Mathematical methods and its applications
Dr. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 11
Method of variation of parameters

Hello friends. Welcome to my lecture on Method of Variation of Parameters. As you might recall this method of variation of parameters was developed by Lagrange to find a particular solution of first order linear differential equation, but we will see that this method can be applied to linear differential equation of higher order as well. This method has an advantage over the method of undetermined coefficients. In that the method of undetermined coefficients could be applied to linear differential equation with constant coefficients while this method is applicable to the linear differential equation with variable coefficients also and moreover we had seen that in the case of method of undetermined coefficients the there was limitations on the form of r x.

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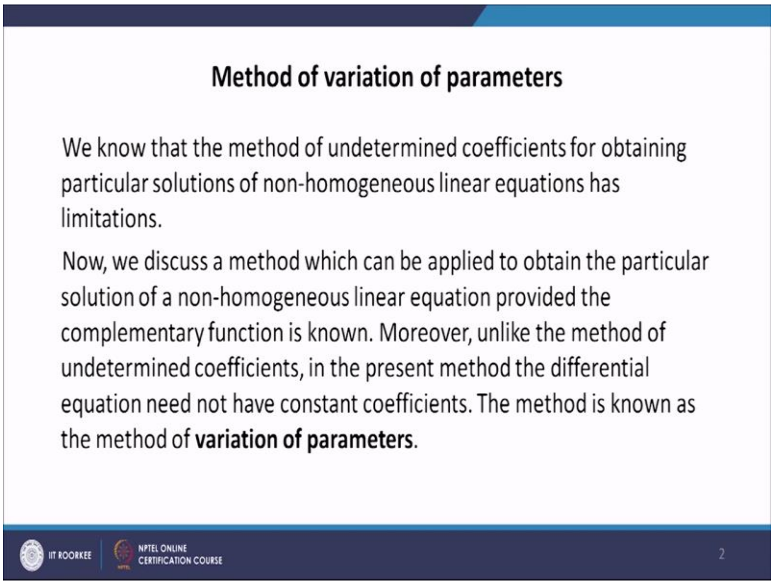

When we had taken up the differential equation of second order as y double dash plus p y plus q p y dash plus q y equal to r x. So, r x had to be of a particular form, but here we shall see that there is no such limitation.

So, were going to discuss the method which can be applied to obtain a particular solution of a non-homogenous linear equation provided we know the complimentary function. So, unlike the method of undetermined coefficients in the present method the differential equation need not have coefficient methods; coefficients as constants. Now the method is known as the method of variation of parameters.

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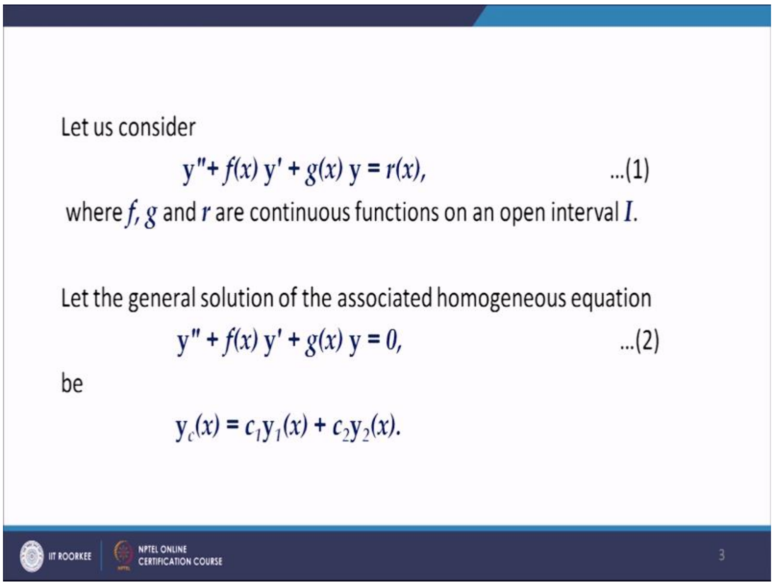

So, let us begin with a second order linear differential equation in the standard form that is the coefficient of y double dash we assume as 1, if it is not 1 we divide it the differential equation by the coefficient by y double dash and make it unity.

So, y double dash plus f x y dash plus g x y equal to r x where f g and r are continuous functions of x on an open interval i. Now in this method we must know the general solution of the associated homogeneous linear differential equation that is y double dash plus f x y dash plus g x y equal to 0 and we know that the general solution of the homogenous equation is nothing, but the complimentary function of the equation 1. So, we must know the 2 independent solutions y 1 and y 2 of the equation 2 so that we have its general solution as c 1 y 1 x plus c 2 y 2 x which is known as the complimentary function of equation 1.

Now we shall be looking for a particular solution of the equation 1 so that which will be called as particular integral. So, that we can write the general solution of equation 1 as y c x plus y p x where y p x will denote a particular solution of equation 1. So, thus we seek a particular solution of equation 1 of the form y p x equal to u x into y 1 x plus v x into y 2 x. So, the constants here in the y c x which is c 1 and c 2 are now replaced by functions of x they are u and d.

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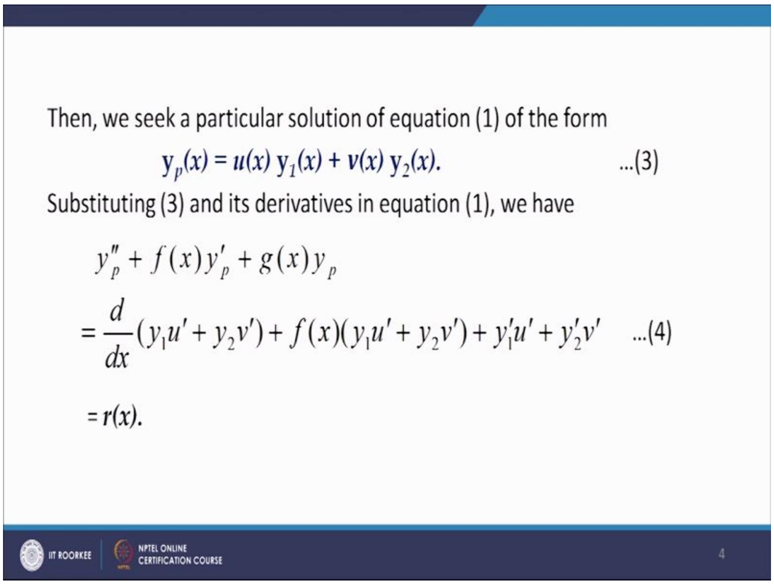

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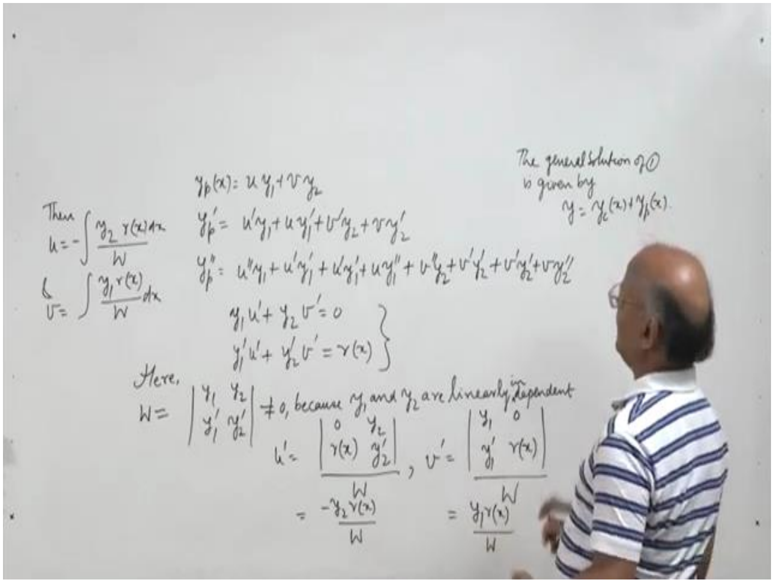

So, our aim will now be to determine the values of the unknown functions u and d. So, that y p x is a solution of equation 1. So, let us substitute y p x in equation 1 and also its derivatives where; so that we get u dash y 1 plus u y 1 dash plus v dash y 2 plus v y 2 dash and y p double dash d v equal to u double dash y 1 plus u dash y 1 dash which is the derivative of u dash y 1 then derivative of u y 1 dash. So, u dash y 1 dash plus u y 1 double dash then derivative of v dash y 2. So, v double dash y 2 plus v dash y 2 dash and then derivative of v y 2 dash. So, v dash y 2 dash and then v y 2 double dash. So, substituting these values of y p y p dash and y p double dash in equation 1, we will arrive at y p double dash plus f x y p dash plus g x y p equal to this.

Now, here we will see that the terms d over d x y 1 u dash plus y 2 v dash, they correspond to second derivative terms here y 1 because when you differentiate y 1 u dash you get y 1 dash u dash then y 1 u double dash then y 2 dash v dash then y 2 v double dash. So, those terms can be combined together and we can write them as d over d x of y 1 u dash plus y 2 v dash now there is a special purpose of writing y double dash plus f x y p dash plus g x y p into this form because in order to determine the unknown functions u x and v x. We need 2 conditions which involve the unknown functions u x and v x. So, one condition that we assume is that y 1 u dash plus y 2 v dash equal to 0. Now this equation does not come out of w, it is prompted by these first 2 terms you see here d over d x y 1 u dash plus y 2 v dash and here f x into y 1 u dash plus y 2 v dash. So, we assume that y 1 u dash plus y 2 v dash equal to 0 then the first 2 terms here will vanish and we shall have y 1 u y 1 dash u dash plus y 2 dash v dash equal to r x.

So, this will be our second condition. So, we will have 2 conditions which involve the first derivatives of u and v and we shall solve them by Cramer’s rule and get the values of u dash and v dash which after the integration will give us the values of the unknown functions u and v.

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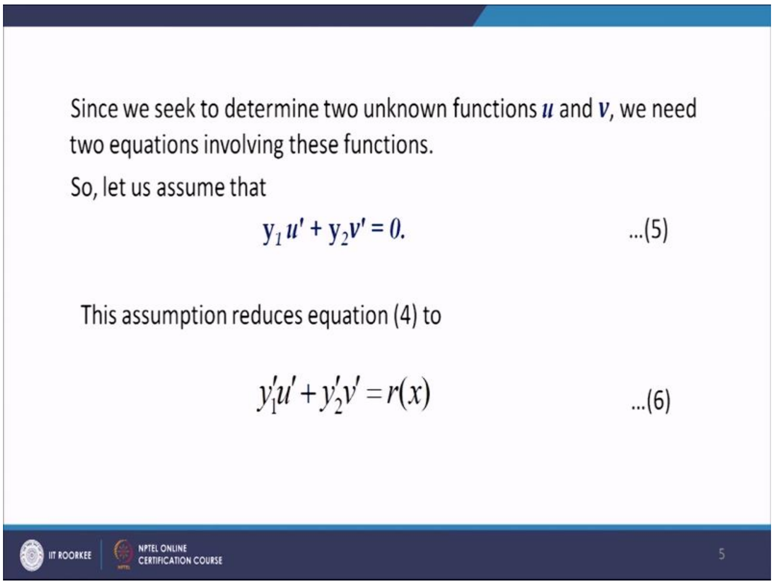

So, what we will get is; so, we seek it we seek to determine 2 unknown functions u and v which involve these 2 functions. So, let us assume that y 1 u dash plus y 2 v dash equal to 0. This assumption reduces equation 4 to y 1 dash u dash plus y 2 v dash equal to r x.

So, now we have 2 equations y 1 u dash plus y 2 v dash equal to 0 and y 1 u dash plus y 2 dash v dash equal to r x we can solve this 2 equations by the Cramer's rule because the determinant of the coefficient matrix here is determinant of the coefficient matrix which we shall denote by w here w is equal to determinant of y 1 y 2 y 1 dash y 2 dash, this determinant is not equal to 0 because y 1 and y 2 are linearly independent, they are part of complimentary function y c x which is a general solution of the homogenous associated homogenous linear equation.

So, because y 1 y 2 are linearly independent w is not equal to 0. So, we can solve these 2 equations by the Cramer's rule and if we solve them by Cramer's rule then u dash will be equal to determinant first column we replace by the 0 and r x of the determinant this and then y 2 y dash y 2 dash divided by w. This is u dash and similarly v dash we can write as y 1 y 2 y 1 dash and then here y 2 y 2 dash column that will be replaced by 0 and r x. So, 0 r x divided by w. So, we will get u dash equal to minus y 2 r x divided by w and v dash we will be getting as y 1 r x divided by w.

Now, I notice that there is a mistake here y because y 1 y 2 are linearly independent. So, now, we integrate u dash and v dash. So, then u will be equal to integral y 2 r x divided by w and then and v is equal to y 1 r x. So, once we have the values of u and v we can write a particular integral y p x which is u x into y 1 x plus v x into y 2 x and then the general solution will be y equal to y c x plus y p x.

Now, let us demonstrate this method by 2 by a differential equation of second order 4 y double dash plus thirty six y equal to cosec 3 x.

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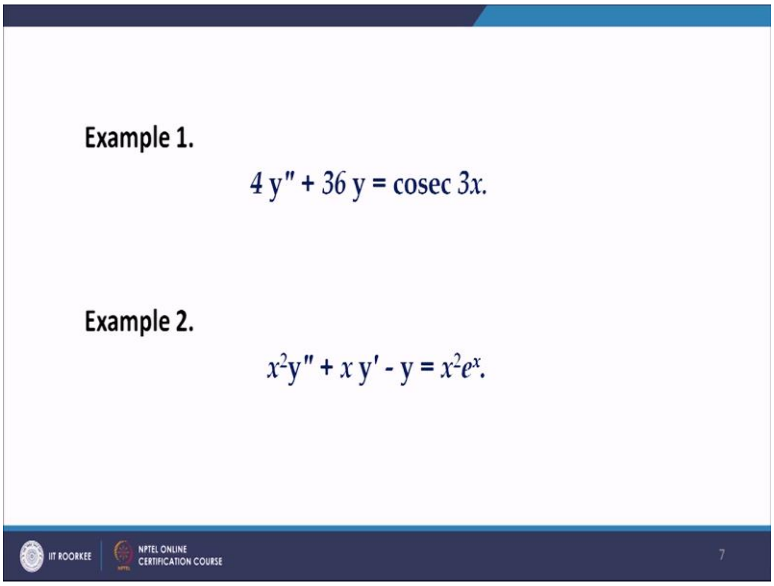

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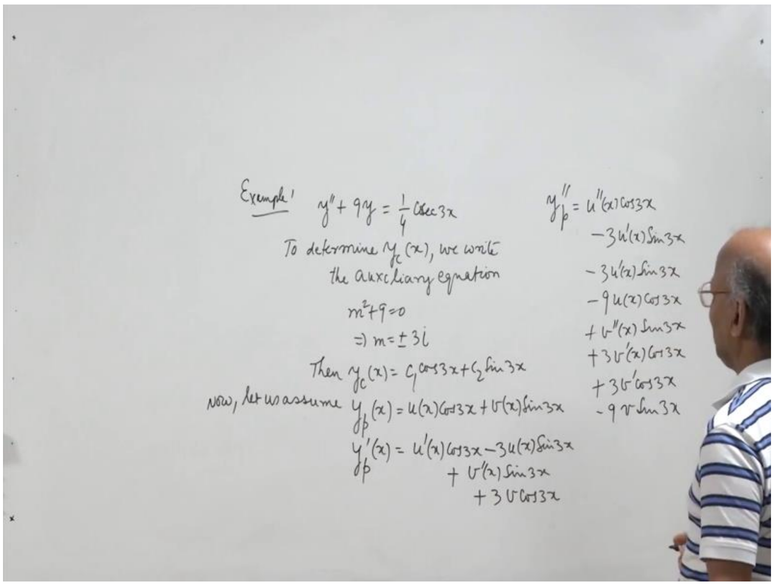

So, we have, we can write this equation in the standard form in the standard form y double dash plus 9 y equal to 1 by 4 cosec 3 x.

So first we have to determine the complimentary function here. So, to determine y c x, we write the Quasi Euler equation m square plus 9 equal to 0 which gives us 2 roots m equal to plus minus 3 i which are complex conjugate of each other. So, we can the complimentary function. So, then y c x will be equal to c 1 cos 3 x plus c 2 sin 3 x.

Now, let us assume y p x equal to u x cos 3 x plus v x sin 3 x. So, then we will differentiate y p x and y p y p double dash we will get. So, y p x dash will be equal to u dash x cos 3 x and then minus 3 times u x sin 3 x and then we will have v dash x sin 3 x plus v 3 times v cos 3 x.

We can find one more derivative y p double dash. So, y p double dash will be equal to u double dash x plus into cos 3 x minus 3 u dash x sin 3 x and then we will have here minus 3 u dash x sin 3 x minus 9 u x cos 3 x and then we shall have v double dash x sin 3 x and then 3 times v dash x cos 3 x plus 3 v dash 3 v dash cos 3 x minus 9 v sin 3 x. So, we can put these values of y p y p dash y p double dash in the equation y double dash plus 9 y equal to 1 by 4 cos six 3 x and then we assume in order to determine as we have shown in the method in order to determine the unknown functions u x and v x we assume that.

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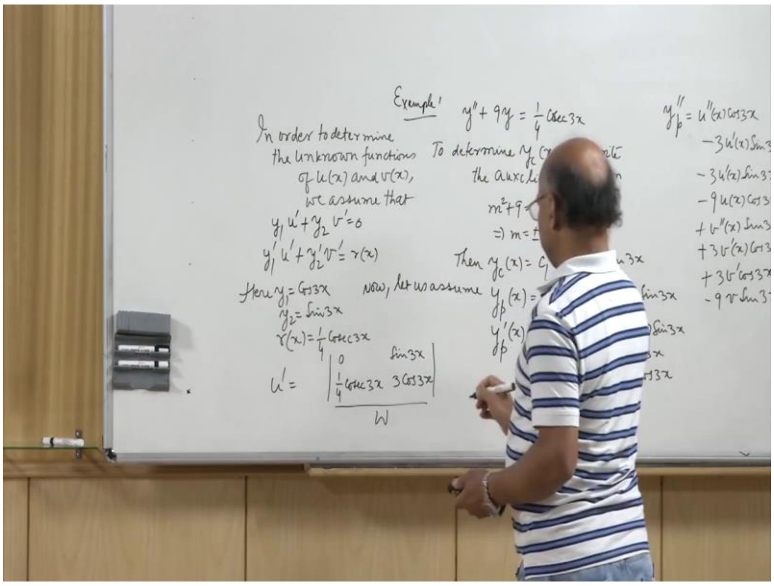

So, y 1 dash y 1 u dash plus y 2 v dash equal to 0 and y 1 dash u dash plus y 2 dash v dash equal to r x. Now here we have y 1 y 2 as this is y 1 y 1 is cos 3 x y 2 is sin 3 x. So, here y 1 we can take as cos 3 x and y 2 as sin 3 x r x is equal to 1 by 4 cosec 3 x. So, having these values we shall have the value of u dash. So, u dash will be equal to we will replace y 1 y 1 dash by 0 r x. So, we will have 0 r x that is 1 by 4 cosec 3 x and then y 2 y 2 dash. So, y 2 will be sin 3 x and y 2 dash will be 3 cos 3 x divided by w which will give you u dash equal to 1 by minus 1 by 4 sin 3 x cosec 3 x sin 3 x cosec 3 x when we multiply we get 1.

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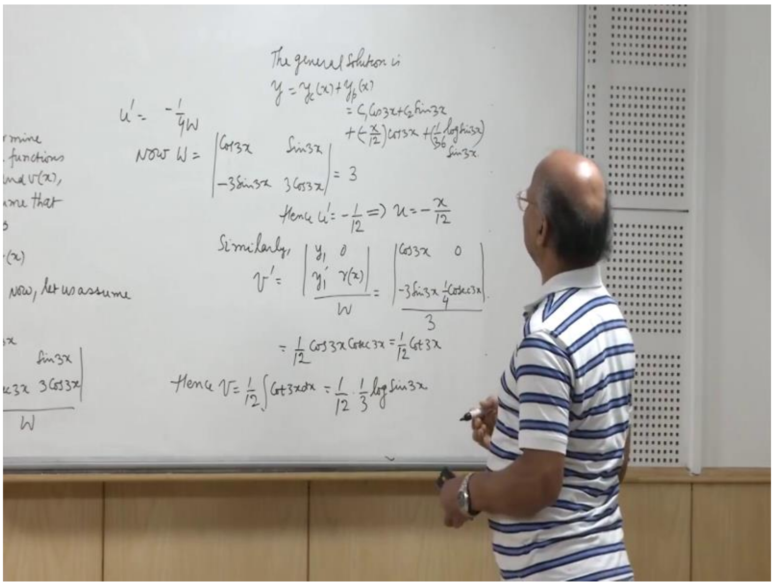

So, 1 by 4 w and now w is equal to w is equal to y 1 y 1 dash. So, y 1 is cos 3 x y 1 dash will minus 3 sin 3 x sin 3 x is y 2 its derivative is 3 cos 3 x.

So, you find the value of this determinant. So, 3 cos square 3 x plus 3 sin square 3 x will be equal to 3. So, we get hence u dash comes out to be minus one by 12 similarly let us find v x sorry; similarly let us find v x. So, v dash will be equal to v dash will be equal to 0 r x we replace y 1 y 1 dash 0 r x divided by w which is equal to y 1 is cos 3 x. So, cos 3 x minus 3 sin 3 x 0 cosec 3 x divided by w is equal to 3. So, we will get cosec 3 x into cos 3 x divided by 3. So, 1 by 3 cos 3 x into cosec 3 x which is one by 3 cot 3 x, let us find v from here.

So, hence v is equal to 1 by 3 integral. So, there is a correction here, it should be r x is 1 by 4 cos cosec 3 x. So, we shall have here one by 4 cos 3 x into cosec 3 x. So, this is 1 by 12. So, this is 1 by 12 cot 3 x. So, this will be 1 by 12 here integral cot 3 x d x.

Now this is equal to integral when we make this will be one by 12 into 1 by 3 log sin 3 x. So, we get 1 by log because derivative of log sin 3 x is 1 by sin 3 x into cos 3 x into 3. So, we get, so we get this. So, the; if we get the value of v which is 1 by 36 log sin 3 x and the value of u comes out to be this gives you the value of u; u equal to minus x by 12. So, we get the general solution the general solution is y equal to y c x plus y p x y c x is equal to c 1 cos 3 x plus c 2 sin 3 x and y p x is u x into cos 3 x. So, u x into cos 3 x means minus x by 12 cos 3 x and v x it comes out to be 1 by 36; 1 by 36 log sin 3 x this is v x into sin 3 x.

So, we get the general solution of the given differential equation. Now let us see similarly, I can tackle the equation number 2, there is a slight change in this. We will have to in the case of example 2, we will have to determine the complimentary function, let us see how we determine this?

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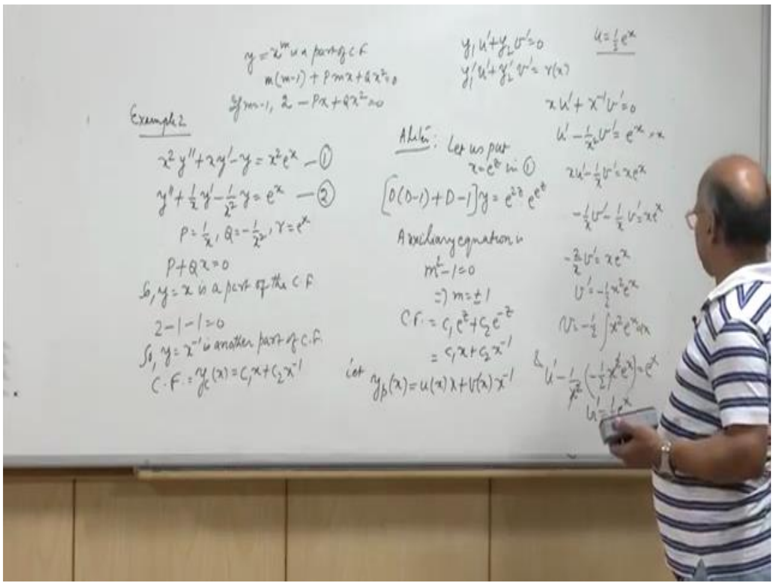

So, we have x square y double dash plus x y dash minus y equal to x square e to the power x. So, let us first bring this equation into the standard form. So, y double dash plus one over x into y dash minus one over x square into y equal to e to the power x we have. So, here p is equal to one by x q is minus one over x square and r is small r x small r is e to the power x small r x now we can see that p plus q x equal to 0 p plus q x equal to 0 means y equal to x is a part of the complimentary function.

So, we have got one solution; one integral involved in the complimentary function let us look at the other integral involved in the arbitrary in the other integral involved in the complimentary function that also let us see we can be obtained by the method of inspection we know that y equal to x to the power m is a part of the complimentary function provided m into m minus one plus p m x plus q x square equal to 0 . So, if i put here m equal to minus one then i got what will get what minus one into minus two. So, we get 2 minus p x plus q x square equal to 0 if m is equal to minus one let us see whether this condition is satisfied here. So, 2 minus p x 2 minus p x will be equal to one here 2 minus one and then q into x square. So, we get minus one equal to 0.

So, 2 minus p x my plus q x square is equal to 0. So, y equal to minus x sorry x to the power minus one is another part of the complimentary function. So, we have got both the integrals involved in the complimentary function. So, we can write general solution of the associated homogenous equation o we can the complimentary function y c x as c 1 e x plus c 2 x to the power minus 1. Now it was easy to get y; y equal to x, but y equal to x to the power minus 1 was not easy to get. So, we can there is another method by which we can find the complimentary function let us note that the given equation is Quasi Euler equation. So, the given equation is Quasi Euler equation. So, we can get solve it by the by the method of Quasi Euler equation and then we can find the complimentary function there.

So, let us put say for example, in the other method; let us put x equal to e to the power z in equation 1; in 1 then because it is a Quasi Euler equation, it will reduce to linear differential equation with constant coefficients. So, we will get d into d minus 1 plus d minus 1 y equal to x equal to e to the power z. So, e to the power 2 z into e to the power e to the power z we are just interested in finding the complimentary function here. So, the auxiliary equation will be this is d square minus d plus d minus. So, we get m square minus 1 equal to 0 which gives you m equal to plus minus 1 and so the complimentary function here will be c 1 e to the power z plus c 2 e to the power minus z because the 2 roots are real and distinct. So, we have c 1 e to the power z plus c t e to the power minus z now e to the power z is x. So, we get c 1 x plus c 2 x to the power minus 1.

Now, from here on we can apply the method of variation of parameters having obtained the complimentary function of the given differential equation. So, we shall assume that y equal to let us assume y p x equal to u x into y 1 y one is x plus v x into y 2 y 2 is x to the power minus 1 and then when we substitute y p x y p dash x and y p double dash x in equation 2 this in this standard form in equation 2 writing the equation in the form 2 is very essential when we apply the method of variation of parameters because r x will have to be taken as e to the power x not as x square e to the power x because the equations have been derived in the in a in a in the manner where the coefficient of y double dash is taken as unity.

So, the 2 equations involving the unknown functions u x and v x you might recall are y 1 u dash plus y 2 v dash equal to 0 y 1 dash u dash plus y 2 dash v dash is equal to r x . So, these are the 2 equations that come relate the unknown functions derivatives of unknown functions u and v. So, put the values of y, y 1, y 2, y 1 is your x. So, x u dash and y 2 is your x minus x to the power minus 1. So, x to the power minus 1 v dash equal to 0 and then y 1 dash y 1 is x. So, its derivative is 1. So, we get u dash and then y 2 dash will be minus 1 upon x square into v dash equal to r x is equal to e to the power x. Now you can solve this equation these 2 equations for the values of u dash and v dash. So, we know that if you multiply this equation we can apply Cramer's rule or we can solve it by otherwise also.

So, we multiply this equation by x and then subtract what do you get? So, if you multiply this by x you get x u dash minus 1 by x v dash equal to x c x from this equation then we subtract this 1. So, what do we get? Minus 1 by x v dash minus 1 by x v dash equal to x c x from this equation we subtract that equations we get this. So, minus 2 by x v dash equal to x e to the power x or v dash will be equal to minus 1 by 2 x square e to the power x and. So, we can obtain y minus half integral x square e to the power.

Now, you know how to solve this how you can integrate by parts and get the value of integral x square e to the power x now once we have the value of v dash as minus half x square e to the power x we can get u dash. So, u dash will be and u dash will be equal minus 1 by x square v dash. So, we will get minus one by x square v dash is minus half x square e to the power x equal to x c e to the power x from this equation u dash minus 1 by x square v dash equal to e to the power x.

So, this x square will cancel and we will get u dash plus half e to the power x equal to e to the power x. So, e to the u dash equal to 1 by 2 e to the power x, so its integral can be obtained easily. So, u is equal to 1 by 2 u to the power x. So, u can be obtained directly and u v can be obtained from here. Once we have values of u and v, we put them here. So, we will know y p x and once we have y p x y will be equal to y c x plus y p x and we will have the general solution of the equation given in example 2. With that I would like to conclude my lecture.

Thank you very much for your attention.