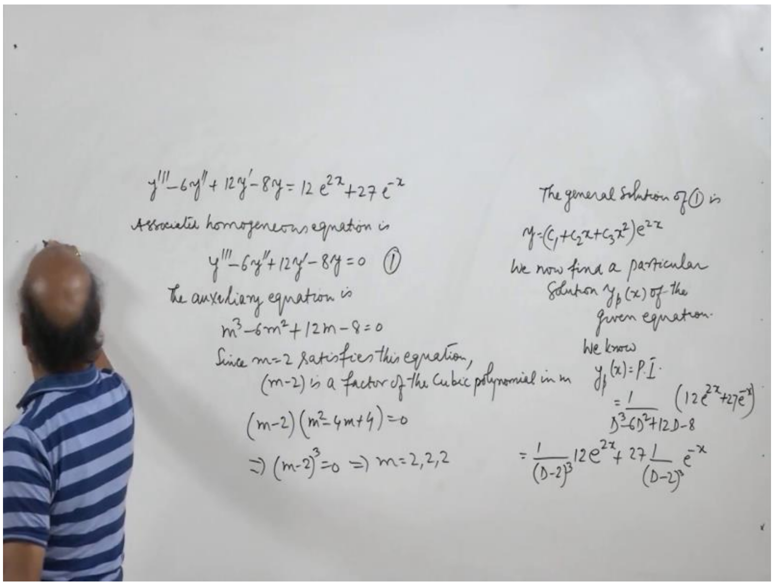

So, first example we will do is a is a is a linear differential equation of third order with constant coefficients. So, y triple prime, y triple prime minus 6 y double prime plus 12 y dash minus 8 y equals 12 e to the power 2 x plus 27 e to the power minus x.

Let us first write the general solution of the associated homogenous linear equation, So, associated homogenous equation is, y triple prime minus 6 y prime, plus 12 y prime, 6 y double prime minus 12 y plus 12 y prime minus 8 y equal to 0. So, this is the associated homogenous linear differential equation of third order, in order to solve this equation, we write the corresponding auxiliary equation. The auxiliary equation is m cube minus 6 m square, plus 12 m minus 8 equal to 0. Now we can notice that this is a cubic equation in m. So, we shall have to fine a value of m which satisfies this equation; obviously, m equal to 1 does not satisfies, but m equal to 2 satisfies this equation since m equal to 2 satisfies this equation m minus 2, is a factor of the cubic polynomial in m.

So, we will have m minus 2, this is a factor of this equation, m minus 2 and then we have m minus m square minus 4 m plus 4. We can easily check this that the other factor is m square minus 4 m plus 4. And this gives you m minus 2 whole cube equal to 0. So, we have m equal to, so this value of m that is 2 across thrice. So, it is a case of repeated roots and therefore, the general solution of equation 2 this is equation 2 or rather I would write it as equation 1, general solution of equation 1 is y equal to c 1 plus c 2 x plus c 3 x, square e to the power 2 x. Now we shall find a particular solution of the given equation, so we shall we now find a particular solution y p x of the given equation.

So, y p x we know, y p x is nothing, but the particular integral p i. And this is equal to 1 over d cube minus 6 d square plus 12 d minus 8 operating on 12 e to the power 2 x plus 27 e to the power minus x. Now this is equal to 1 over d minus 2 whole cube, 12 e to the power 2 x, plus 27 1 over d minus 2 whole cube, operating on e to the power minus x.

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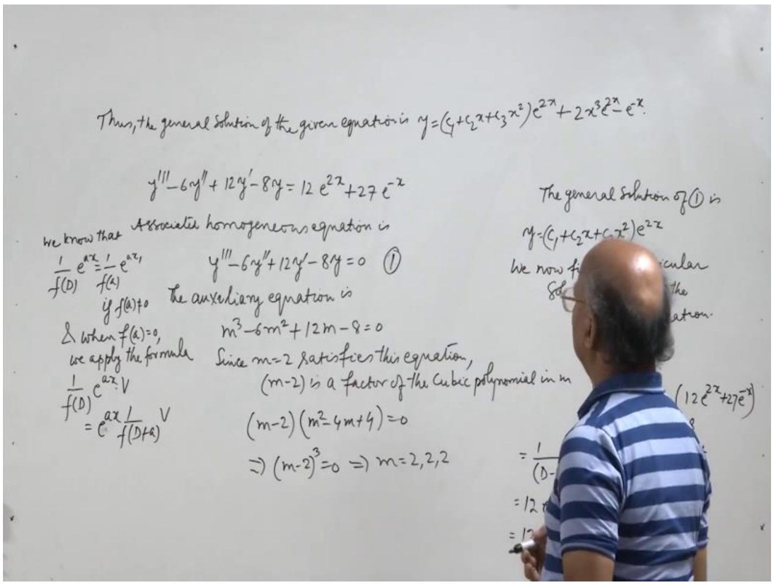

Now let us apply the formula we know that 1 over f d e to the power a x is equal to 1 over f a e to the power a x. If f a is non 0, and when f a becomes 0, we apply the formula

1 over f d e to the power a x into v. Where v is any function of x as 1 over equal to e to the power a x 1 over f d plus a operating on v. So, we apply this formula.

Now, here we see that this 12 is a constant, when we replace b by 2 here d minus 2 becomes 0. So, we shall apply the formula 1 over f d for a x into v equal to 8 for x 1 over f d plus a d. So, this is e to the power 2 x 1 over d plus 2 minus 2. So, we have d cube operating on 1, and here we shall have 27 d will be replaced by minus 1. So, we will have minus 3 to the power 3. Now here know one d reference represents d over d x. So, 1 over d represents integral with respect to x. So, we have to integrate one thrice with respect to x. So, once we when we integrate it ones we get x, when we integrate it again we get x square by 2 and when we integrate it again we get x cube by 6. So, 12 e to the power 2 x x cube by 6 and here we have this is e to the power minus x we have to write. So, this we have minus 20. So, this is minus 27. So, minus e to the power minus x we have.

So, this is 2 x cube; 2 x cube e to the power 2 x minus e to the power minus x. And thus the general solution is y equal to solution of the homogenous equation that is c 1 plus c 2 x plus c 3 x square e to the power 2 x plus y b x. Which is 2 x cube e to the power 2 x minus e to the power minus x. So, this is the general solution of the equation given in example one. Now let us see how we will solve equation 2.

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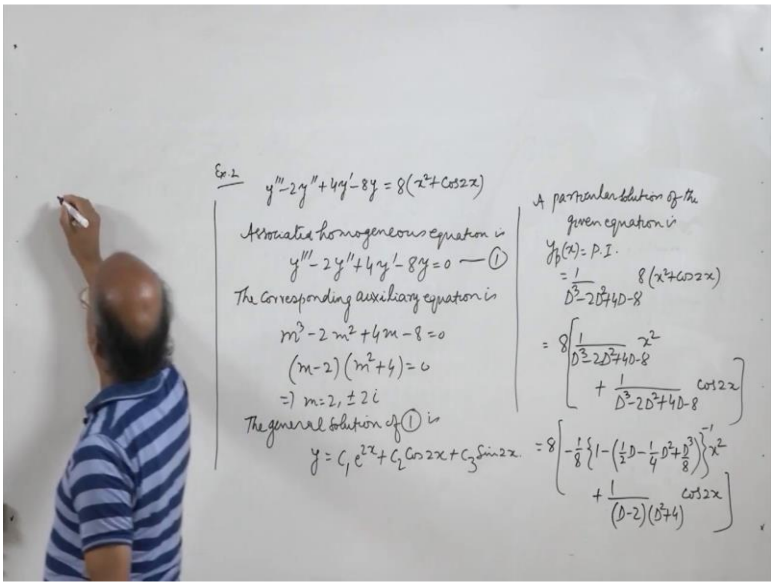

So, let us look at equation 3, equation example 2. This is y triple dash minus 2 y double dash plus 4 y dash, minus 8 y equal to 8 times x square plus cos 2 x. So, we will again find the general solution of the associated homogenous linear differential equation, So, associated homogenous equation is y triple prime minus 2 y double dash plus 4 y dash minus 8 y equal to 0.

Now, in order to find the general solution of this we write the corresponding auxiliary equation, the corresponding auxiliary equation is m cube minus 2 m square plus 4 m minus 8 equal to 0. Now it is easy to solve because we see that it is factors are m minus 2 into m square plus 4 equal to 0. So, we get m equal to 2 and plus minus 2 y. So, one root is real which is 2 and the other 2 roots are complex conjugate they are 2 i and minus 2 i. So, the general solution is general solution of the equation 1, corresponding that here alpha is equal to 0, 0 beta is equal to 2. So, we have e to the power alpha x that is e to the power 0 that is 1. So, we have c 2 cos 2 x plus c 3 sin 2 f. This is the general solution of the associated homogenous equation or you can say this is the complementary function of the given equation.

Now, let us find the particular integral a particular solution of the given equation is y p x equal to p i particular integral which is 1 over d cube, minus 2 d square, plus 4 d minus 8, operating on 8 x square plus cos 2 x. So, I can write it as 8 times 1 over d cube minus 2 d square plus 4 d minus 8 operating on x square plus 1 over d cube minus 2 d square plus 4 d minus 8 operating on 8 cos 2 x. So, 8 I can write outside. So, now, here f d is d cube minus 2 d square plus 4 d minus 8. 1 over f d we have to expand in descending powers of d.

So, we will write 8 times minus 1 over 8 and then we write 1 minus 4 d by 8, 4 d by 8 we shall write as minus 4 d by 8. So, 1 by 2 d and 9 minus 1 by 4 d square and then d cube by 8 raise to the power minus 1 operating on x square. So, what we do is we have to write it in the ascending powers of d. So, we write 1 over minus 8 outside. So, that we have this expression 1 minus this quantity divided by 8. So, 1 by 2 d minus 1 by 4 2 by our 1 by 4 d square plus d cube by 8 and here what we do is we replace d square by minus a square.

So, minus 4 so; that means, here we have d square into d square is replaced by minus 2 square; so minus 4. So, minus 4 d d square by minus a square there is minus 4. So, we have plus 8 and here we have 4 d minus 8. So, this becomes 0. So, this becomes 0. So, what we do is, we factorise it as this factors are d minus 2 d square plus 4. So, d minus 2 d square plus 4 cos 2 x. Because of this factor d square plus 4 when we replace d square by minus 4 it is becoming 0.

So, what we do is we operate by d plus 2 and 1 over d plus 2 in the numerator and denominator. So, we operate by that. So, so let me write it separately we. So, let us find the expression for this first.

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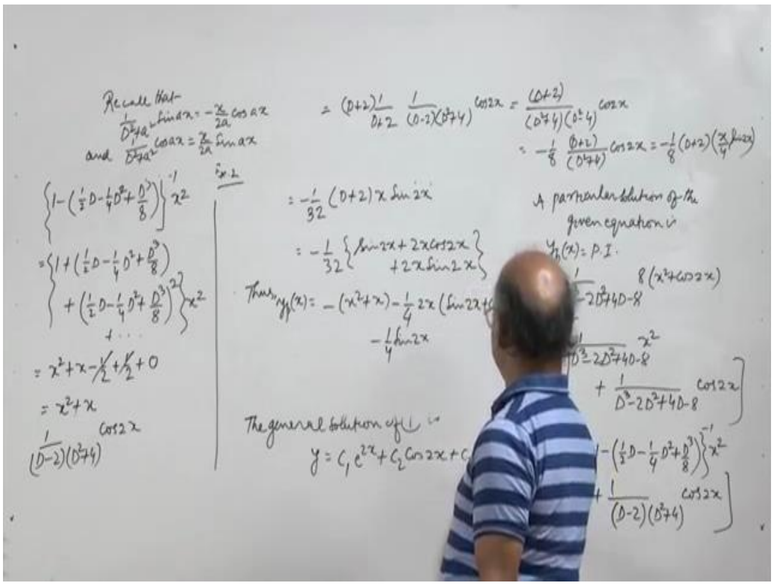

So, 1 minus 1 by 2 d minus 1 4 d square plus d cube by 8 raise to the power minus 1 x square. So, this let us expand. So, we get 1 plus 1 by 2 d minus 1 by 4 d square plus d cube by 8, 1 plus x square 1 minus when we express 1 minus x to the power minus 1 we get 1 plus x plus x square. So, we get 1 by 2 d minus 1 by 4 d square plus d cube by 8 raise to the power 2 and so on this operating on x square.

So, this will give you x square 1 by 2 d of x square d of x square is 2 x; so 2 x by 2. So, we get x here, we get x here and then when 1 by 4 d square d square of x square is 2; so minus 2 by 4. So, we get minus half d cube of x square is 0. So, we get 0. So, we get 1 by 2 d of x square that is 1 by, So, this is 1 by 4 when we squared this 1 by 4 d square d square of x square is 2 2. So, 2 by 4, we get 1 by 2. The other terms contain powers of d higher than 2. So, there they will give you 0. So, we will just get this. So, this will cancel with this we get x square plus x. So, now, let us find out 1 over d minus 2 into d square plus 4 cos 2 x.

Let us recall that; what we will do is we operate by d plus 2, and 1 over d plus 2, d plus 2 and 1 over d plus 2 are inverse operators. So, their total effect is nil; so this 1 over d minus 2 and then d square plus 4. Now this is how much? This is d plus 2 I think I will reshuffle this d square plus 4, and then d square minus 4, cos 2 x. Now d square is in this factor d square is replaced by minus a square we get minus 4 minus 4 minus 8. So, minus 1 by 8 d plus 2 over d square plus 4 cos 2 x.

Now, let us recall that 1 over d square plus a square sin a x is equal to minus x by 2 a cos x and 1 over d square plus a square cos a x is equal to x by 2 a sin x. So, what we will get 1 over d square plus 4 cos 2 x will be x by 4 sin 2 x. So, minus 1 by 8 d plus 2 and we shall get x by 4 x by 2 a, a equal to 2. So, x by 4 sin 2 x. Now what we will do? So, this is equal to minus 1 by 32 d plus 2 operating on x sin 2 x. So, this is equal to minus 1 by 32, d of x and 2 x. When you d means d over d x d over d x operating on x sin 2 x will give you sin 2 x plus 2 x cos 2 x and then we will get 2 x sin 2 x. So, let us get this 2, let us put these values there. What we will get is thus y p x is equal to 8 multiplied by minus 1 by 8 is minus 1. So, we get minus x square plus x. And then 8 times minus 1 by 32; so minus 1 by 4, minus 1 by 4, and then we have 2 x times sin 2 x plus cos 2 x. Then we have 8 times minus 1 by 32. So, minus 1 by 4 sin 2 x.

Now, when we get this y p x to the general solution of the homogenous equation, that the general solution of homogenous equation is c 1 e to the power 2 x plus c 2 cos 2 x plus c 3 sin 2 x. Then minus 1 by 4 sin 2 x will get absorbed in this part of the general solution of 1 c 3 sin 2 x. So, we shall not consider this.

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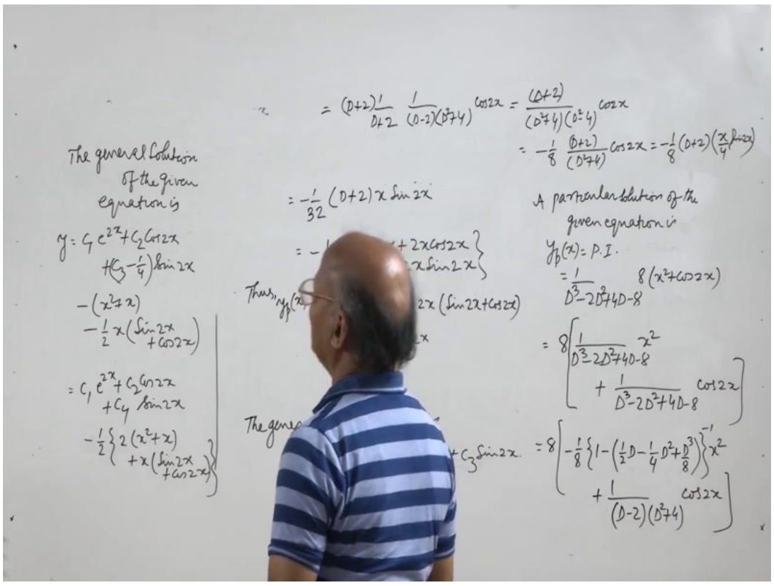

And therefore, we shall write the general solution of equation 1 as, So, the general solution of the given equation is y equal to c 1 e to the power 2 x, plus c 2 cos 2 x plus c 3 sin 2 x, c 3 will be actually c 3 minus 1 by 4 we can write it as c 3 minus 1 by 4 sin 2 x, and then we have minus x square plus x minus 1 by 2, into x sin 2 x plus cos 2 x. So, c 3 minus 1 by 4 can be replaced by a new arbitrary constant, and we write c 1 e to the power 2 x plus c 2 cos 2 x plus say suppose some other constant we can write c 4 sin 2 x. And then we can write combine this whole thing and write minus 1 by 2, times 2 x square plus x. And then we can write plus x times sin 2 x plus cos 2 x.

So, this is the general solution of the equation given in example 2. Now will problem given in example 3 is a is a bit different one. So, let us try that one also let us see how we solve that third question.

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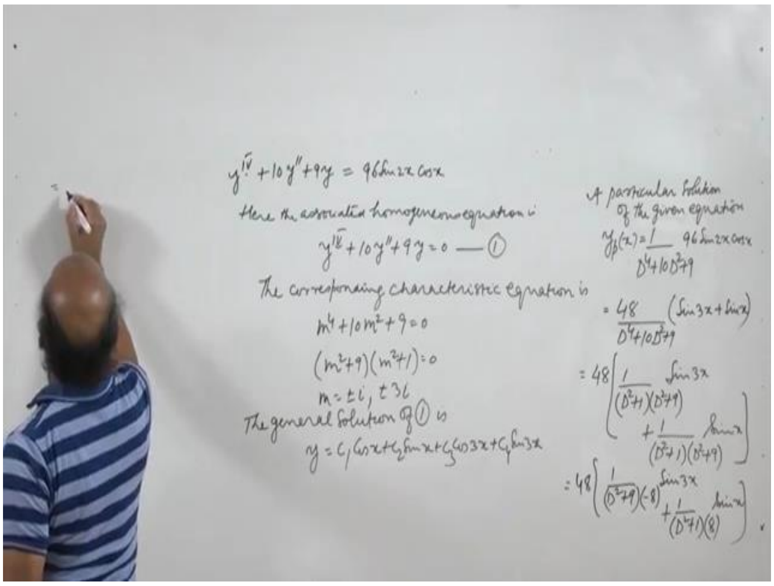

So, in third question is y 4, 4th derivative of y with respect to x 10 y double dash plus 9 y, is equal to 96 sin 2 x cos x. So, here the associated homogenous equation is y 4, plus 10 y double dash plus 9 y equal to 0. Let us write the corresponding characteristic equation or can say auxiliary equation. They are same things. So, the corresponding characteristic equation is m 4 plus 10 m square plus 9 equal to 0. Now it is it can be solved easily the factors are m square plus 9, into m square plus 1. You can take m square equal to t then it, becomes second degree second degree is a quadratic equation in t and you can solve it.

So, replace t by m square you get this now m is equal to plus minus i plus minus 3 i are it is roots. So, the general solution of the equation 1 is y equal to c 1 cos x plus c 2 sin x plus c 3 cos 3 x, plus c 4 sin 3 x. Let us now find particular integral or we can say particular solution of the given equation. 1 over d 4 plus 10 d square plus 9, operating on 96 sin 2 x cos x. Now the formula that we have studied, there they will none of those formulas can be applied to get the particular integral here, because the formula that we have done is that 1 over f d operating on sin a x 1 over f d operating on cos a x those formulas. So, here what you do is we write it as 48 times 1 over d 4 plus 10 d square plus 9. Now 2 times sin 2 x cos x can be written as sin 3 x plus sin x sin c plus sin d twice sin c plus d by 2 into cos c minus d by 2. So, we have this and this.

Then can be written as so, 48 times let us apply this operator 1 over d 4 plus 10 d square plus 9 on each of the terms here. So, we will write it as factors let us write 1 over d square plus 1 d square plus 9 operating on sin 3 x 1 over d square plus 1 d square plus 9 operating on sin x. So, this is equal to 48, now 1 over d square plus 9, cannot be operated on sin 3 x because d square plus 9 becomes 0 when d square is plus by minus 3 square. So, 1 over d square plus 1 is operated on sin 3 x. So, 1 over d square plus 9 this d square is replaced by minus nine. So, we get minus 8 here and we get sin 3 x and we have 1 over d square plus 9, is operated on sin x first. So, d square is a plus by minus 1 square. So, we get 1 over d square plus 1 into 8 sin x.

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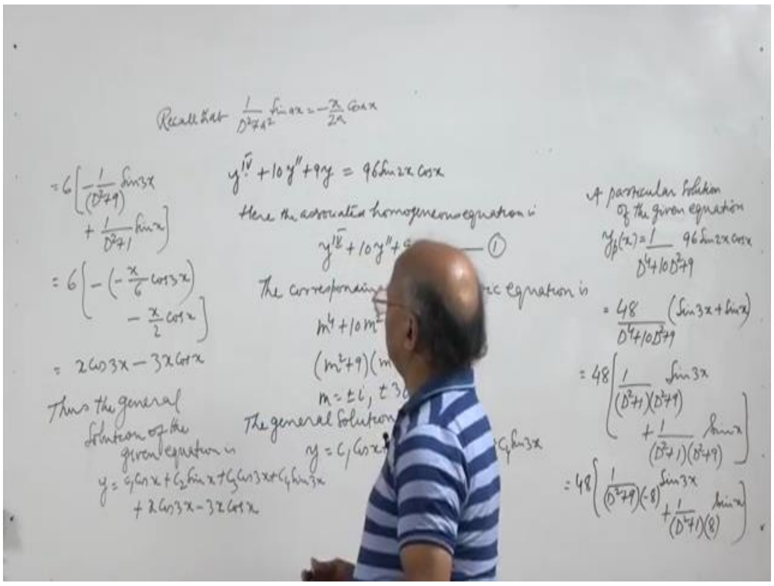

So, this is equal to 6 times 1 over minus 1 over d square plus 9, sin 3 x plus 1 over d square plus 1, operating on sin x. Now let us recall that 1 over d square plus a square, sin a x is minus x by 2 a cos x. So, this is equal to 6 times minus into minus x by 6 because a is 3 here, cos 3 x and here we get minus x by 2 cos x. So, this is x cos 3 x, minus 3 x cos x. And thus the general solution is y equal to c 1 cos x c 2 sin x c 3 cos 3 x c 4 sin 3 x plus x cos 3 x minus 3 x cos x. So, this is the solution of the general solution of the equation given in example 3.

With that I will like to conclude my lecture. In my next lecture we will discuss about the partial differential equations.

Thank you for your attention.