Mathematical methods and its applications
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Lecture – 12
Solution of second order differential equations by changing dependent variable

Hello friends. Welcome to my lecture on Solution of Second order Linear Differential Equation by Changing the Dependent Variable.

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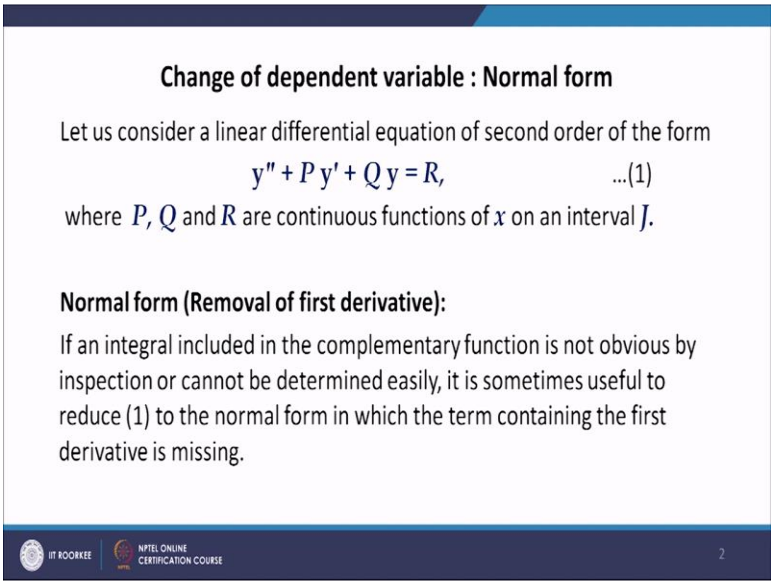

So, we begin with a differential equation linear differential equation of second order in the standard form. That is y double dash plus P y dash plus Q y equal to R where P Q R are continuous functions of x on an interval say J. In this method what we do is when in my previous lecture, we studied how to find the solution of this second order linear differential equation with variable coefficients. When one integral included in the complimentary function is known.

Now, there are cases of second order differential equations where, one integral included in the complimentary function is not easy to find. So, if an integral included in the complimentary function is not obvious by instruction or we cannot determine it easily, then what we will do is we reduce the given differential equation of second order to the normal. Form by normal we mean we will reduce it to a form where the first order derivative term is missing. So, how we do it let us see in the second slide. Let us put y equal to u v in the equation one.

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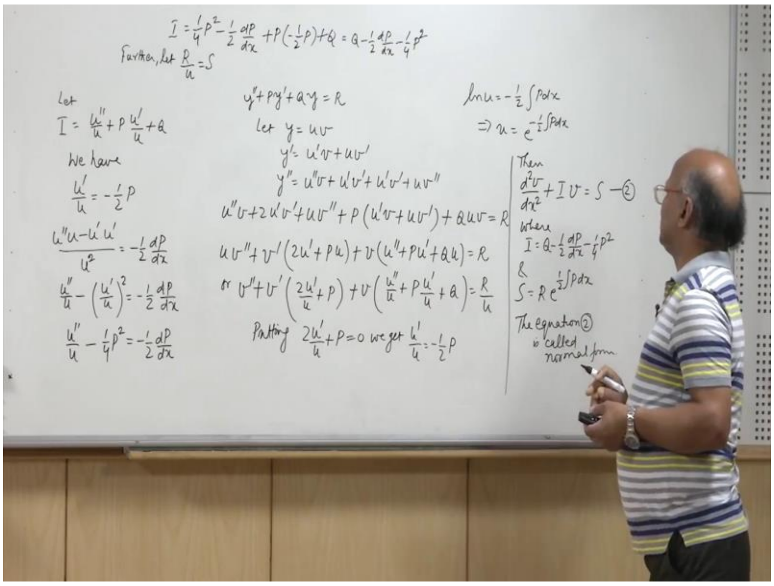

So, we will put y equal to u in equation 1. Then what do we get? So, we have y double dash plus P y dash plus Q y equal to R. Let us put y equal to u v then y dash is equal to u dash v plus u v dash, y double dash will be equal to y double dash v plus u dash v dash n plus u dash v dash plus u v double dash. Let us replace these values in the differential equations 1.

So, y double dash plus P y dash plus Q y equal to R will then become u double dash v plus 2 u dash v dash plus u v double dash plus P into y dash. So, u dash v plus u v dash plus Q into u v equal to R. Now let us write it as a second order differential equation in v. So, we have u v double dash, first we write this term u v double dash, and then we write v dash times. So, v dash times what we get, 2 u dash plus P u and then we write the coefficient of v, the coefficient of v is u double dash, plus P u dash plus Q u equal to R.

Let us divide it by u the coefficient of v double dash. So, R we can write v double dash plus v dash times 2 u dash by u plus P plus v times u double dash by u plus P times u dash by u plus Q equal to R by u.

Now, we are going to reduce this second order differential equation in v to the normal. Form normal form means where the first order derivative term is missing. So, let us put 2 u dash by u plus P equal to 0. So, putting we get u dash by u equal to minus half P, when we integrate this with respect to x, what we get is l n u equal to minus half integral P d x. So, this will give you u equal to e to the power minus half integral P d x.

Now, let us say suppose. So, this term is now 0, with this choice of u, now let us put this as say I. So, if I call I let I be equal to u double dash by u plus P times u dash by u plus Q. Then let us simplify this expression for I, we can see here that u dash by u equal to we know that u dash by u equal to minus half P.

So, if we differentiate with respect to x. What do we get u double dash into u minus u dash into u dash divided by u square equal to minus half, d P by d x? Or if you divide by u square you get u double dash by u minus u dash by u whole square equal to minus half d P by d x now u dash by u equal minus half P. So, let us put here. So, u double dash by u is equal to minus 1 by 4 P square equal to minus half d P by d x. So, let us replace the value of u double dash by u here. And the values of u dash by u and let us find a simplified expression for I.

So, I will be equal to 1 by 4 P square, minus half d P by d x P times u dash by u is minus half P plus Q. So, this is minus half P square that is 1 by 4 P square. So, we get minus 1 by 4 P square. So, this Q minus half d P by d x minus 1 by 4 P square; so we get a simplified expression for I, and let us denote this R by u by x. So, let us say further let R by u equal to s. So, then we shall have d square P by d x square plus I into v equal to S. Where I is again I write I equal to Q minus half d P by d x minus 1 by 4 P square, and S is equal to R by u and u is e to the power minus half integral P d x.

So, R times e to the power half integral P d x. Now let us note here that this equation this equation is called as the normal form. Let us say let me call it as equation 2, the equation 2 is called the normal form. Now we can integrate this equation or we can find a solution of this general solution of this equation provided R is a constant. If R is a constant, then this will be linear differential equation of second order with constant coefficients R I is a some constant divided by x square. Which if you assume that I is some constant divided by x square then multiplying the equation by x square it will reduce to a quasi-Euler equation. And you know how and we know how to solve a quasi-Euler equation. So, we have a catch here.

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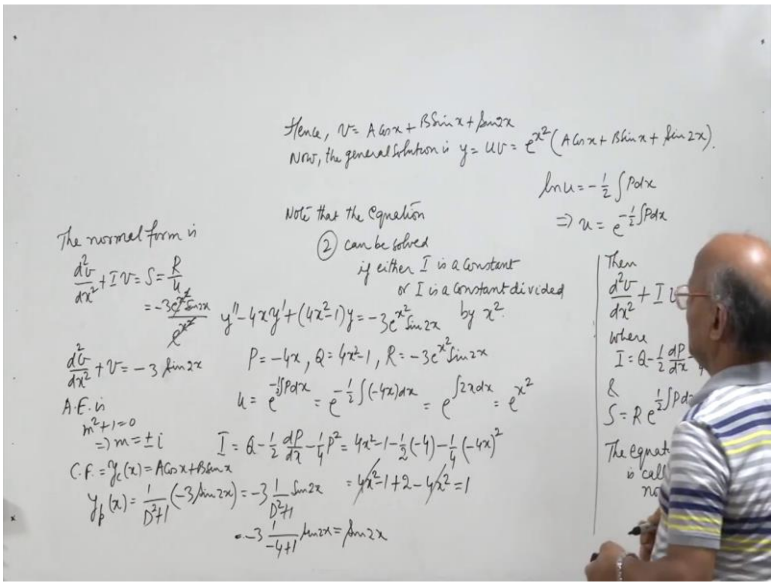

So, let us note that the equation 2 can be solved if either I is a constant or I is a constant divided by x square.

So, yes just as we said in the beginning, that only a special class of second order linear differential equation can be solved with variable coefficients. So, there is a catch here. Such equations where I is either a constant or a constant divided by x square can be solved by reducing the given equation to the normal form. So, let us see how this is these to this the normal form. We are given the value of or we have the value of I and S and this is what we have noted.

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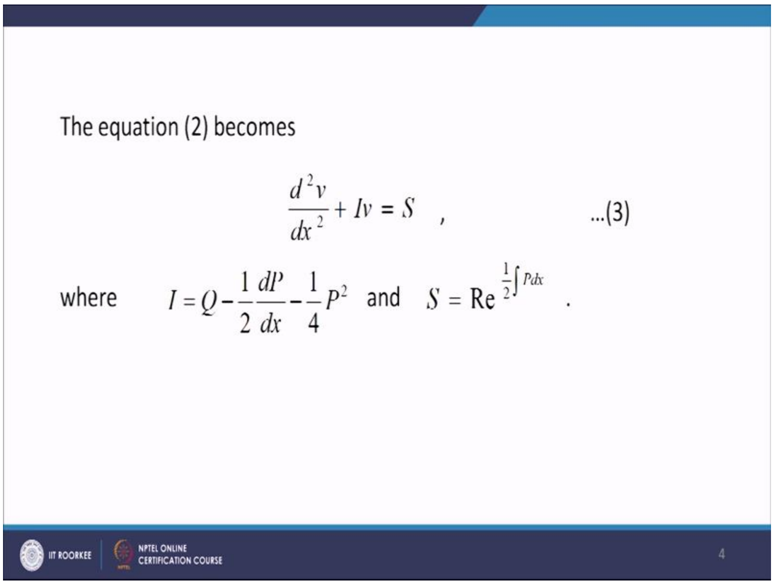

Just now if the value of I is a constant or a constant divided by x square, then the constant becomes readily integrable this form is said to be the normal form.

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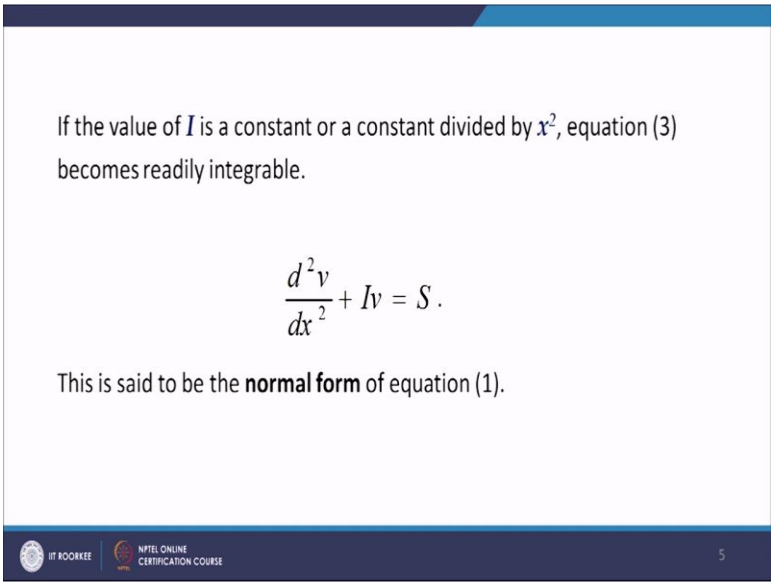

Now, let us take an example and see how we will use this method.

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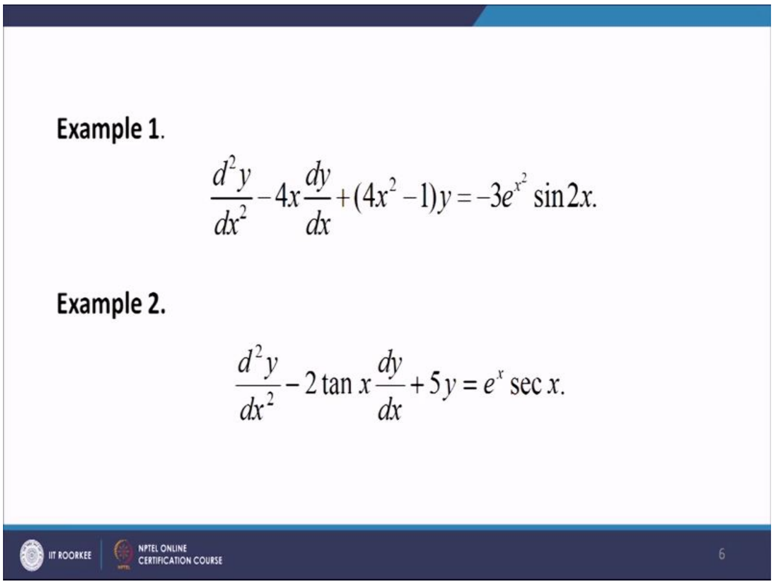

So, let us say d square y over d x over that is y double dash. We have y double dash minus 4 x y dash plus 4 x square minus 1 into y equal to minus 3 e to the power x square sin 2 x. So, when we compare it with the standard form we have the value of P equal to minus 4 x Q equal to 4 x square minus 1 and R equal to minus 3 e to the power x square sin 2 x. So, let us find first when we reduce this equation to the normal form the value of u, u is equal to e to the power minus half integral P d x; so e to the power minus half integral minus 4 x d x. So, this is e to the power integral 2 x d x, and which gives you e to the power x square.

So, we know the value of u. Now let us find I; I is equal to Q minus half d P by d x minus 1 by 4 P square. So, Q is 4 x square minus 1 minus half d P by d x. When you find d P by d x here you get minus 4 and then minus 1 by 4 P square. So, you get minus 4 x square whole square. So, what we get is 4 x square minus 1, and here we get plus 2 and here we get minus 4 x square.

So, this cancels with this 2 minus 1 is 1. So, we get I equal to a constant. And therefore, we can see from this article that this given equation is readily integrable. The normal form is now d square v over d x square plus I into v equal to x. So, I is equal to one and S is R over u. So, S is R over u, and R over u is minus 3 e to the power x square, sin 2 x divided by u is e to the power x square.

So, e to the power x square gets cancelled. And we get minus 3 sin 2 x. So, we get d square v by d x square, I is 1; so plus v equal to minus 3 sin 2 x. Now this is a linear differential equation of second order with constant coefficients. So, we can write auxiliary equation, auxiliary equation is m square plus 1 equal to 0. So, we get m equal to plus minus 1 plus minus I. So, complimentary function y c x is equal to A cos x plus B sin x. Where A and B are arbitrary constants. And particular integral y P x equal to 1 over d square plus 1 where d represents d over d x operating on minus 3 sin 2 x. So, this is minus 3 times 1 over d square plus 1 operating on sin 2 x. When we replace d square by minus 2 square the denominator does not become 0. So, this is minus 3 1 over minus 2 square that is minus 4 plus 1 sin 2 x. So, this is minus 3 minus 3 cancel, and you get sin 2 x.

So, this way we get the value of v s. So, hence v is equal to y c x plus y P x; so A cos x plus B sin x plus sin 2 x. Now general solution is y equal to u into v, that is u is e to the power x square, so e to the power x square into A cos x plus B sin x plus sin 2 x. So, this is the general solution in the case of example 1.

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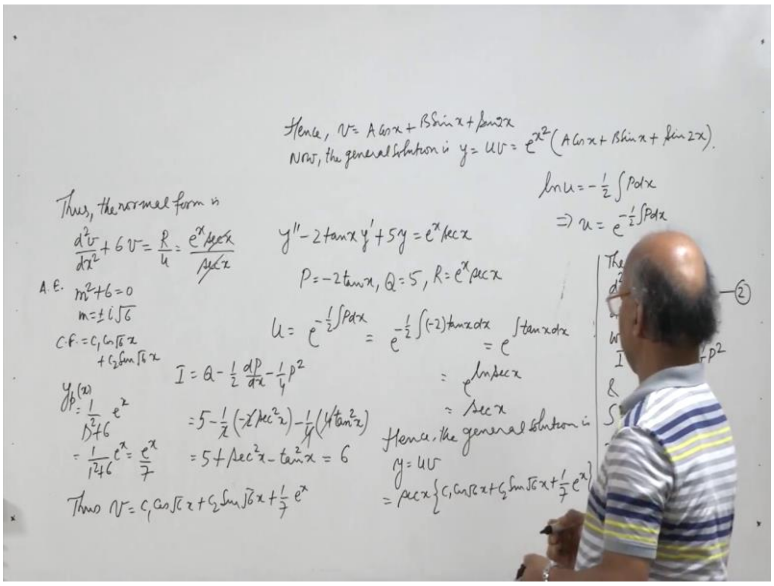

Let us take one more example to make the things clear. So, we have y double dash in example 2. Minus 2 tan x plus y into y dash plus 5 by equal to e to the power x into sec x. So, here P is equal to minus 2 tan x. Q equal to 5 and R equal to e to the power x into sec x. Now u is equal to y over, formula e equal to u equal to e to the power minus half integral P d x. So, this is e to the power minus half integral minus 2 tan x d x, which is e to the power integral.

Now, integral of tan x is log sec x. So, we have e to the power l n sec x. So, this is equal to sec x. So, we have found the value of u. Let us now determine I I equal to Q minus half d P by d x minus 1 by 4 P square. So, Q is 5 minus half d P by d x will be minus 2 sec square x the derivative what tan x is sec square x. Minus 1 by 4 P square P square is 4 tan square x. So, this will cancel with this, and we have 5 this will cancel with this we have 5 plus sec square x minus 10 square x. Sec square x equal to 1 plus 10 square x. So, this is 1 sec square x minus 10 square x is one. So, this is 6. So, I is again a constant and therefore, we can easily integrate the normal form that is d.

So, thus we have the normal form as I into v that is 6 into v equal to S. S is R by u. So, S is equal to R by u which is equal to or e to the power x sec x, divided by sec x. So, we cancel this. And now this is easy to integrate auxiliary equation is m square plus 6 equal to 0. It is our auxiliary equation. So, m equal to plus minus I root 6, and therefore, complimentary function is c 1 cos root 6 x plus c 2 sin root 6 x, and particular integral y P x is equal to 1 over d square plus 6 operating on e to the power x.

Now, we can apply 1 over f d e to the power a x formula because here when d is replaced by a, a is 1 here, f d does not become 0; so 1 over 1 square plus 6 e to the power x. So, we have e to the power x by 7. So, we have thus v equal to c f. So, c 1 cos root 6 into x plus c 2 sin root 6 into x plus 1 by 7 e to the power x. This is and we multiply it by u to get the general. Solution u is sec x. So, hence the general solution is y equal to u into v equal to sec x times c 1 cos root 6 x plus c 2 sin root 6 x plus 1 by 7 e to the power x.

So, this is the general solution of equation given in the case of example 2. In my next lecture I will discuss how to solve second order linear differential equation with variable coefficients by changing the independent variable. In this article we have changed the dependent variable from y to v in that next lecture; we shall be changing the independent variable from x to z. Now, how we get the relationship between x and z: that we shall see in the net lecture.

Thank you for your attention.