Mathematical methods and its applications

Dr. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 10
Method of reduction for second - order Linear differential equations

Hello friends. Welcome to my lecture on Method of Reduction for Ordinary Second-order Linear Differential Equations. It is very well known that the second-order linear differential equations with variable coefficients cannot be solved in general. Only very special types of linear differential equation with variable coefficients can be solved. So, we shall be looking at some special types of linear differential equations of second-order with variable coefficients whose solutions general solution can be found.

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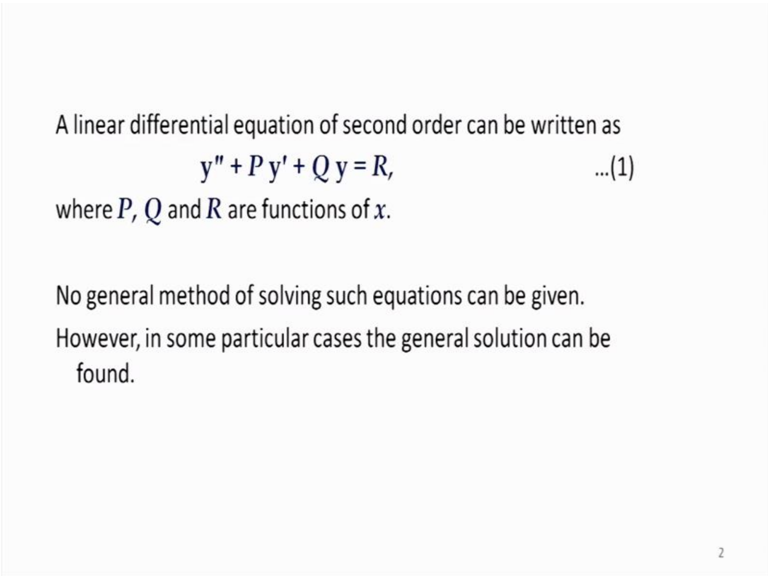

Suppose, we have a linear differential equation of second-order with variable coefficients; we can write the equation in the standard form as y double dash plus P y dash plus Q y equal to R. In this standard form, by standard form, I mean that the coefficient of y double dash here is unity. If it is not unity, you can divide the direct equation by the coefficient of y double dash and arrive at the standard form y double dash plus P y dash plus Q y equal to R. And here P, Q, R are functions of x - continuous functions of x. Now, no general method of solving such equations as I said no general method of solving such equations can be given in some particular cases.

However, the general solution can be found. So, we shall be discussing some particular cases of these second-order linear differential equations where we can find the general solution.

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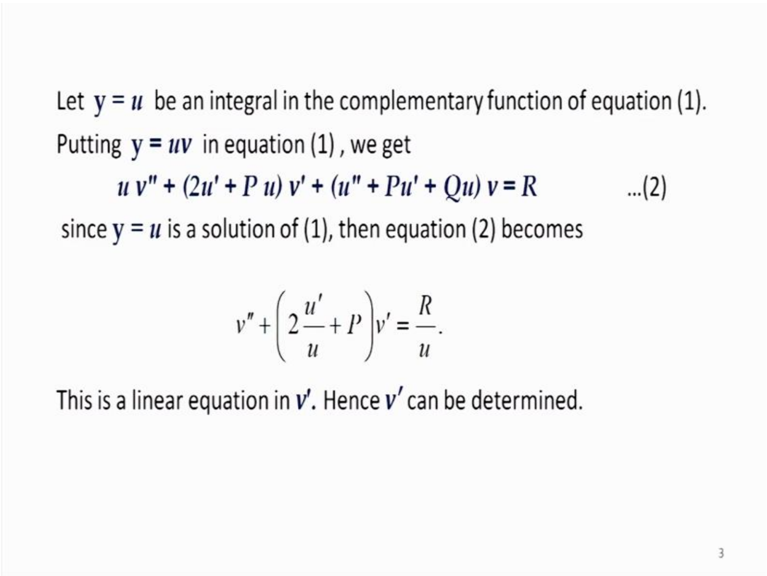

So, suppose we here make an assumption that the one integral in the complementary function of the equation 1 is known that is that complementary function as you might recall the is the general solution of the associative homogenous equation that is y double dash plus P y dash plus Q y equal to 0. Since, it is a second-order equation; it is general solution that is the complementary function will involve two independent functions of x say u and v.

Suppose, we know one integral R that is y integral v mean one solution, one solution involved in the complementary function of equation one so that means, that u will satisfy the homogeneous equation y double dash plus P y dash plus Q y equal to 0 that is u v dash plus P u dash plus Q u will be equal to 0. Now, let us then we shall assume that y equal to u into v, u is known to us, v is the function of x, which we are looking for. So, y equal to u in to v is the general solution of equation 1. So, if it is a solution of equation 1, then when we substitute y equal to u v in equation 1. Let us see what do we get.

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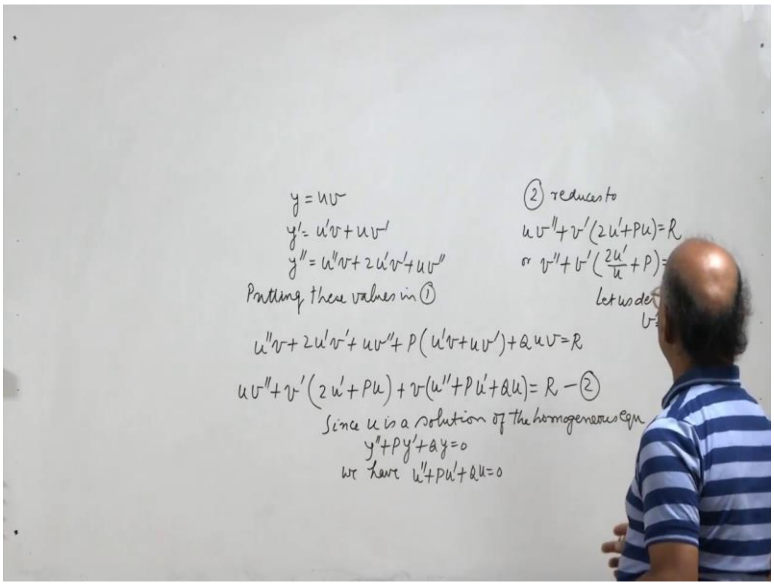

So, y equal to u into v gives you y dash equal to u dash into v plus u v dash and then y double dash will be equal to u double dash v plus 2 u dash v dash plus u v double dash. Now, let us put the values of y dash and y double dash in equation 1. So, then putting these values in equation 1, we get y double dash that is u double dash v plus 2 u dash v dash plus u v double dash plus p times u dash v plus u v dash plus Q y is u into v is equal to R.

Now, let us see. So, we will write it as a second-order differential equation in v. So, we shall first write u v double dash then we shall write v dash times 2 u dash plus P u. Now, then we shall have v times u double dash plus P u dash plus Q u equal to R. Now, since u is the solution of the associated homogeneous equation, we have since u is a solution of the homogenous equation y double dash plus P y dash plus Q y equal to 0. We shall have u double dash plus P u dash plus Q u equal to 0. So, our equation let me call it as equation number 2, this two reduces to, so two reduces to u v double dash plus v dash times 2 u dash plus P u is equal to R. Now, dividing this equation by u, we arrive at this equation. So, we arrive at v double dash plus 2 dash y u plus p into v dash plus R by u.

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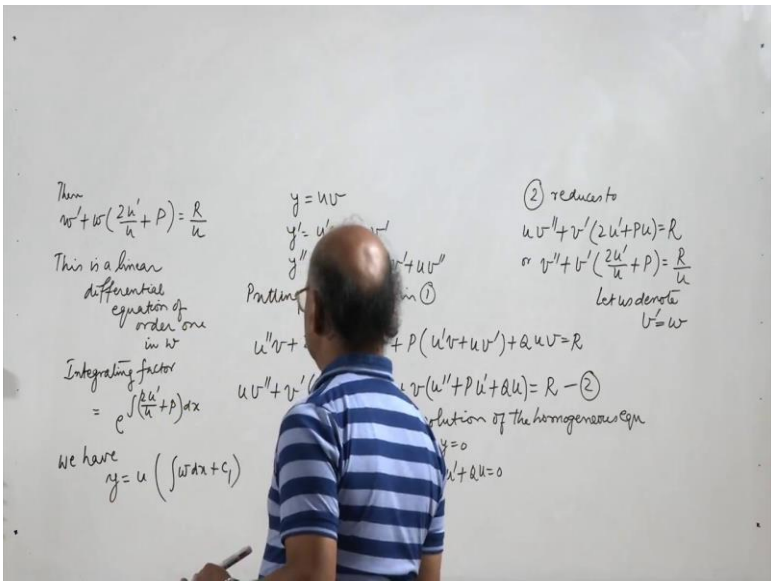

Now, let us denote v dash by w, v dash is dv over dx. So, dv over dx is another function of x let us write it as w. Then we shall have w dash plus w times 2 u dash by u plus p equal to R by u. Now, this is a linear differential equation of order one in w. So, from second-order linear differential equation, see we are looking for the solution of second-order linear differential equation y double dash plus P y dash plus Q y equal to R and we have come down to a first order linear differential equation. So, we know how to solve this linear differential equation of order one.

So, we find here integrating factor will be e to the power integral 2 u dash by u plus p dx after finding the integrating factor we can easily solve this equation in w. So, having w obtained then we will be knowing dv over dx. So, we shall again integrate that and will get the value of v. So, once we know v, we can multiply it by u and we will know the general solution y. Now, one arbitrary constant we shall have here. So, we shall have the final solution as y equal to u times v is integral w dx plus c 1. So, c 1 is one constant the other constant of integration will come from w when we obtain w here when we obtain w here one constant of integration will occur here.

So, one constant of integration will occur integration in w, and another constant of integration is this. So, therefore, we will have two constants of integration. So, once we know w, we can integrate it and get the general solution y.

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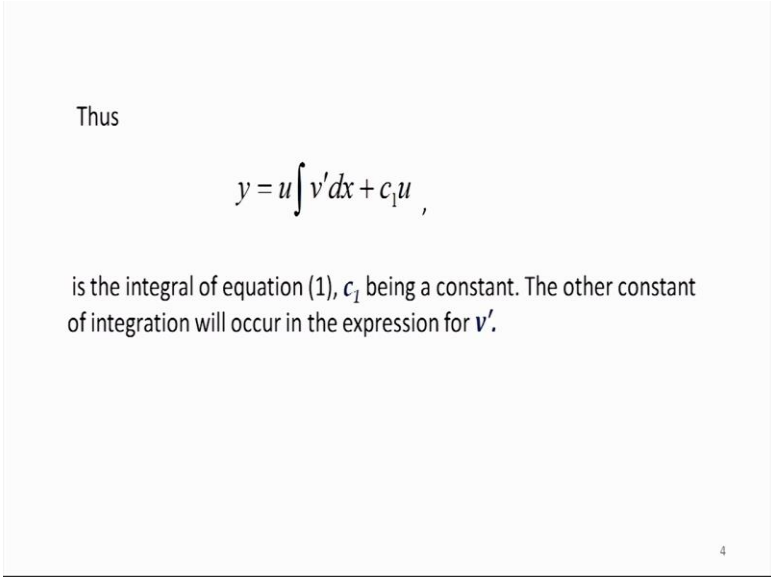

Now, let us see how we can use this method to, so this is what we have y equal u w is v dash v dash d x plus c 1 u, c 1 is a constant, the other constant of integration will occur in the expression for v dash. Now, let us look at see how to obtain one integral included in the complementary function.

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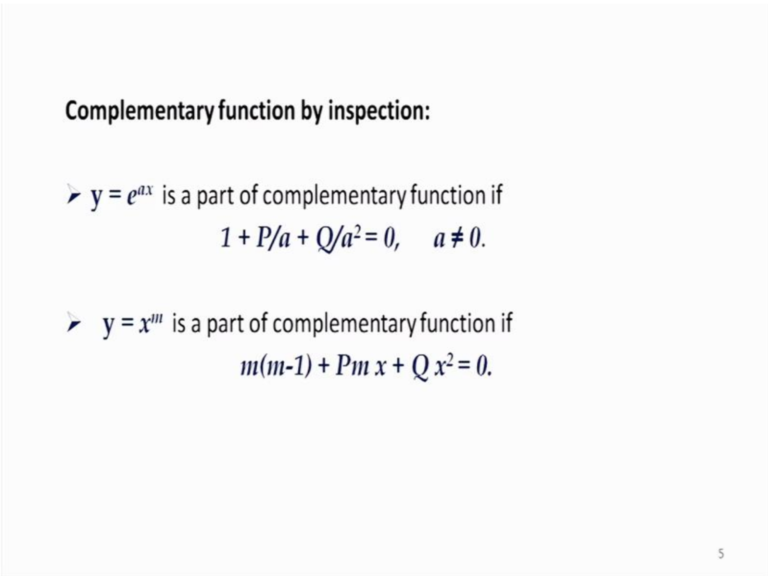

So, sometimes the one integral included in the complementary function can be found by inspection. Let us see, these formulas tell us that y equal to e to the power a x is a part of the complementary function if 1 plus p by a plus q by a square is equal to 0, a is not equal to 0. So, if the coefficients P and Q, which are functions of x satisfy this equality that is 1 plus P by a plus Q by a square is equal to 0 for some a, then you can take u to be equal to e to the power a x. And you can start with the equation solution y equal to e to the power a x into v, and solve the problem.

Now, if you find that for certain value of m, m into m minus 1 plus P m x plus Q x square turns out to be 0, then y equal to x to the power m is a part of the complementary function. And you can take u equal to x to the power m, and take the general solution the solution of the given equation to be y equal to x to the power m into v, and you can proceed to find the value of v. Now, let see some particular cases here which generally occur in the solution of linear differential equation with variable coefficients.

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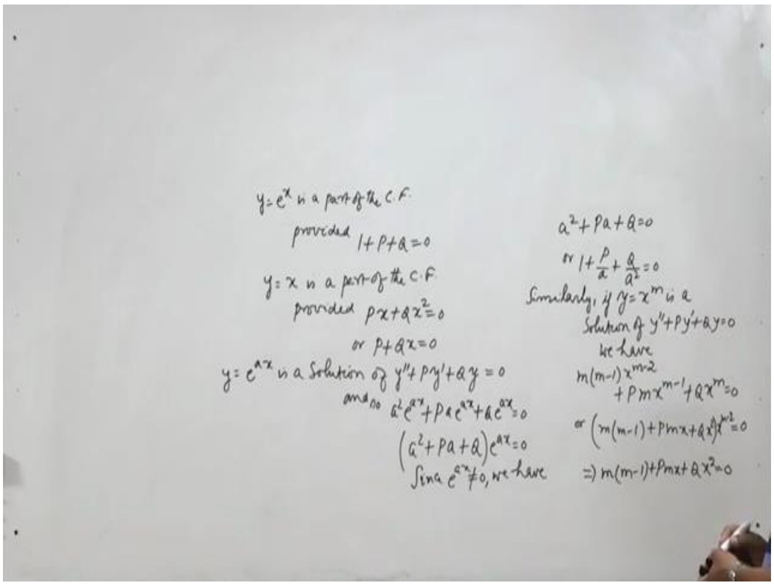

So, some particular cases which are of interest to us or for example, y equal to e to the power x, if you take a equal to 1 is a part of the complementary function C. F provided 1 plus P plus Q equal to 0. And when y equal to x, again let us take m equal to one is a part of the complementary function provided m equal to 1, we are taking. So, m into m minus 1 is 0, then P x plus Q x square is equal to 0 or we can say P plus Q x equal to 0, because when x is not 0, we can divide by x; so P plus Q x equal to 0. So, if you find that after multiplying Q by x and adding it to P, what we get is 0, then you can start with u equal to x, u will be a part of the complementary function. Now, let us see how we arrive at these formulas, when y equal to e to the power a x will be a part of the complementary function provided 1 plus P by a plus Q by a square equal to 0. How do we get this formula?

So, let us see, y equal to e to the power x is a solution of y double dash plus P y dash plus Q y equal to 0. So, y equal to e to power a x when we substitute in this equation it should be it is satisfy this equation. So, y double dash will come out to be a square e to the power a x plus P y dash will be a times e to the power a x and Q times e to the power a x equal to 0. Now, this I can write it as a square plus P a plus Q times e to the power a x equal to 0, now e to the power a x is never 0. So, we have a square plus P a plus Q equal to 0 or we can interpreted as 1 plus P by a plus Q by a square equal to 0. So, 1 plus P by a Q by a square equal to 0 must be satisfied if y equal to e to power a x is a part of complementary function.

Similarly, we can show the other one. Similarly, if y equal to x to the power m is a solution of y double dash plus P y dash plus Q y equal to 0, we will have y double dash means m into m minus 1 x to the power m minus 2, and then p times m into x to the power m minus 1. And then we have Q times x to the power m equal to 0. Or I can write it as m into m minus 1 plus P m x plus Q x square into x to the power m minus 2 equal to 0. See whenever x is not 0, x to the power m minus 2 will not be 0, so this implies m into m minus 1 plus P m x plus Q x square must be 0. So, m into m minus 1 plus P m x plus Q x square must be 0, if y equal to x to the power m is a solution of the associated homogeneous equation that is y double dash plus P y dash plus Q y equal to 0 now.

So now, this is so we can obtain the function u by checking these two conditions.

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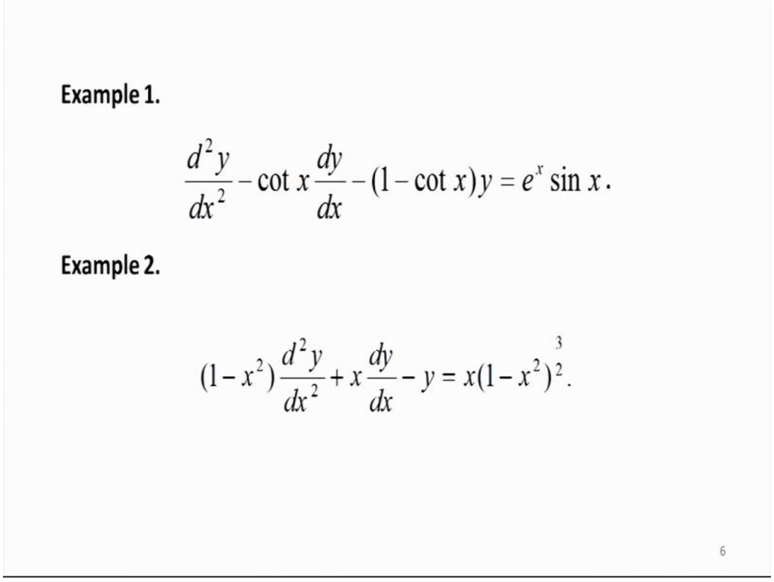

Let us now go to example, and see how we apply this method.

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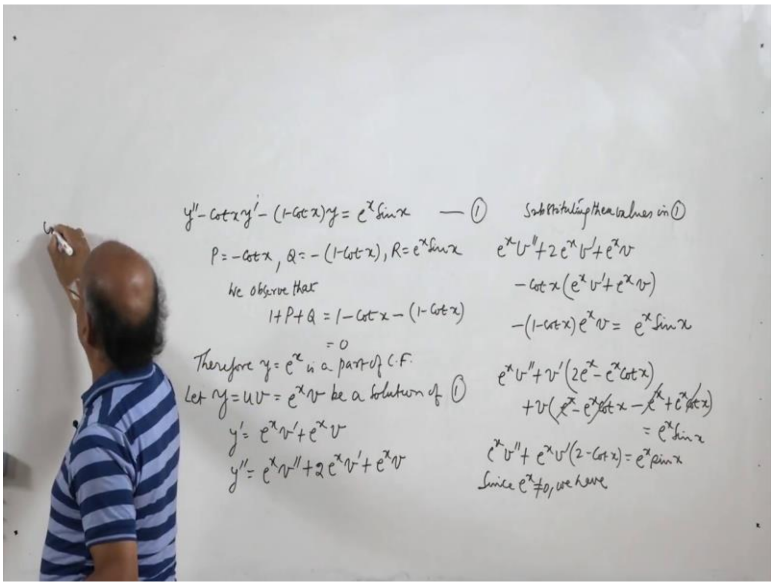

So, let us take up the second-order differential equation which is already given in the standard form, y double dash minus cot x into y dash minus 1 minus cot x into y equal to e to the power x sin x. It is already in the standard form, we do not have to do anything. We can straight away check and find u. So, here P is equal to 2 minus cot x, if you compare it with the standard form y double dash plus P y dash plus Q y equal to R. Q is equal to minus 1 minus cot x, and R is equal to e to the power x sin x. So, we can see we observe that 1 plus P plus Q comes out to be 0, and therefore y equal to e to the power a x that is a is 1 here. So, y equal to e to the power x is a part of the complementary function. So, it will play the role of u.

So, let us say, let y equal to u into v, we are looking for the function v; u is equal to e to the power x, v solution of let me call it equation 1. The given equation one then we shall have to find y dash and y double dash. So, y dash will come out to the e to the power a x into v dash plus e to the power x into v, and y double dash will be equal to e to the power x v double dash two times e to the power x into v dash plus e to the power x into v.

So, let us substitute these values of y and y dash y double dash in equation 1. So, substituting these values in equation 1, we shall have e to the power x v double dash plus 2 e to the power x v dash plus e to the power x into v minus cot x times y dash. So, e to the power x into v dash plus e to the power x into v minus 1 minus cot x into y is equal to u into v. So, e to the power x into v is equal to e to the power x sin x. Let us write it as a second-order differential equation in v. So, retain e to the power x v double dash, we write first then we write the terms in v dash.

So, v dash times 2 e to the power x minus e to the power x cot x. Then we write the term in v, the term in v is e to the power x. And the term here is minus e to the power x cot x; and here we have minus e to the power x plus e to the power x cot x equal to e to the power x sin x. So, this cancels the coefficient of v is equal to 0. So, we have e to the power x v double dash plus e to the power x into v dash 2 minus cot x equal to e to the power x sin x. Now, since e to the power x is never 0, we can divide by e to the power x. We have v double dash minus plus 2 times v dash times 2 minus cot x equal to sin x.

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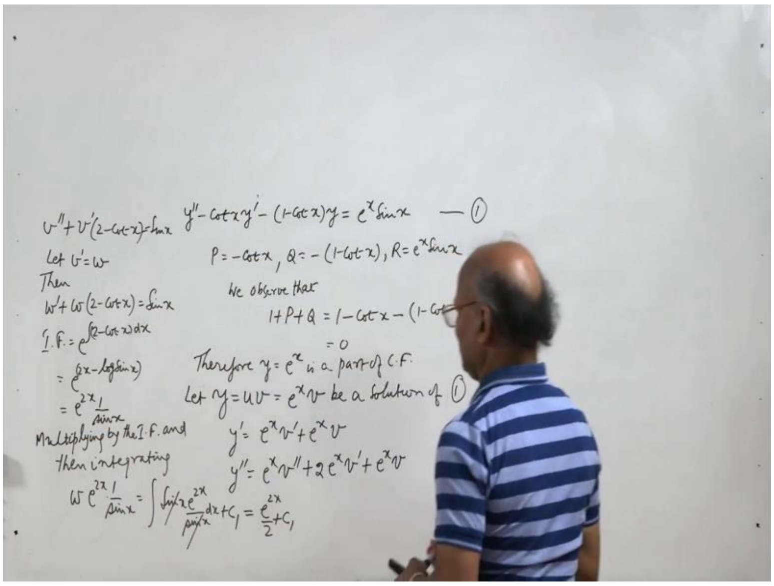

Now, let us denote v dash y w then we have w dash plus w times 2 minus cot x equal to sin x, it is a linear differential equation of order one in w. So, integrating factor is e to the power integral 2 minus cot x dx, this will be e to the power 2 x minus log sin x. So, this is e to the power 2 x and e to the power minus log sin x means e to the power log 1 over sin x. So, this is 1 over sin x, so this is the integrating factor. Now, we multiply this equation by the integrating factor, and integrate with respect to x, so multiplying by the integrating factor, and then integrating. We shall have w into e to the power 2 x into 1 upon sin x equal to sin x into e to the power 2 x upon sin x dx plus a constant let us say c 1. So, this will cancel with this, and we will get e to the power 2 x by 2 plus c 1. So, this will give the value of w.

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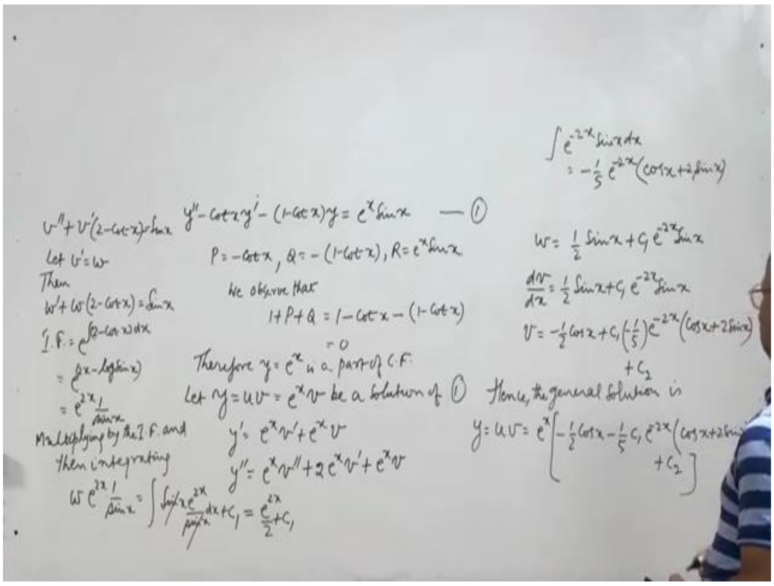

So, thus we get w as we can multiply the equation by sin x upon e to the power 2 x. So, 1 upon 2 sin x c 1 times e to the power minus 2 x sin x. So, this is d v by d x. Now, we can integrated with respect to x and obtain the value of v, v is minus 1 by 2 cos x plus c 1 times integral of e to the power minus 2 x sin x. When you integrate e to the power minus 2 x into sin x dx integration by parts you do then you will get this as minus 1 by 5 e to the power minus 2 x cos x plus 2 sin x you can. So, when you integrate by parts, e to the power minus 2 x sin x dx, what you get is minus 1 by 5 e w to the power minus 2 x cos x plus 2 sin x. So, let us put that here; so minus 1 by 5 e to the power minus 2 x cos x plus 2 sin x plus a constant c 2.

Now, hence the general solution is y equal to u into v which is e to the power x into v. So, v is a minus half cos x minus 1 by 5 c 1 e to the power minus 2 x cos x plus 2 sin x plus c 2. So, this is the general solution of the equation given in example 1.

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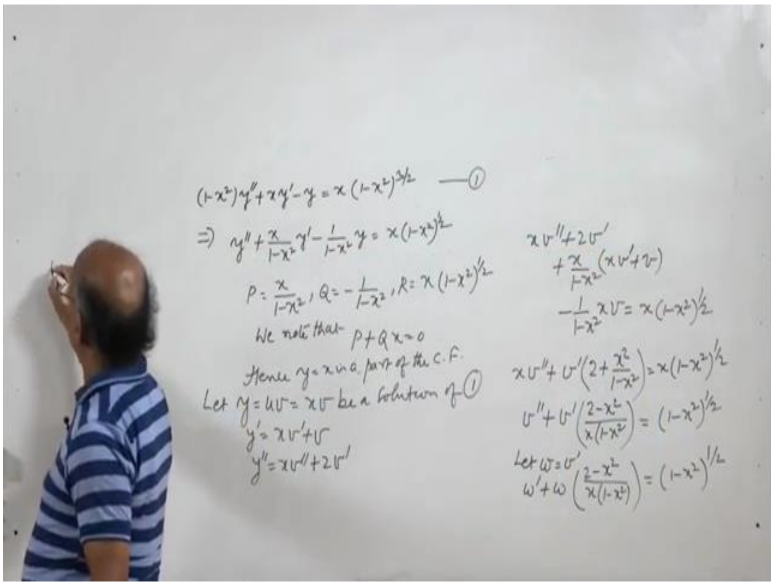

Now, let us take another example to show the working of this method. Now, in the example 2, we are given 1 minus x square y double dash plus x y dash minus y equal to x into 1 minus x square to the power 3 by 2. Now, first we will bring this equation to this standard form. So, we divided the entire equation by 1 minus x square. So, we get y double dash plus x upon 1 minus x square into y dash minus 1 upon 1 minus x square into y equal to x into 1 minus x square raise to the power half. So, thus P is here is equal to x over 1 minus x square, Q is minus 1 over 1 minus x square, R is x 1 minus x square to the power half.

So, we note that clearly P plus Q x equal to 0, and hence y equal to x is a part of the complementary function. So, we can take u as x, and then we assume, so let y equal to u into v that is x into v be a solution of the given equation. We can take it again as 1, then y dash will be x v dash plus v y double dash will be x v double dash plus 2 v dash. Substituting these values in this equation in the standard form, so y double dash x v double dash plus 2 v dash plus x over 1 minus x square y dash is x v dash plus v and then minus 1 over 1 minus x square into y is x into v is equal to x times 1 minus x square to the power half.

So, we can collect the coefficients of v double dash v dash and v, x v double dash v dash plus we get plus we dash times 2 plus x square upon 1 minus x square. And then we get v x upon 1 minus x square, and here v x upon 1 minus x square they cancelled out, and we get x times 1 minus x square to the power half. Now, divide by x, we get v double dash plus v dash this is to minus 2 x square plus x square, so 2 minus x square divided by x into 1 minus x square. And then we have 1 minus x square to the power half now this is a linear differential equation of order one in v dash.

So, let w v equal to v dash, then we have w dash plus w times 2 minus x square upon x times 1 minus x square and then equal to 1 minus x square to the power half.

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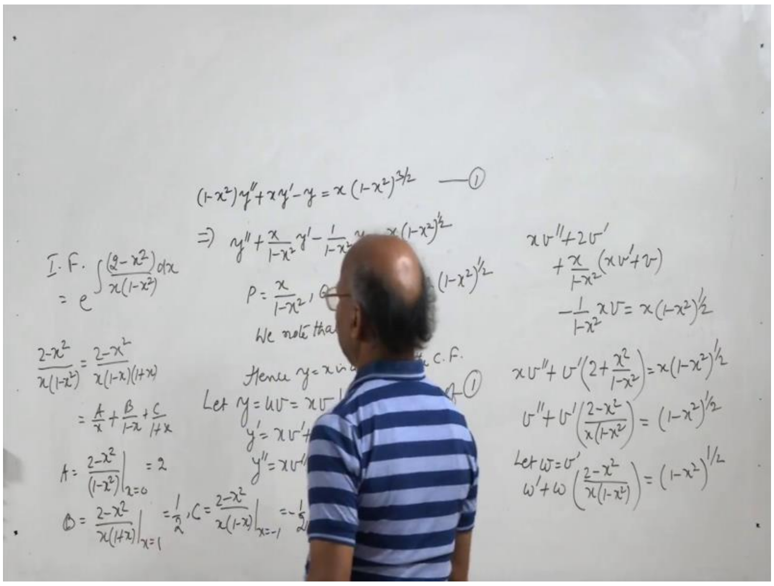

So, this first order linear differential equation in w, we will find the integrating factor here. So, integrating factor is e to the power integral 2 minus x square upon x into 1 minus x square. Now, in order to integrate 2 minus x square over x into 1 minus x square, we need to break it into partial fractions. So, 2 minus x square upon x into 1 minus x square can be written as 2 minus x square upon x into 1 minus x into to 1 plus x. Then we can write the partial fractions as a over x plus b over 1 minus x plus c over 1 plus x.

Now, the values of a b c can be found directly from the given identity. So, a will be equal to 2 minus x square upon remove x from here, we have 1 minus x into 1 minus x. So, 1 minus x square evaluated at x equal to 0. And this will give you 2. Similarly, we remove 1 minus x from here, so 2 minus x square x into 1 plus x and v evaluated at x equal to 1. So, we get 2 minus 1 that is 1, 1 upon 2. And similarly, we get c, c as 2 minus x square upon, now we will have a x equal to minus 1. So, we will put x into 1 minus x at x equal to minus 1. And this will give you 2 minus 1, so that is 1, 1 upon minus 1 upon 2.

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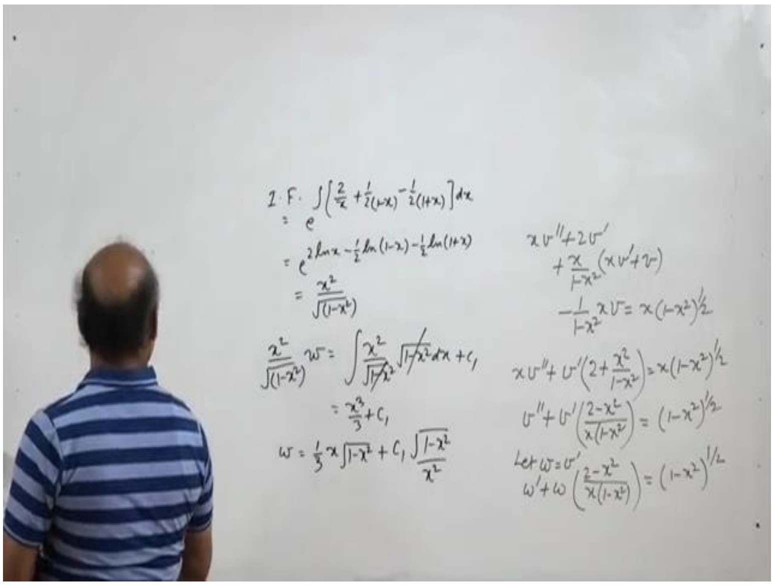

So, with these values of a, b, c, we can now then find the integrating factor the integrating factor will be, so having found the values of a, b, c then we have integrating factor as e to the power integral 2 over x and then we have v as 1 by 2. So, 1 by 2 and then we have minus half 1 plus x the value of c, so this dx. And this will be e to the power two l n x then minus 1 by 2 ln 1 minus x minus 1 by 2 ln 1 plus x. So, this is nothing but x square upon under root 1 minus x square. So, we multiply this. So, having found integrating factor, we multiply the integrating factor to the differential equation here. And then integrate, so we will have x square upon under root 1 minus x square into w equal to integral x square upon under root 1 minus x square into under root 1 minus x square dx plus a constant of integration c 1. So, this cancels with this, we get x cube by 3 plus c 1.

So, this gives you w equal to 1 by 3 x under root 1 minus x square plus c 1 under root 1 minus x square divided by x square, and this is nothing but dv by dx.

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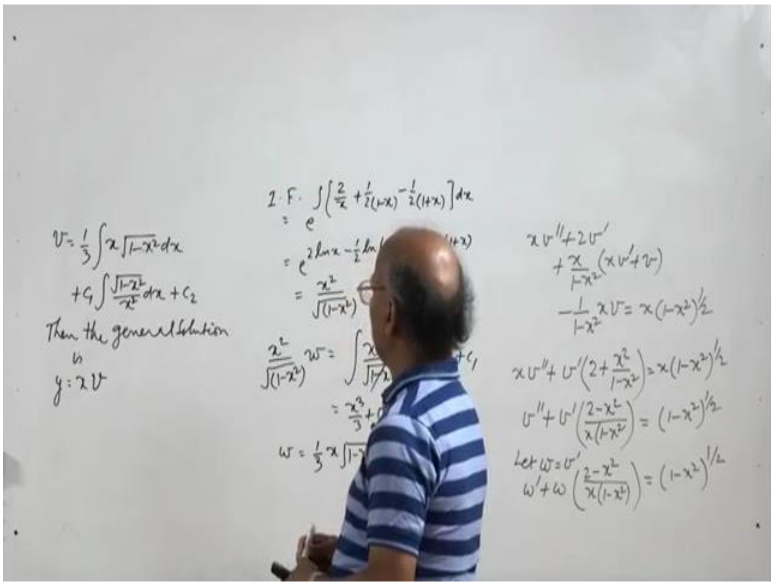

So, thus dv by dx is, so v is equal to this is dv by dx, so v is equal to 1 by 3 integral x under root 1 minus x square dx plus c 1 integral under root 1 minus x square divided by x dx plus c 2. Now, you know how to determine the integrals of x into under root 1 minus x square and under root 1 minus x square by x square. So, find these integrals, substitute here and then general solution will be y equal to u into v, u is x here, so x times v, so x into v. We can put the value of v here. Since, we has two arbitrary constants c 1 and c 2, we will get general solution of the equation given in example 2. With that, I will like to conclude my lecture.

Thanks.