Mathematical methods and its applications
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Lecture – 13
Solution of second order differential equations by changing independent variable

Welcome friends, to my lecture on Solution of Second order Differential Equations by Changing Independent Variable. Now we are going to discuss another method sometimes this method is found to be very useful, and in this method we change the independent variable.

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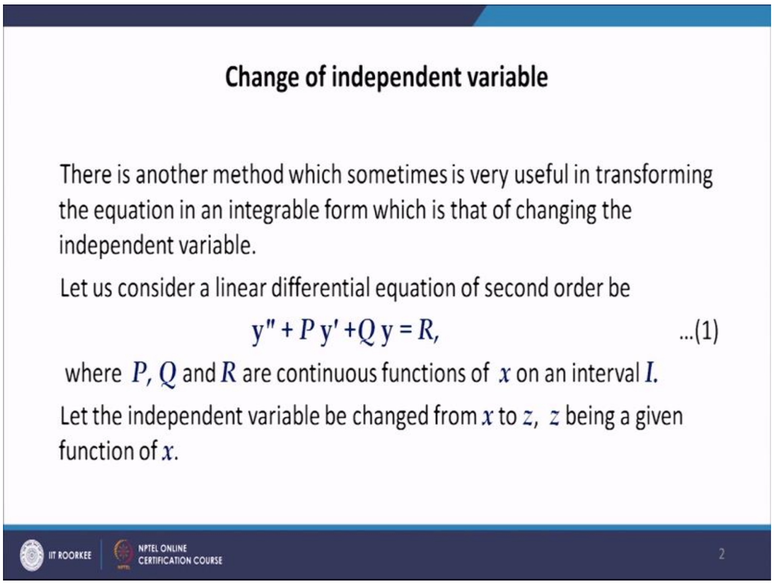

So far the differential equations that we are considering they are of second order linear differential equation with variable coefficients where x is the independent variable. So, in this method we will change the independent variable from x to say another independent variable that is z. So, let us consider a linear differential equation of second order, that is y double dash plus P y dash plus Q y is equal to R where P Q R are continuous functions of x on an on some interval i. We will be changing as i said we will be changing the independent variable from x to z. The relationship between x to z will be found from the fact that i the changed equation it becomes readily integrable.

Once we have that aim in mind that the changed equation has i, mean is readily integrable. From there we shall be able to derive the relationship between x and z.

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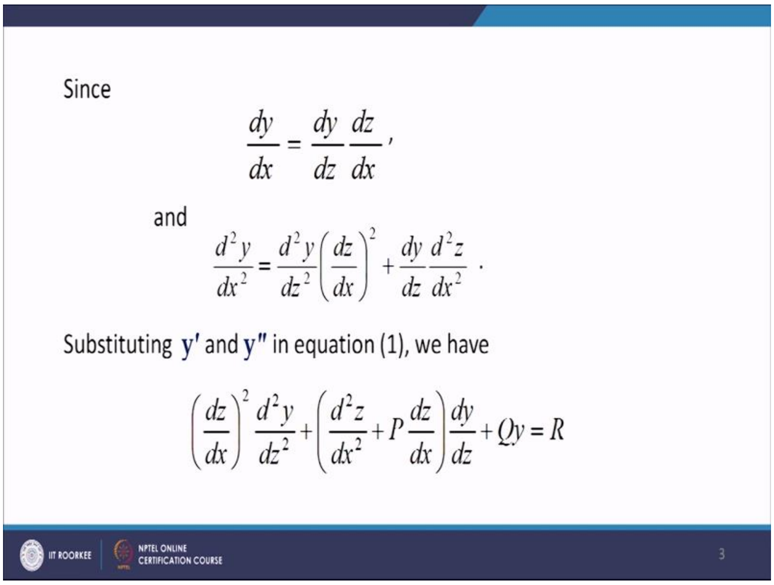

So, let us see what we do since x is related to z, we can write d y by d x equal to d y by d z over into d z by d x. And d square by d x square if you find d square by d x square.

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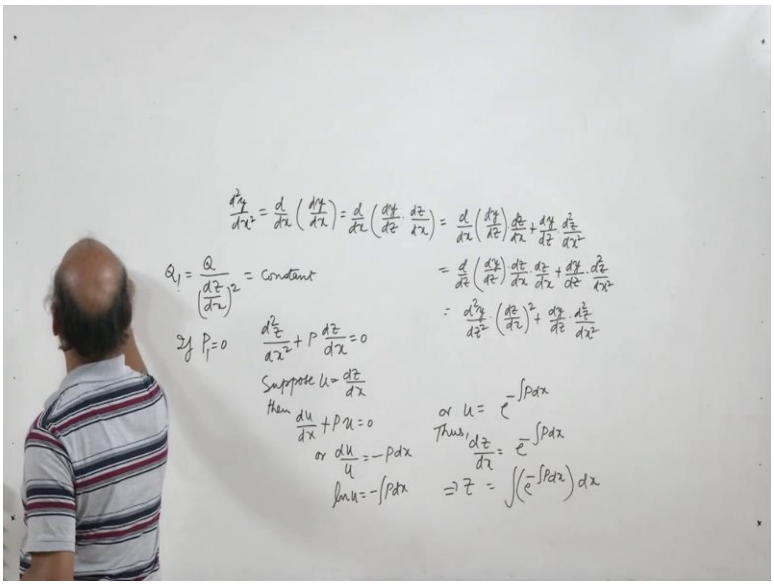

Then d square by d x square will be equal to d over d x of d y by d x which is equal to d over d x of d y by d z into d z by d x. So, differentiating this product of functions of x, we shall be having d over d x of d y by d z into d z by d x plus d y by d z into d square z by d x square.

Now, this can be further expressed as d over d z of because z is a function of x, we can write it as this into d z by d x, into d z by d x plus d y by d z, into d square z by d x square. So, this will be equal to d square y over d z square into d z by d x whole square, plus d y by d z into d square z y d x square. So, the second derivative of y with respect to x is equal to d 2 y over d z 2 into d z by d x whole square plus d y by d z into d square z by d d x square.

Now, substituting these values of y dash and y double dash in the equation 1, we shall have this expression d square y over d z square into d z by d x whole square plus d square x by d x square plus P d z by d x into d y by d z plus Q y equal to R. So now, let us denote.

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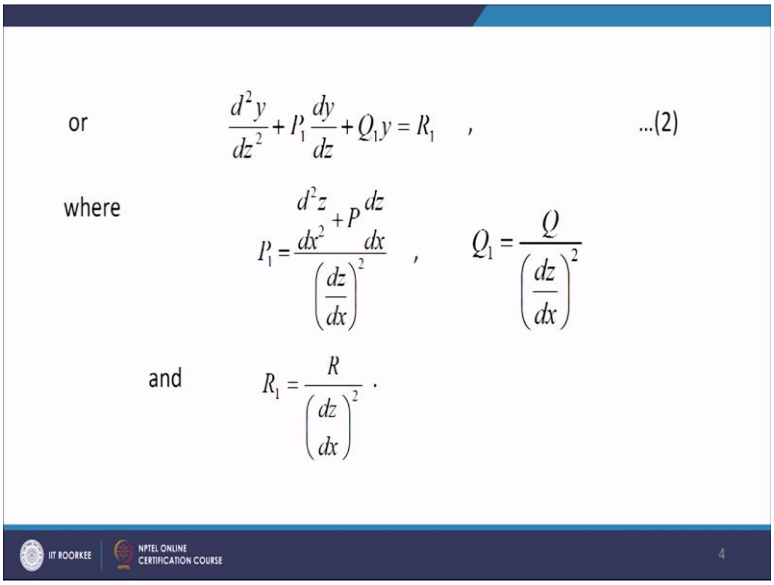

Let us write this equation in the form d square y over d z in this standard form d square y by d z square plus P 1 d y by d z plus Q 1 by y equal to R 1 where we write P 1 for this expression d square z by d x square plus P d z by d x over d z by d x whole square. And Q 1 we write for d Q over d z by d x whole square and R 1 denote R over d z by d x whole square.

So, these P 1 Q 1 R 1, P 1 Q 1 R 1 are some functions of x. And can they can be easily expressed as functions of z by the relationship between z and x which we have yet to establish.

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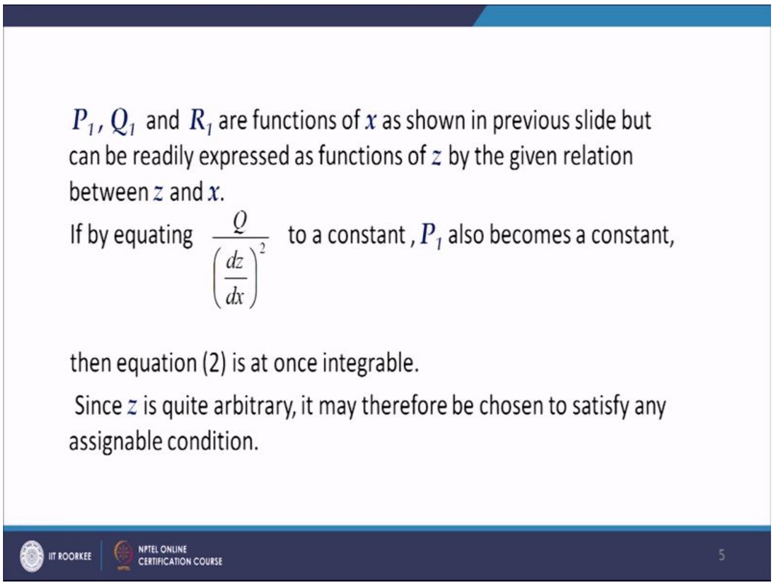

So, now what we do is let us discuss various possibilities, which will give us a function of z i mean z as a function of x and with those possibilities we shall be able to integrate the equation 2. So, the first possibility is that let us equate Q over d z by d x whole square. Q over d z by d x whole square is Q 1. So, let us equate Q 1 to a constant. So, if we do that. So, what do we have Q 1 equal to Q y d z by d x whole square. Suppose it is some constant. Then if it is so happens that P 1 also becomes a constant. If P 1 also becomes a constant then the equation 2 will be a linear differential equation of second order in the independent variable z, the dependent variable y with constant coefficients, and we have seen earlier how we solve a second order linear differential equation with constant coefficients.

So, if the function if z is a function of x, in such way that Q over d z by d x whole square is a constant. Then if P 1 also becomes a constant the equation 2 is at once integrable. So, this is one possibility where we can find the general solution of equation 2. Now since z is quite arbitrary. It can be chosen to satisfy any assignable condition s 1 condition that we put is that Q over d z by d x whole square is equal to a constant.

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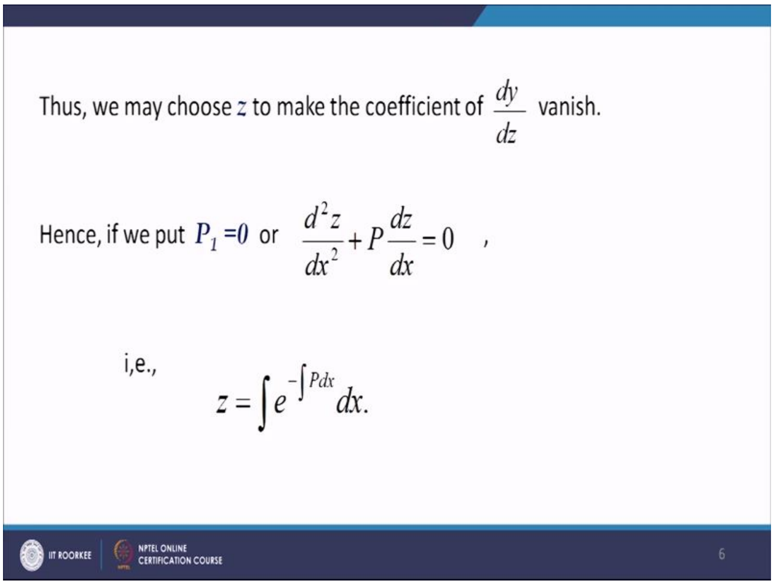

Let us look at other possibilities, in the other possibility what we do is we choose z in such a way that the coefficient of d y by d z in the equation 2 vanishes. So, coefficient of d y by z in the equation 2 is P 1. So, let us put P 1 equal to 0. So, if P 1 equal to 0 we get d square z by d z square, plus P d z by d x equal to 0. So, let us say suppose, u is equal to d z by d x. Then this a question the question can be expressed as d u by d x plus P into u equal to 0. We can separate the variables u and x are now let us integrate. So, integrating both sides we have l n u equal to minus integral P d x are u equal to e to the power minus integral P d x.

So, thus d z by d x is equal to e to the power minus integral P d x. Now integrating again with respect to x we get z as integral e to the power minus integral P d x d x. So, if we put the coefficient of d y by d z, that is P y equal to 0 we get z equal to integral of e to the power minus integral P d x d x, and after that if Q 1 with this choice of z as a function of x if it. So, happens that Q 1 becomes a constant then it is a second order linear differential equation with constant coefficients it will be d square y over d z square plus a constant time by equal to R 1. So, it will be a linear differential equation of second order with constant coefficients, and therefore we can solve it we can find the general solution of this equation.

Now, the other possibility is that suppose Q 1 is a constant divided by z square. Then z square times d square y by d x d square plus a constant times y will be equal to R 1 times z square. And, So, we will have quasi Euler equation and we know how to solve quasi Euler equation. So, we can again get the solution of this differential equation. So, if P 1 is taken equal to 0 and after that P 1 equal to 0, we get z as a function of x, with this choice of z as a function of x if Q 1 either becomes a constant or Q 1 is a constant divided by z square then we can find the general solution of equation 2, and hence we can find the general solution of equation 1.

So, that is one more possibility and so we can see here if the equation 2 will reduce to this form. If the curve given, comes out to be a constant or a constant divide by z square the equation 3 becomes readily integrable.

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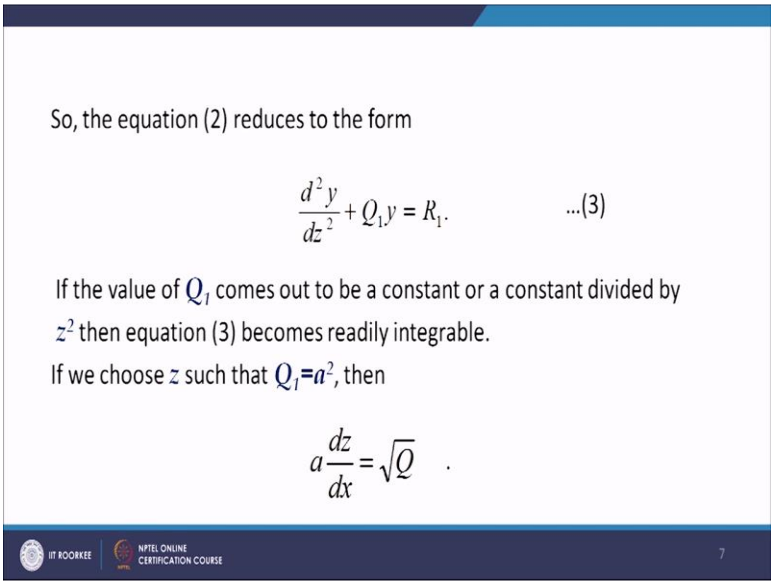

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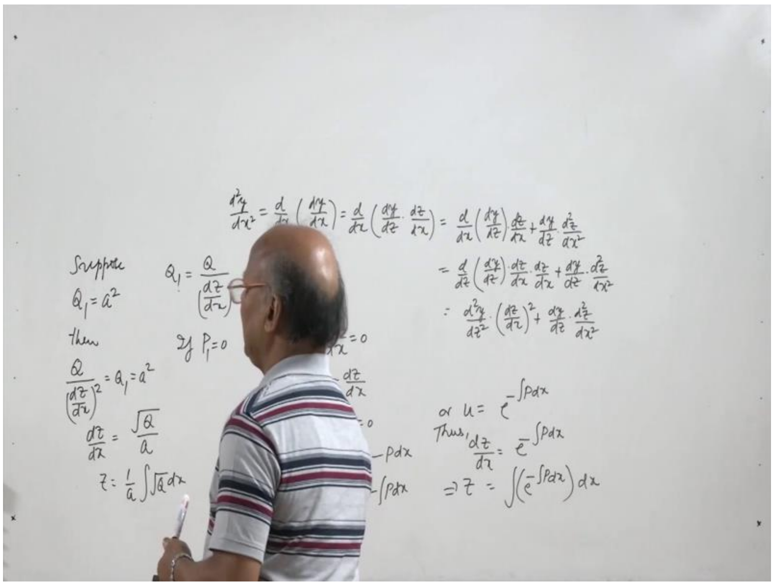

Now, let us see when we put Q 1 as a constant, say suppose Q 1 we take as some constant a square. We just want z as a function of x. So, this constant can be chosen according to our requirement. So, we are we will be choosing Q 1 equal to a square. So, then a Q upon d z by d x whole square, this is equal to Q 1 and Q 1 is a square. So, we shall have d z by d x equal to Q upon a square R or root Q divided by a, or we can say z is a z is equal to 1 upon a integral root Q d x. We will get z as a function of x.

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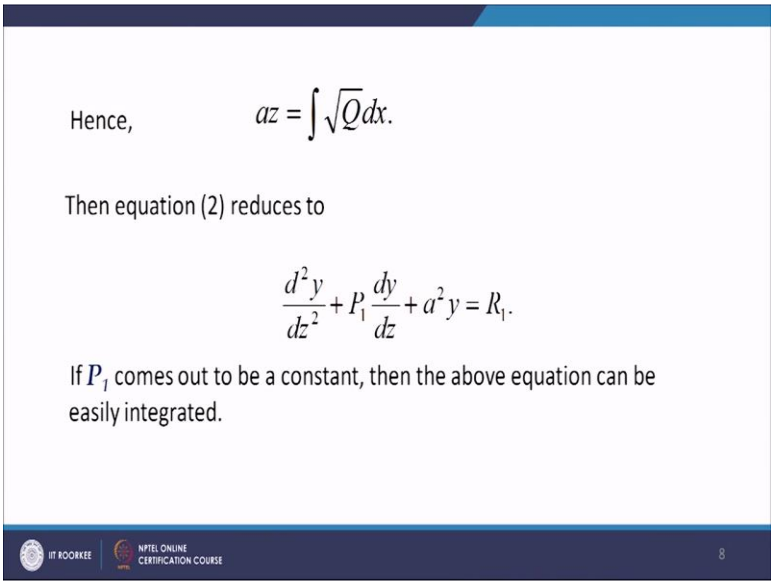

Now, this is what we will get and the question will reduce to d square y over d z square plus P 1 d y by d z plus a square y equal to R 1. If P 1 is a constant then the question can be easily integrated because it will be a linear differential equation of second order, with constant coefficients. Now let us see some examples on this. So, let us consider the question given in example 6, first we will write this equation in the standard form. So, the coefficient of d square y by d x square you will get unity which means that we will divide the equation by x to the power 6.

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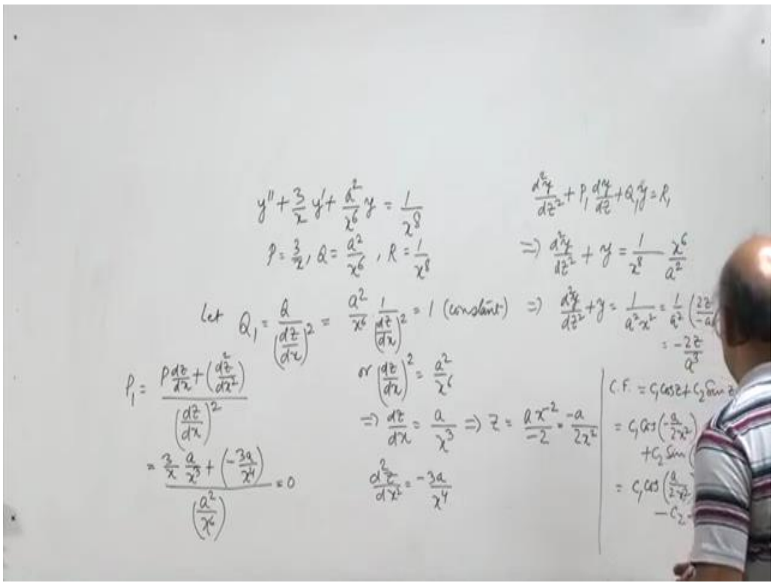

So, we shall have y double dash plus 3 by x y dash, plus a square by x to the power 6, into y equal to 1 by x to the power 8. So, when we compare this equation with the standard form, we get P equal to 3 by x Q equal to a square by x to the power 6 and R equal to 1 by x to the power 8.

So, let us first explore the possibility when we put Q as a constant. So, let us put Q as a constant. So, let Q be equal to some constant, say let us take it as 1. Then what we Q by is Q by d z by d x whole square we put as a constant. So, wait i think wait a minute sorry. So, let us let Q 1 be equal to Q 1 is equal to Q by d z by d x whole square, after we change the variable from x to z we get this Q 1. So, after we change the variable from x to z we will get Q 1 as this.

So, this u let us put as equal to this is equal to a square by x to the power 6, into 1 by d z by d x whole square. Let us put it as a constant say 1. We can take any value of the constant if you do not take here one you take some other constant then the relationship between x and z will be changed accordingly. So, R d z by d x whole square equal to a square by x to the power 6, which implies that d z by d x equal to a by x cube. And this gives you when we integrate with respect to z we get z equal to a x to the power minus 2 divided by minus 2.

So, this is a upon minus a upon 2 x square. Now let us find the value of P 1. P 1 equal to P d z by d x plus d z by d x whole square, d square z by d x square upon d z by d x whole square. So, P is equal to 3 by x, d z by d x is a by x cube, plus d square z by d x square we can find from here. So, d square z by d x square we differentiated once more see this is a times x to the power minus 3. So, we have minus 3 a x to the power minus 4; so minus 3 a by x to the power 4. So, we get minus 3 a over d z by d x whole square which is a square by x to the power 6.

Now, here we have 3 a by x to the power 4 here minus 3 a by x to the power 4. So, this is equal to 0. So, when Q 1 is assumed as a constant, which is which we have taken here as 1, P 1 also comes out to be a constant which is 0. So, we have the equation in in the independent variable as d square y over d z square, plus P 1 d y by d z plus Q 1 into y equal to R 1. This becomes d square y over d z square plus Q 1 we have taken as 1. So, y and R 1 is R over d z by d x whole square. So, R 1 is 1 over x to the power 8 R 1 divided by d z by d x whole square. So, divided by a square over x t the power 6. So, we get here x to the power 6 over a square.

So, we get here d square y over d z square plus y equal to 1 by a square x square, now we have to change the variable independent from x to z, and the relationship is z equal to minus a by 2 x square. So, 1 by a 1 by x square is 2 z over minus a. So, this is 1 by a square 2 z over minus a. 1 by x square equal to 2 z over minus a. So, this will be minus 2 z over a cube. Now this is a second order linear differential equation with constant coefficients. So, we can or write the complimentary function for this. Here the auxiliary equation will be m square plus 1 equal to 0.

So, the roots will be complex conjugate m equal to plus minus i. So, complementary function will be c 1 cos z plus c 2 sin z. And z we see is equal to minus a by 2 x square. So, c 1 cos minus a by 2 x square plus c 2 sin minus a by 2 x square, cos minus theta is equal to cos theta, sin minus theta is minus sin theta. So, we can write it as c 1 cos a by 2 x square, and here minus c 2 sin a by 2 x square, minus c 2 can be replaced by another constant, say c 2 dash and we can write c 1 cos a by 2 x square plus c 2 dash sin a by 2 x square.

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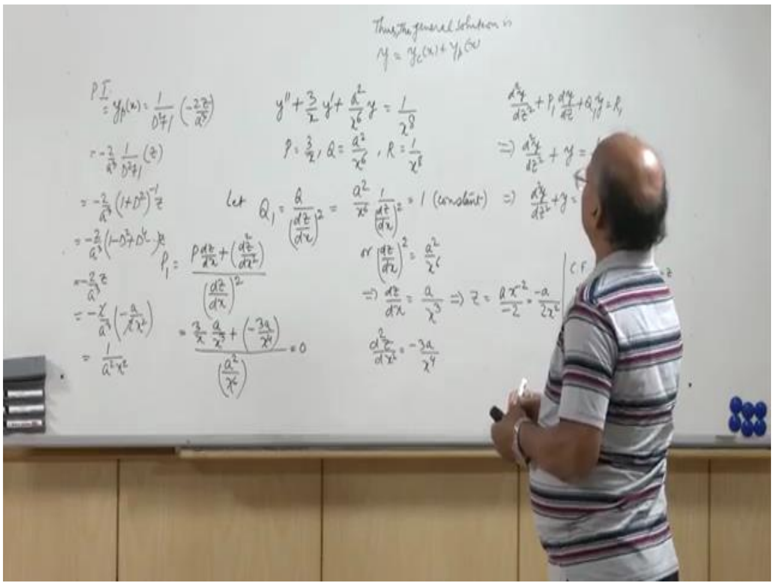

Now, a particular integral let us find, So, this y P x y P z let us first find y P x y P x will be 1 over d square plus 1 1 over d square plus 1 acting on minus 2 z by a cube. So, this is equal to minus 2 by a cube 1 over d square plus 1, acting on z. Now this is a polynomial z is a polynomial of degree 1. So, 1 over z d square plus 1 we shall expand as in a in the form of binomial expansion. So, minus 2 by a cube 1 plus d square raise to the power minus 1 operating on z. When we write binomial expansion for this we shall have 1 minus d square plus d 4 minus d 6 and so on. But the second derivative of z is 0. So, we will get minus 2 by a cube 1 minus d square plus d 4 and so on, operating on z which will give you minus 2 by a cube minus 2 by a cube d square z d 4 z all are 0. So, we have minus 2 z by a cube and z is equal to minus a by 2 x square.

So, this is minus a by 2 x square. So, this will give you 1 by a square x square, and thus the general solution is y equal to y c x plus y P x, this is y c x and this one is y P x. So, we take the sum of the 2 and it writes it equal to y. So, that is the general solution this case now let us discuss one more problem on this to make it more clear. So, let us take up another problem here in the example 2, we are given the equation x d square y by d x square minus d y by d x minus 4 x cube by equal to 8 x cube sin x square. So, again we divide the equation by x to make the coefficient of y double dash unity.

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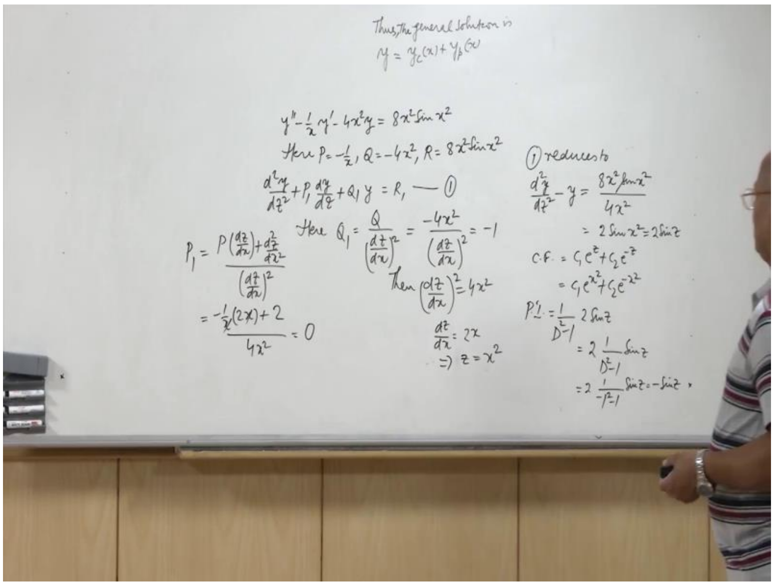

So, we shall have y double dash minus ,1 over y dash minus 4 x square into y equal to 8 x square sin x square. So, here P is equal to minus 1 over x Q equal to minus 4 x square and R is equal to 8 x square sin x square. So, again we try the method where we put Q 1 as a Q 1 Q y Q equal to a sorry Q 1 equal to a constant.

So, after we change the variable from x to z we get the equation d square y by d z square plus P 1 d y by d z plus Q 1 y equal to R 1. So, here Q 1 is Q y d z by d x whole square. So, let us put Q 1 as a constant. Say Q 1 will be equal to minus 4 x square upon d z by d x whole square. So, Q 1 we can we can take as a constant let us take the constant as minus 1 to then it will be simple. So, let us take Q 1 equal to minus 1, then d z by d x whole square will be equal to 4 x square. So, which will give us d z by d x equal to 2 x. One can take d z by d x equal to minus 2 x also that will give you other relation between x and z. So, the equation will change accordingly. So, we are taking positive here sin here. So, d z by d x equal to 2 x this gives you z as x square. So, does z has x x square let us find the value of P 1.

So, d z by d x is 2 x P here is minus 1 over x and d square z by d x square is 2 d z by d x whole square is 4 x square. So, again this cancels with this minus 2 plus 2 is 0. So, we get P 1 equal to 0 and thus the equation 1 reduces to d square y by d z square P 1 is 0 Q is minus 1. So, minus y equal to R 1 R 1 is R over d z by d x whole square. So, a 8 x square sin x square divided by d z by d x whole square which is 4 x square. So, we shall have 2 sin x square. So, complementary function here is this is auxiliary equation is i mean x minus 1 equal to 0. So, m is equal to plus minus 1. So, the both the roots of the auxiliary equation are real and distinct and therefore, we will have c 1 e to the power z plus c 2 e to the power minus z and z is equal to x square; so c 1 e to the power x square plus c 2 e to the power minus x square.

Now, here this in order to find the particular integral, the right hand side of this equation which is linear differential equation with constant coefficients, in the right hand side x has to be changed to z. So, we have 2 sin z because z is equal to x square. So, particular integral will be 1 by d square minus 1 d d here represents d over d z. So, 1 over d square minus 1 operating on 2 sin z and this will be equal to 2 times 1 over d square minus 1 operating on sin z.

So, when 1 over d square plus alpha s square operates on sin a at z we replace d square by minus a square. So, this will be 2 times 1 over minus 1 square minus 1 sin z this is minus sin z.

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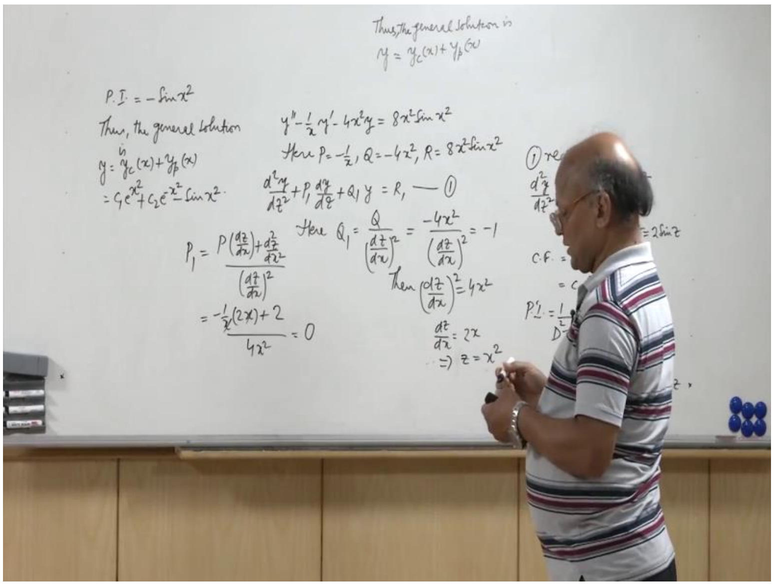

So, we have got the particular solution as minus sin x square. So, thus the general solution is the other solution is y equal to c 1 e to the power x square plus c 2 e to the power minus x square minus sin x square. So, that is how we solve this equation given in example 2.

So, we have seen in our in the past 3 lectures, we have seen how we can solve a linear differential equation of second order with variable coefficients, when it is given in a when it when we are given some special kinds of such equations, as i said earlier such i mean a general solution is not a well known for all second order linear differential equation with variable coefficients only a special class of such equations can be solved. So, we have seen those equations which can be solved by in our past 3 lectures.

Now in my next lecture we shall be discussing, how to find the general solution of a higher order homogenous linear differential equation, with constant coefficients.

Thank you for your attention.