Mathematical methods and its applications
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Lecture – 14
Solution of higher-order homogenous
linear differential equation with constant coefficients

Hello friends. Welcome to my lecture on Solution of Higher-order Homogenous Linear Differential Equations with Constant Coefficients. So far we considered second order homogenous linear differential equation with constant coefficients. Now we shall generalize the idea to higher-order homogenous linear differential equation with constant coefficients.

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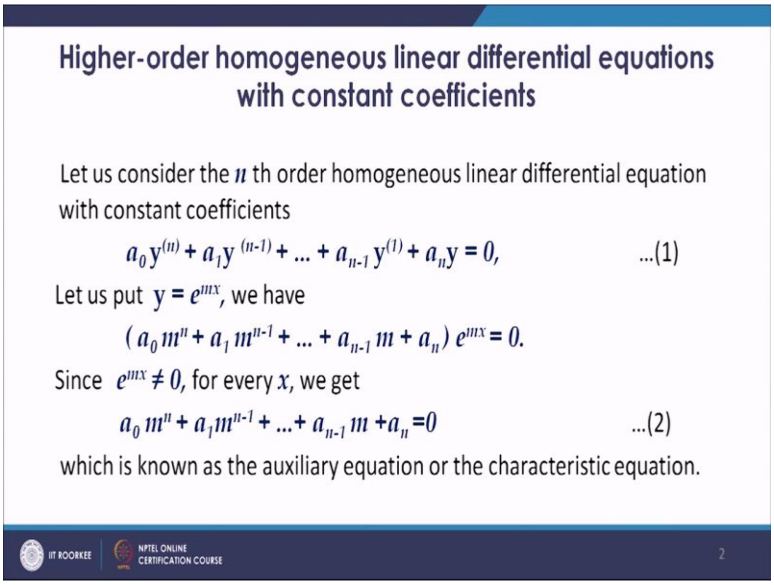

So, let us consider the nth order homogenous linear differential equation with constant coefficients a naught y n, y n is nth derivative of y with respect to x, plus a 1 y n minus 1 a to y n minus 2 and so on a n minus 1 y 1 plus a n y equal to 0. Where a naught a 1 a 2 a n minus 1 a n are constant coefficients and y r denotes the rth derivative of y with respect to x.

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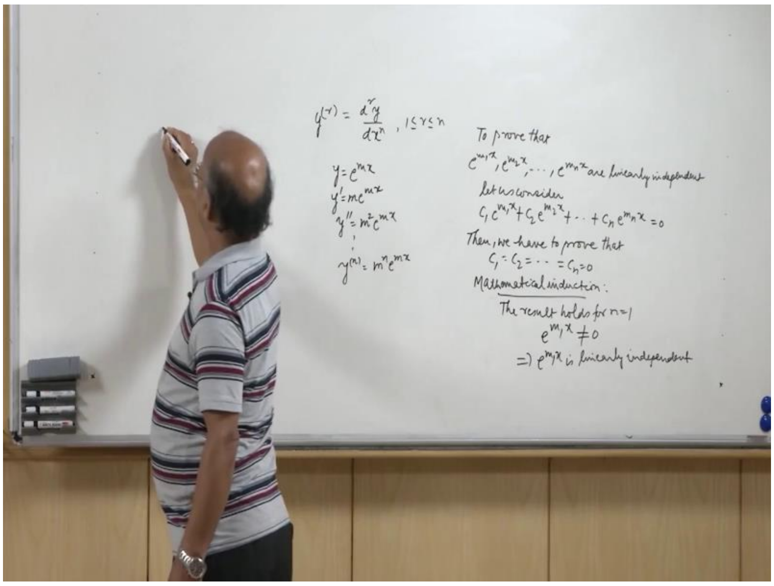

So, one less than or equal to r less than or equal to n, this denotation for the derivatives of y with respect to x; now, let us put y equal to e to the power m x. We have seen in the case of the second order linear differential equation with constant coefficients that in the case of first order linear differential equation it turned out that exponential solution is a solution of that homogenous linear differential equation of first order. So, we tried why not let us see for exponential solutions of second order. And it indeed turned out that the second order linear differential equation have exponential functions as solutions.

So, let us put y equal to e to the power m x in equation 1. So, then what do we have if you y equal to e to the power m x then y dash is m times e to the power m x, y double dash is m square times e to the power m x and so on nth derivative of y gives you m to the power n e to the power m x. So, when you put y equal to e to the power m x in equation 1, but you get is a naught m to the power n plus a 1 y m to the power n minus 1 and so on, n minus 1 m plus a n times e to the power m x is equal to 0.

Now, it is very well known that e to the power m x is never 0 for l e x belonging to r. So, we will get a naught m to the power n plus a 1 m to the power n minus 1 and so on n minus 1 m plus n equal to 0 which is known as the auxiliary equation or the characteristic equation. So, this equation will have n roots counting the multiplicity. So, those n roots if those n roots are all real and distinct. So, there are 3 possibilities, the n roots of the equation auxiliary equations are all real and distinct. Then there is the second case where the some of the roots may be equal and then the third possibility where some of the roots may be complex. So, let us discuss each case one by one.

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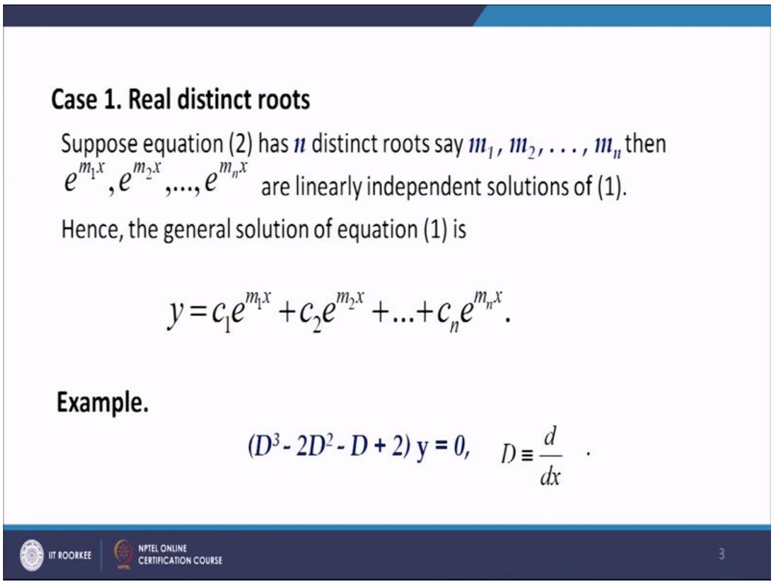

So, suppose the n roots of the auxiliary equation are m 1 m 2 m n and they are all distinct. Then the corresponding solutions of the equation 1 will be e to the power m 1 x e to the power m 2 x and so on, e to the power m 1 x and they are all linearly independent because if you take to prove that these n solutions e to the power m 1 x e to the power m 2 x and so on e to the m 1 x, they are linearly independent l I.

Let us consider c 1 e to the power m 1 x plus c 2 e to the m 2 x and so on, c n e to the power m 1 x equal to 0. Then we have to prove that c 1 c 2 c n are equal to 0. We can prove this by mathematical induction. Suppose if we can prove this by mathematical induction. So, the result clearly holds for n equal to 1, result holds for n equal to 1, because when n is equal to 1 e to the power m 1 x is never 0. So, the result holds for n equal to 1. So, this implies that e to the power m 1 x is linearly independent.

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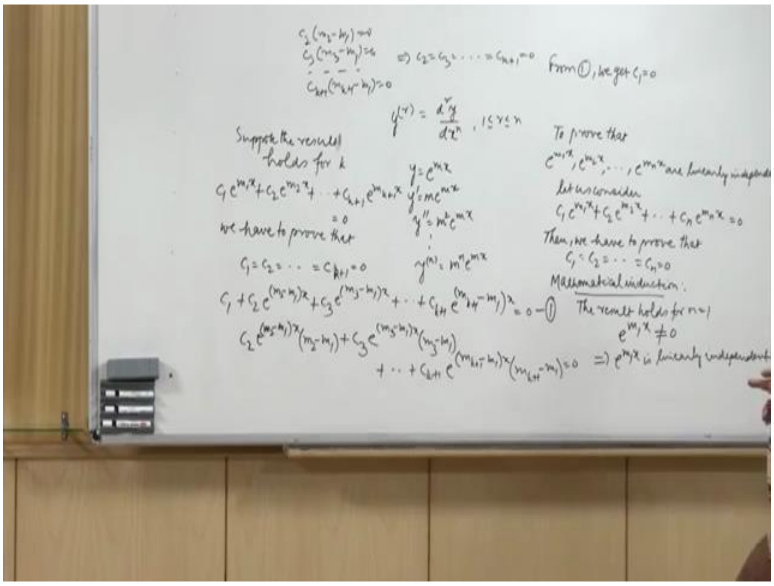

So, suppose the result holds for k functions that are then we have to show that the result holds for k plus 1 function. So, c 1 e to the power m 1 x plus c 2 e to the power m 2 x and so on c k plus 1 e to the power c 1 e to the power m 1 x plus c 2 e to the power m 2 x and so on c k plus 1 e to the power m k plus 1 x equal to 0, we have to prove that c 1 c 2 c k plus 1 equal to 0.

Now, since e to the power m 1 x is never 0, we can divide this equation by e to the m 1 x. So, dividing we have c 1 e to the plus c 2 e to the power m 2 m 1 into x, plus c 3 e to the power m 3 minus m 1 into x n; so on c k plus 1 e to the power m k plus 1 minus m 1 into x equal to 0. Now the result holds for c 1 c 2 c k plus 1. So, whenever and the k functions is equal to 0. Now we can differentiate this with respect to x, let us differentiate this with respect to x, then we will get c 2 e to the power m 2 minus m 1 x, into m 2 minus m 1 plus c 3 e to the power m 3 minus m 1, into x into m 3 minus m 1 and so on c k plus 1 e to the power m k plus 1 minus m 1 into x into m k plus 1 minus m 1 equal to 0.

So, e to the power m 2 minus m 1 x e to the power m 3 minus m 1 x e to the power m k plus 1 minus m 1 x are functions, where these m 2 minus m 1 m 3 minus m 1 m k plus 1 minus m 1 they are all distinct, and we have assumed the result for k functions. So, these are k functions. So, they are linearly independent. So, their coefficients must be 0. So, c 2 times m 2 minus m 1 equal to 0, c 3 times m 3 minus m 1 equal to 0, and so on. C k plus 1 m k plus 1 minus m 1 equal to 0. Since m 2 is m 1 m 2 m 3 and so on m k plus 1 are all distinct, this implies that c 2 c 3 and so on c k plus 1 are all 0s, and from this equation when you put c 2 c 3 c k plus 1 equal to 0, we get So, this equation let us call it as 1. So, from one we get c 1 equal to 0.

So, c 1 c 2 c 3 and so on c k plus 1 equal to 0. So, the result holds for k by mathematical induction it holds for all k.

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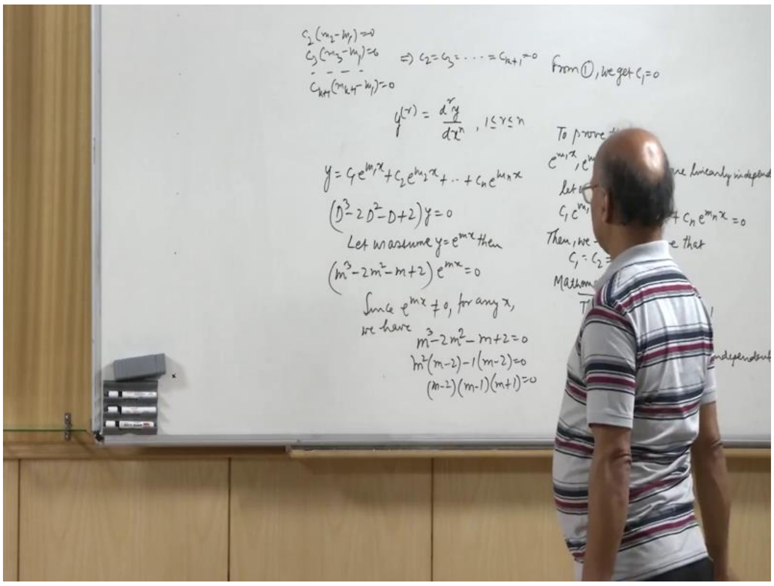

Here the n functions e to the power m 1 x e to the power m 2 x and for e to the power m 1 x are linearly independent. And therefore, the general solution we can write as, So, general solution can be written as y equal to c 1 e to the power m 1 x plus c 2 e to the m 2 x and so on, c n e to the power m 1 x when the roots of the auxiliary equation are all real and distinct. So, let us say for example, D cube minus 2 D square minus D plus 2 y equal to 0 D cube minus 2 D square minus D plus 2 y equal to 0, D here represents D over D x. So, it is a third order linear differential equation with constant coefficients. So, let us assume y equal to e to the power m x, then we get m cube minus 2 m square minus m plus 2 e to the power m x equal to 0.

Since e to the power m x is not 0, for any x, we have the auxiliary equation m cube minus 2 m square minus m plus 2 equal to 0. Now we can easily factorize this equation. So, m is square times m minus 2 minus 1 times n minus 2. So, the factors are m minus 2 and m square minus 1.

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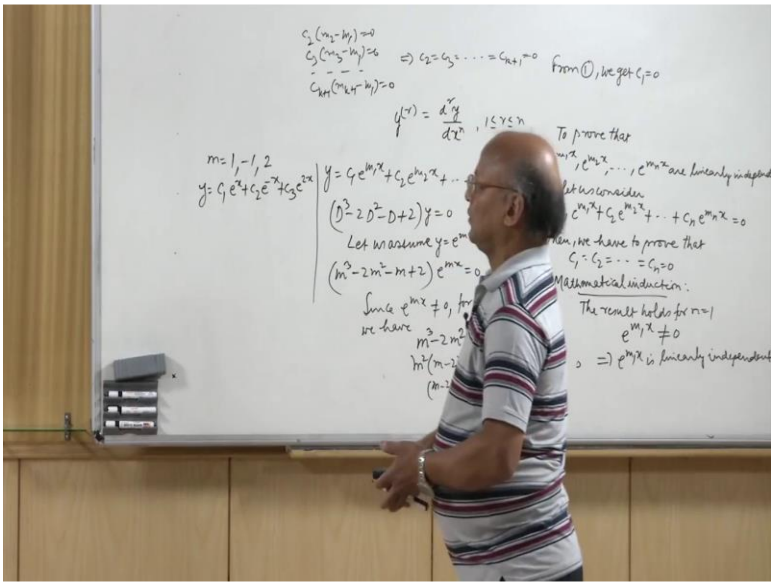

So, we can write m minus 1 into m plus 1 equal to 0. The 3 roots of the auxiliary equation are 1 minus 1, 1 minus and 2. So, these are the auxiliary equation as real and distinct roots. So, with general solution will be y equal to c 1 e to the power x c 2 e to the power minus x plus c 3 e to the power 2 x in this case. Now let us move to second possibility in the second case, we have real multiple roots, when certain root occurs more than one.

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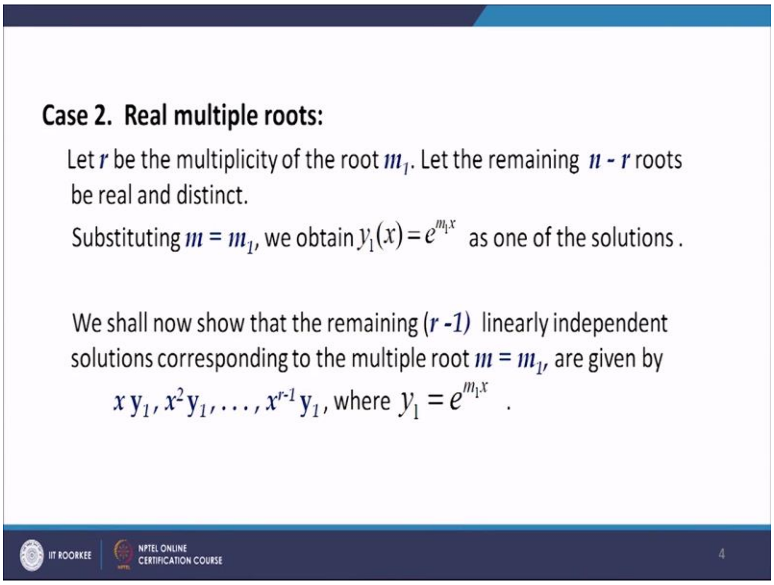

So, let us let r be the multiplicity of a root say m equal to m 1 and let us assume that the remaining n minus r roots be real and distinct.

So, corresponding to the distinct roots n minus r, n minus r roots the corresponding n minus r real and distinct roots we can write the corresponding part of the complementary function like we wrote in the case of case 1. So, when we have r roots which are equal how we will find the corresponding part of the complementary function let us see that here. So, assume that m equal to m 1 occurs r times m equal to m 1 occurs r times. So, one solution of the auxiliary equation is one solution of the equation 1 will be y 1 x equal to e to the m 1 x let us try to find the other r minus 1 linearly independent solutions of the equation 1.

So, we shall now show that the remaining r minus 1 linearly independent solutions corresponding to the multiple root m equal to m 1 are given by x into y 1, that is x into e to the power m 1 x x square into y 1 that is x square into e to the power m 1 x and so on, x to the power r minus 1 e to the power m 1 x.

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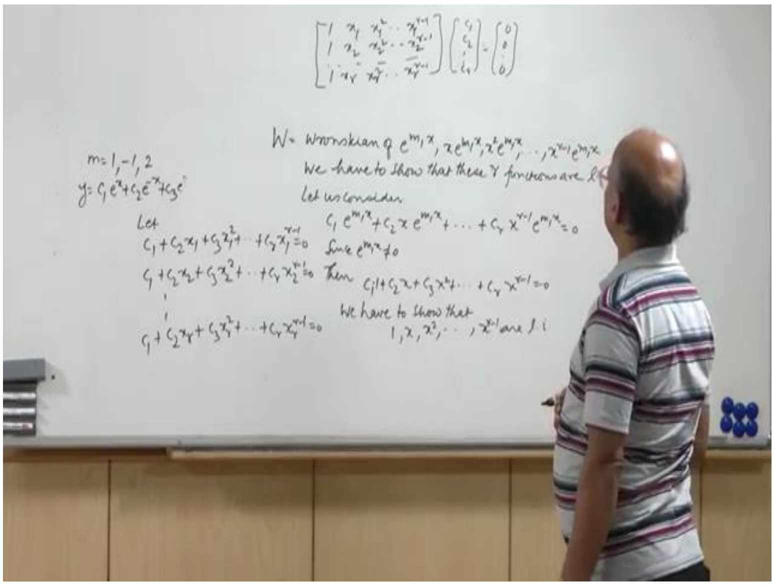

Now, let us note that the Wronskian of these solutions letter W, W is the Wronskian of W is the Wronskian of e to the m 1 x x e to the power m 1 x x square e to the power m 1 x and so on x to the power r minus 1 e to the power m 1 x. So, let us first note that the Wronskian of these r functions is not equal to 0. So, they are linearly independent. Let us see how we will prove this; we have to show that these n r functions are linearly independent. So, let us consider their combination c 1 e to the power m 1 x c 2 x e to the power m 1 x, and so on c r x to the power r minus 1 e to the power m 1 x equal to 0. We have to prove that c 1 c 2 c r are all 0s. Since e to the power m 1 x is never 0, let us divide this equation by e power m 1 x. So, we get then c 1 plus c 2 x plus c 3 x square and so on c r x to the power r minus 1 equal to 0. So, we get this equation.

Now, we have to prove that this equation gives us c 1 c 2 c 3 and so on c r equal to 0. Now, in other words we have to show that the functions 1 x x square x to the power r minus 1 are linearly independent. We have to show that 1 x x square x to the power r minus 1 are linearly independent. So, this is a polynomial in x of degree r minus 1. So, and we need to find these r coefficients. So, let us take n points r points x 1 x 2 x r r points x 1 x 2 x r which satisfy this equation. So, then we shall have let us say let c 1 plus c 2 x 1 c 3 x 1 square and so on c r x 1 r minus 1 equal to 0. We have r distinct point let us take r distinct points x 1, x 2, x r, these satisfy this equation. So, then c 1 c 2 x 1 c 3 x 1 square c r x 1 r minus 1 c 1 plus c 2 x 2 c 3 x 3 square 2 square and so on, c r x 2 r minus 1 equal to 0, and so on c 1 plus c 2 x r plus c 3 x r square and so on c r x r r minus 1 equal to 0.

So this can this is a homogenous system of linear equations, we can write it as 1 x, 1 x 1 x square x 1 to the power r minus 1 1 x 2 x 2 square x 2 to the power r minus 1 and so on, 1 x r x r square x r to the power r minus 1 c 1 c 2 and so on c r equal to 0. Now these homogenous system linear differential equations, so this will have trivial solution provided the determinant of this matrix is nonzero, and the determinant of this matrix is actually nonzero, because it is Vandermonde determinant whose value is equal to determinant of this matrix 1 x, 1 x 1 square x 1 to power r minus 1.

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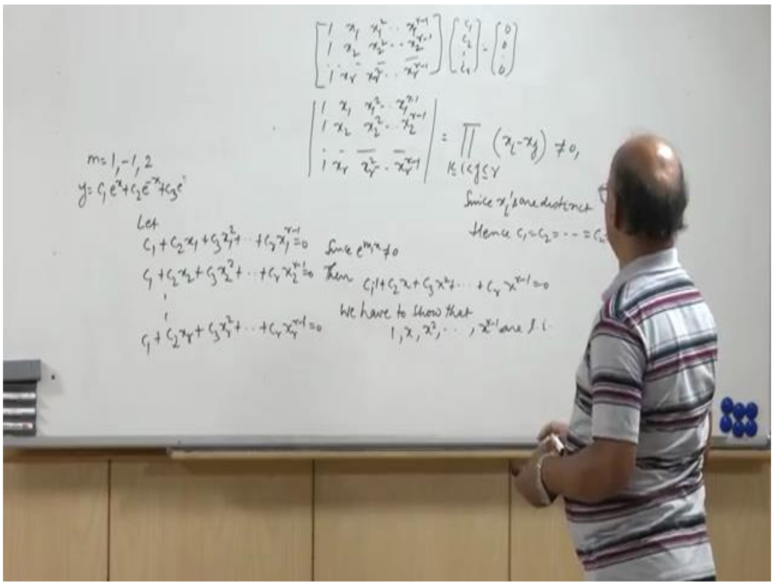

This is equal to product of 1 less than or equal to I less than j less than or equal to r x I minus x j. So, since x i’s are all distinct; since x i’s are all distinct the value of it is value is nonzero.

Since x is are distinct. So, this determinant of this nonzero and therefore, c 1 c 2 c n are equal to 0, hence these n functions 1 x x square x to the power r minus 1 are linearly independent. And So, the n functions here they are linearly independent, y 1 x y 1 x square y 1 x to the power r minus 1 y 1. So, the proof of W not equal to 0 is rather difficult. So, we have proved it by the other method.

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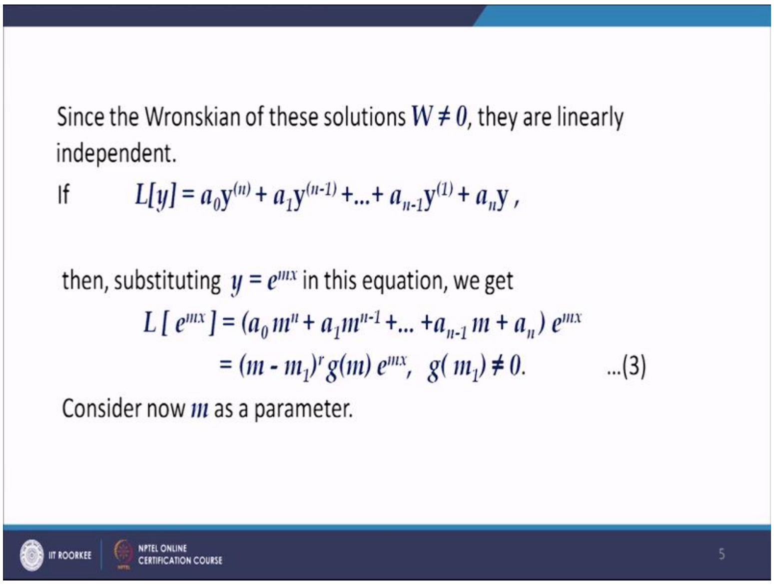

Now, let us see if we let us substitute by if you if you define this as l y. If you define the left hand side of equation 1 by l y, then what we will have if you put y equal to e m x in this then l e m x will be equal to a naught m to the power n a 1 m to the power n minus 1 and so on, a n into e to the power m x. And since m 1 occurs r times as a root of this characteristic equation we have m this can be written as m minus m 1 to the power r into g m, into e to the power m x where g m is not gm 1 is not equal to 0 e to the power m x is never 0. So, m 1 occurs r times means the remaining expression g m into e to the power m x must not be 0. So, g m 1 is not equal to 0.

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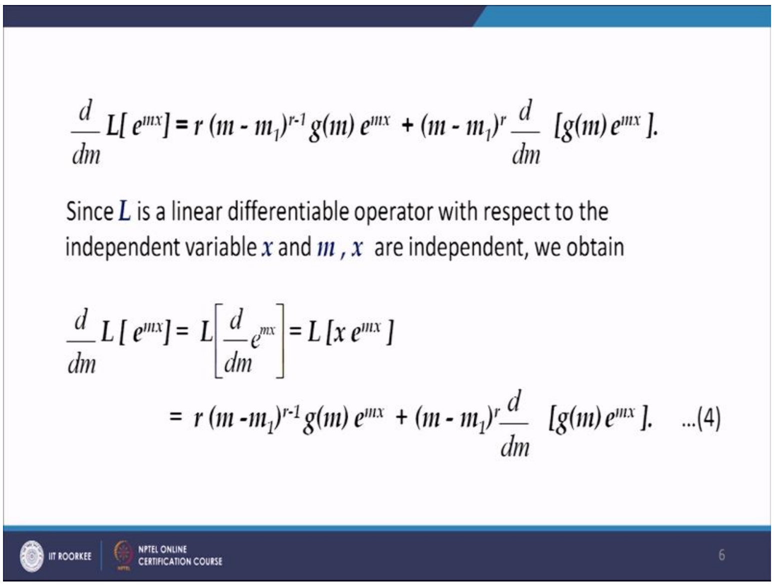

Now, let us consider m as a parameter. So, then D over D m l e m x let us differentiate this equation with respect to m. When we differentiate this equation with respect to m what we get is D over D m l e m x equal to r times m minus m 1 to the power r minus 1 g m e to the power m x m minus m 1 to the power r D over D m of g m e to the power m x. Now since l is a linear differentiable operator, we have taken l s this l s this. So, it is a linear differentiable operator. So, with respect to the independent variable x and m and x are independent to each other.

Therefore, D over D m of l e to the power m x can also be written as l D over D m of e m x. And when we differentiate e m x with respect to m we get x e to the power m x. So, l x e to the power m x is equal to this. And this when it says at x m equal to m 1 this when it says that m equal to m 1. So, x e to the power m 1 x is also a solution of the equation 1. So, since the right hand side vanishes at m equal to m 1 x e to the power m 1 x is also solution of the differential equation 1.

Again differentiating question 4 with respect to m this equation you again differentiate with respect to m then we will have r into r minus 1 m minus m 1 to the power r minus 2 g m e to the power m x and the other terms will be these ones, 2 r m minus m 1 to the power r minus 1 D over D m of g m e to the power m x and then m minus m 1 to the power r D square over D m square g m e to the power m 1 x again we put r m equal to m 1. So, this vanishes because r is greater than 2. So, this vanishes, and this also vanishes and this also vanishes. So, x square e to the power m x is also a solution of equation 1. So, in this manner we continue and when we differentiate r minus 1 times after r minus 1 times we differentiate.

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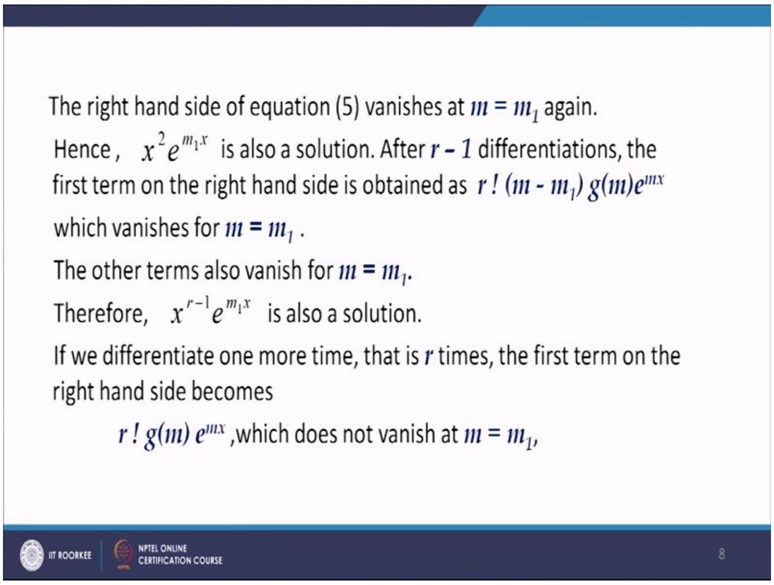

The first term on the right hand side is obtained as first term here. If you differentiate it, if you differentiate this here we have differentiated, when we see here if you differentiate r minus 1 times m minus m 1 to the power are this expression m minus r minus 1 times. Then the first term on the right side will be obtained as r factorial m minus m 1 D g m into e to the power m x, which vanishes at m equal to m 1 the other terms will also vanish because their powers of m minus m 1 will be more than or 1, and here. So, x to the power r minus 1 e to the power m 1 x will also be solution.

Now, if we differentiate one more time with respect to m, that is r times the first term on the right hand side will be now, r factorial g m e to the power m x which does not vanish at m equal to m 1. Since g m 1 is nonzero. So, it shows that x to the power r e to the power m 1 x is not a solution. And thus we find that e to the power m 1 x, x e to the power m 1 x and so on, x to the power r minus 1 e to the power m 1, x are linearly independent solutions corresponding to the multiple root m equal to m 1,

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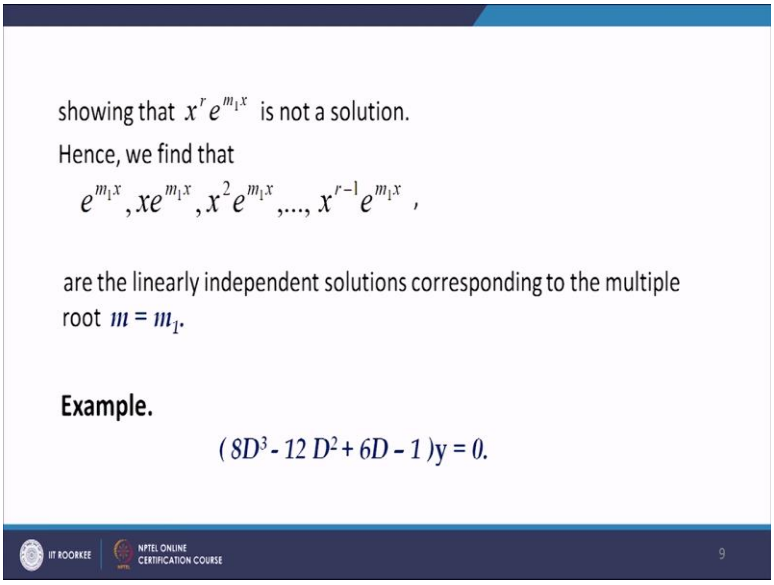

Let us show in the case of r multiple roots, r roots of the equal, we this is how we will find the independent solutions.

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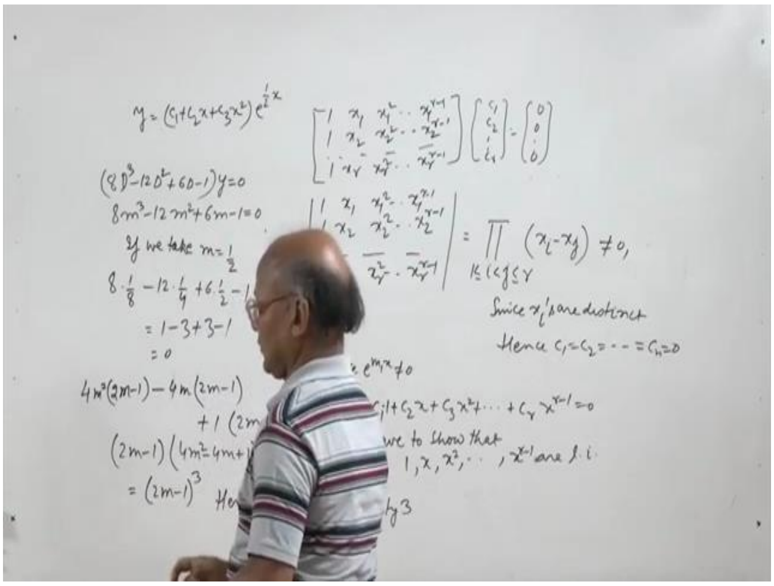

So, let us consider the equation at 8 D cube, 8 D cube minus 1 2 D square, and then we have 6 D minus 1, y equal to 0. So, the auxiliary equation will be 8 m cube minus 1 2 m square plus 6 m minus 1 equal to 0. 8 m cube minus 1 2 m square plus 6 m minus 1 equal to 0.

Let us find; solve this cubic equation. So, here we notice that if we take m equal to half, then we get 8 into 1 by 8 minus 1 2 into 1 by 4 plus 6 into 1 by 2 minus 1 equal to 1 minus 3 plus to plus 3 minus 1 equal to 0. So, m equal to half is a root of this equation. So, we can put m let us put 2 m minus 1 2 m minus 1 is a factor of this equation, 2 m minus 1. So, we can write 4 m square 4 m square means 8 m 8 m cube minus 4 m square. And then we will take 4 m square 8 m not 8 m 4 m into 2 m minus 1. So, we get minus 8 m square m 4 m square minus 1 2 m square minus plus 4 m. So, we get how much we require 8 m cube minus 4 m square minus 8 m square minus 1 2 m square plus 4 m plus 4 m is minus 4 m; so 2 m minus 1.

So, this is equal to this, how we have factorized this. So, this is 2 m minus 1 into 4 m square minus 4 m plus 1. These are the 2 factors and this can be this is square of 2 m minus 1. So, this is 2 m minus 1 whole cube. So, here m equal to half is a root of multiplicity 3. And, general solution is y equal to c 1 plus c 2 x plus c 3 x square e to the power half x in this case.

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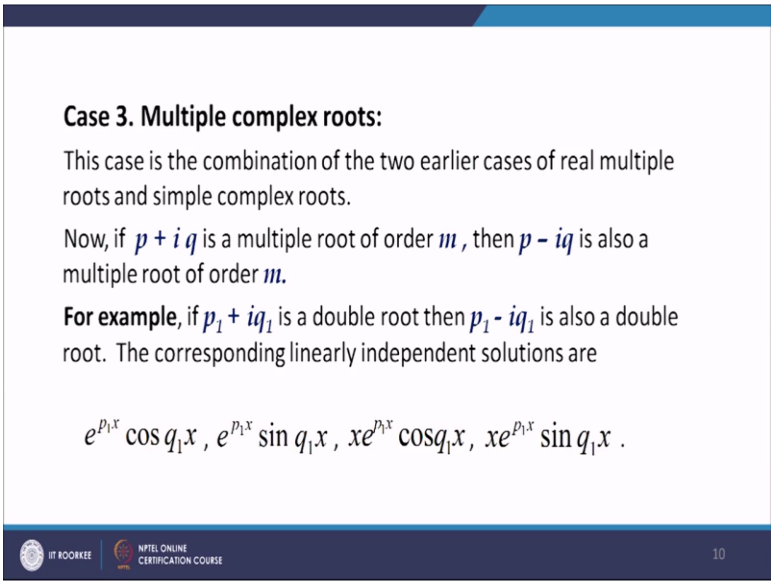

Let us now consider the third case multiple complex roots. This case is a combination of the 2 earlier cases. Suppose the p plus i q is a multiple root of order m then since the coefficients of the given differential equation in 1 are real the p minus /i q will also be a multiple root of order m. So, for example, if p 1 i q 1 is a double root then p 1 minus i q 1 is also a double root.

Now, we have earlier seen that if p 1 plus i q 1 and p 1 minus i q 1 are roots of the auxiliary equation, then the corresponding independent solutions are e to the power p 1 cos q 1 x and e to the power p 1 sin q 1 x. Now p 1 plus i q 1 is a double root, and p 1 minus i q 1 is a double root. So therefore, if y 1 is p 1 plus i q 1 and y 2 is p 2 minus p 2 minus i q 2 y y 1 is p 1 plus i q 1 and y 2 is, sorry if y 1 is e to the power p 1 plus i q 1 into x and y 2 is p 1 minus i q 1 into x. Then the then y 1 plus y 2 y 2 is also a solution and y 1 minus y 2 y 2 i is also solution. So, if p 1 plus i q is a double root and p 1 minus i q 1 is double root then e to the power p 1 q as e to the power then y 1 plus y 2 y 2 and y 1 minus y 2 y 2 y which are e to the power p 1 cos q 1 x and e to the power p 1 x sin q 1 x.

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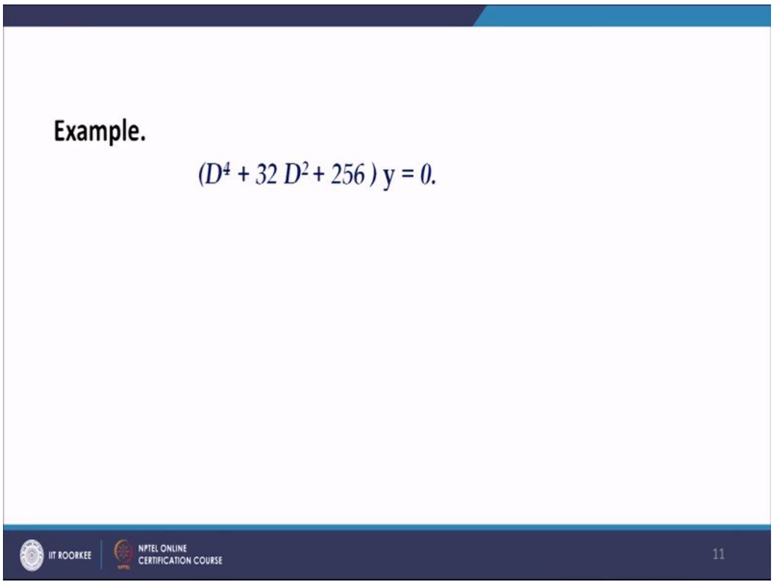

They will also be double roots, and therefore, the corresponding independent solutions will be e to the power p 1 x cos q 1 x x e to the power p 1 x cos q 1 x e to the power p 1 x sin q 1 x, x e to the power p 1 x sin q 1 x.

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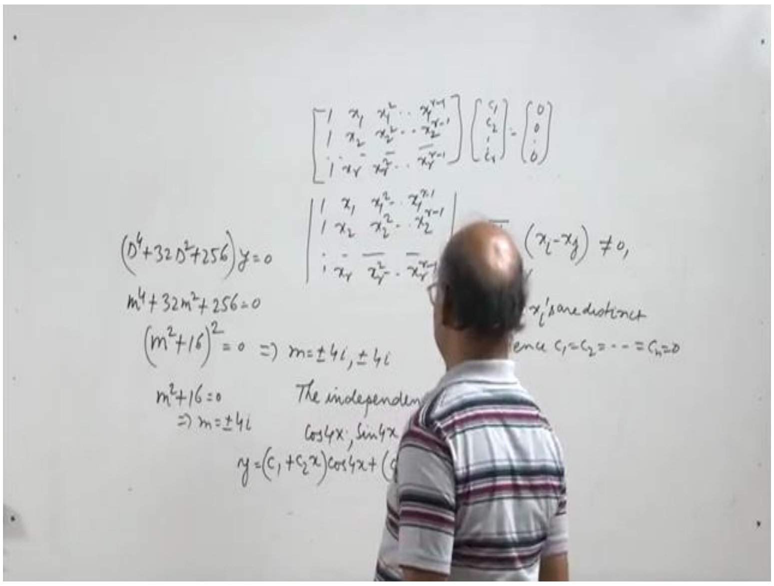

So, we will write the general solution in this case as follows. So, D 4 plus thirty D square 256 suppose this is equal to 0 D 4 plus 32 D square plus 256 y equal to 0. So, the auxiliary equation will be m 4 plus 32 m square plus 256 equal to 0 let us this is equal to m 4 plus 6 m square plus 1 6 whole square equal to 0. If you square this, we get m 4 plus 32 m square plus 256 equal to 0.

So, we have here m square now, m square plus 16 is equal to gives you equal to 0 gives you m equal to plus minus 4 i. So, plus minus 4 i occur twice, because here we have m square plus 1 6 square. So, this gives you m equal to plus minus 4 i and plus minus 4 i. So, 4 occur twice and minus 4 i also occur twice. So, here we will have the independent solutions will be e to the power if p 1 plus i q 1 is a root and p 1 minus i q 1 is a root. Then we have e to the power p 1 x cos q 1 x. So, we have here p 1 is 0. So, cos q 1 x means cos 4 x is a root sin 4 x is a root sin cos 4 x is a solution sin 4 x is a solution. And the other linearly independent solutions are x cos 4 x and x sin 4 x.

So, we will write the general solution as y equal to c 1 cos c 1 plus c 2 x cos 4 x and c 3 plus c 4 x, sin 4 x in this case, where c 1 c 2 c 3 c 4 are arbitrary constants. So, with that we conclude this lecture on solution of homogenous linear differential equations of higher-order.

I thank you for your attention.