Mathematical methods and its applications

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Lecture – 01
Introduction to Linear Differential equations

Hello everyone. In my lecture of today on Mathematical Methods and Applications, I will discuss about the introduction to linear differential equations. Let us first see what do we mean by a differential equation. The there are 2 types of differential equations that are current literature: the ordinary differential equations and partial differential equations. By ordinary differential equations we mean an equation, which involves the derivates of an unknown function with respect to the independent variable.

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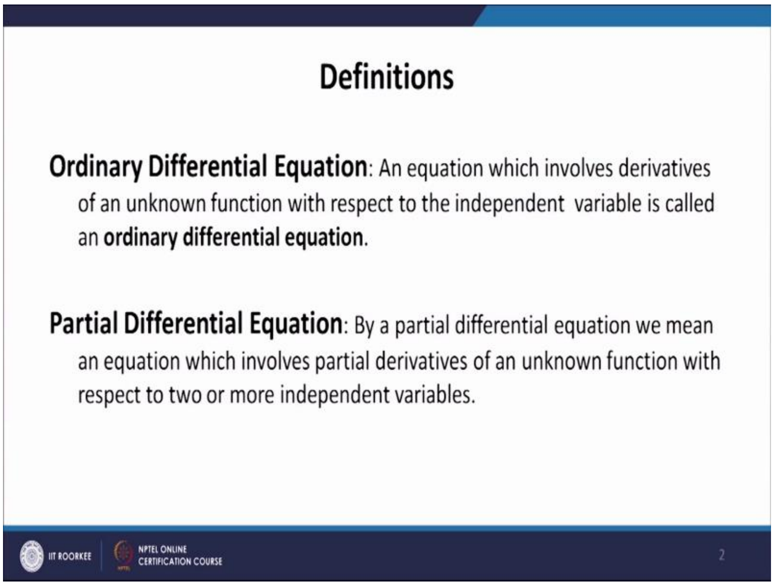

While by a partial differential equation we mean an equation which involves the partial derivatives of an unknown function with respect to the two in are more independent variables for example.

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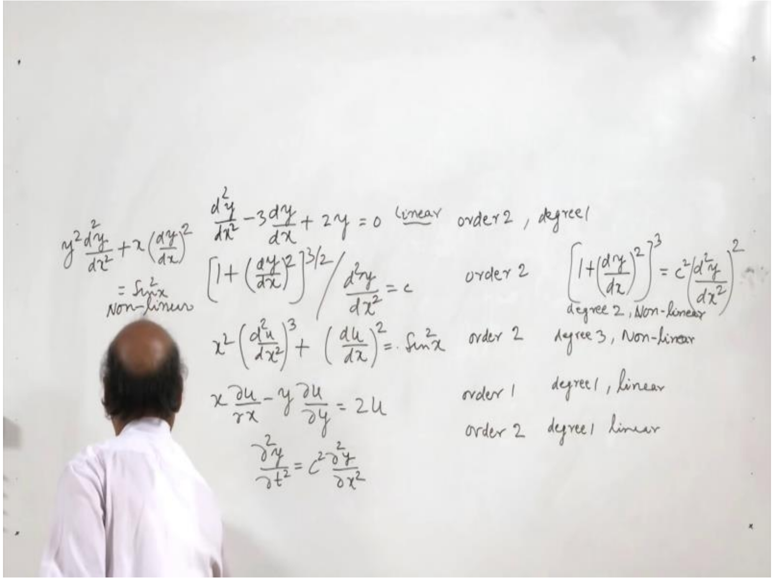

Suppose, let us write d square by over d x square, minus 3 d y by d x plus 2 y equal to 0, 1 plus d y by d x whole square raised to the power 3 by 2 over d square y by d x square equal to a constant, and then I may write as say x square, d square u by d x square, raised to the power 3, plus d square d u by d x whole square equal to sin square x. We can see that these equations involve the derivatives of the unknown functions y and u, with respect to x. So, they are ordinary differential equations, let us look at this differential equation. Now these equations involve partial derivatives of an unknown function u here with respect to x and y, and of an unknown function y here with respect to x and t. So, these two are partial differential equations. So, first three equations are the examples of ordinary differential equations, while the next two are examples of partial differential equations.

Now, let us see what do we mean by the order of a differential equation. The order of a differential equation is the order of the highest derivative that occurs in the differential equation. For example, the first differential equation is of order 2, the second differential equation is of order 2, third differential equation is of order 2, and these partial differential equations are of order 1 and 2. So, the while the degree of a differential equation is the degree of the highest derivative agreeing in the differential equation, after the differential equation has been made free from the radicals and fractions as far as the derivatives are concerned.

For example here the first equation is of degree 1, because the power of the highest derivative d square x by d y x square is 1. The second differential equation contains the radical 3 by 2 and also the fraction. So, to find the degree of this differential equation, we have to first of make it free from radicals and fractions. So, what we do is we write it as; now this equation is a differential equation, ordinary differential equation that is free from are the fractions and radicals, and we see that the power of the highest order derivative that is d square y by d x square is 2. So, it has got degree 2, here the highest derivative is second derivative and its power is 3. So, it is of degree 3, here we have first order derivatives only the order is 1. So, degree is also 1, here we have second order partial derivatives with respect to t and x and the power is 1. So, we have degree 1.

Now, now let us see what do we mean by a linear differential equation. A differential equation is said to be a linear if the dependent variables, and all its derivatives appear in the differential equation to the power 1, it does not have the products of the dependent variable and the derivatives are the functions of the dependent variable are the functions of the derivatives. So, let us say for example, suppose we consider this differential equation, it is a second order differential equation and we see that there is a product of y square that is the dependent variable y, y square product of y square and the derivative of y that is d square by over d square d x square. So, it is a non-linear differential equation. A differential equation which is not linear is said to be a non-linear differential equation.

Now, here if you see this is a linear differential equation, this is a non-linear differential equation, this differential equation is also non-linear and these are linear differential equations. Now general form of a linear differential equation; at linear differential equation in general can be written as an nth order differential equation linear differential equation is of the form.

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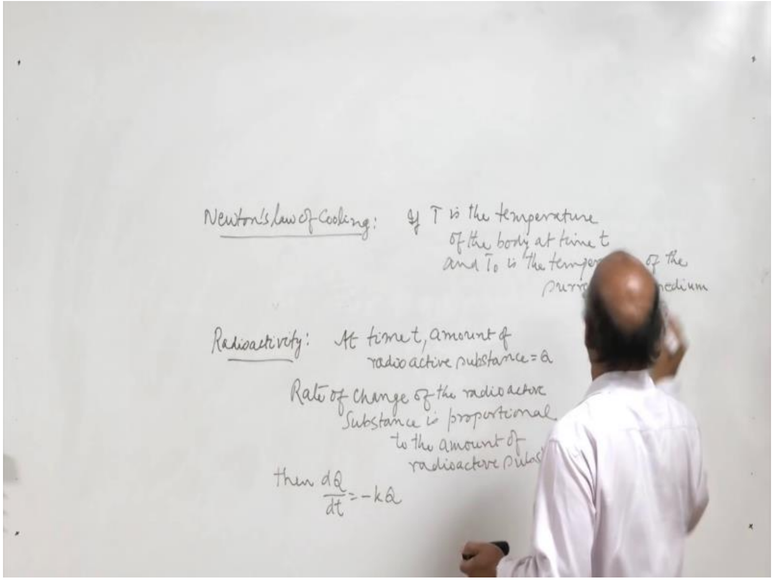

So, in nth order linear differential equation of the form a naught x, plus y to y n plus a 1 x y n minus 1 and so on. A n x y equal to r x, where a naught a 1 x, a 2 x, a n minus 1 x a n x and r x are functions of x, y n denotes the nth derivative of y. So, with respect to x so I mean to say y r is nothing, but d r y over d x r where r varies from 1 to n. So, this is a linear differential equation.

Now, let us see these applications of differential; there are many physical phenomena which involve the ordinary differential equations. So, for example, let us look at the radioactive radioactivity, let us consider a radioactive substance let us say a time t, the amount of radioactive substance is Q. It is known that the rate of change of the radioactive substance rate of substance is proportional to the amount present. Then we shall have the differential equation where k is a constant, d Q by d t represents the rate of change of the radioactive substance with time and Q is the amount present. Now the negative sign is being taken because as time t increases Q decreases. So, these are first order differential equation.

Another example we can take, it is known that if t is the temperature of a body, then the rate of decrease of rate of change of the temperature t of the body is proportional to the difference between the temperature of the body and the surrounding medium. So, if you say t is the temperature of the body, at time t and t naught is the temperature of the surrounding medium; period of change of temperature of the body is equal to minus k times t minus t naught. Where k is again a cons the constant of proportionality, and the negative sign is taken because s t increases capital T decreases.

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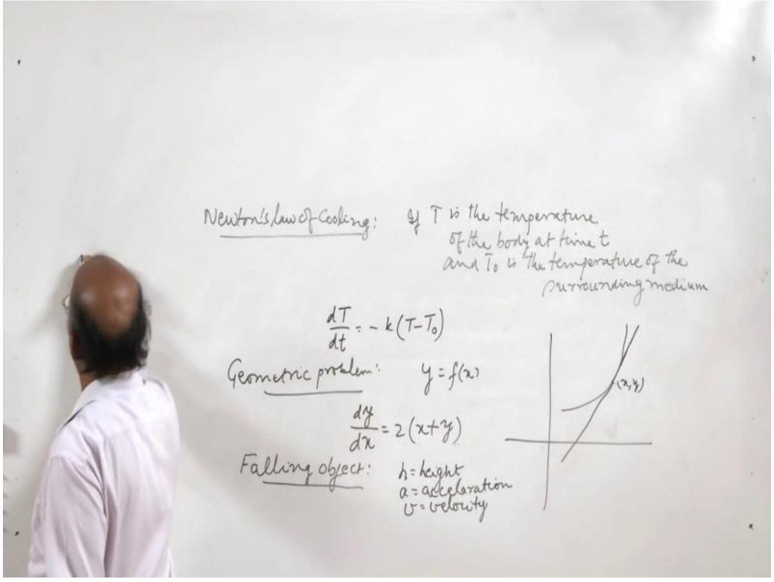

Now, let us look at some geometric problems, let us say we have a function y equal to f x; let us consider function y equal to f x at a point x y, let us say the slope of the tangent to the curve at the point x y is this is twice the sum of the coordinate section y; so d y by d x. So, if this slope of the tangent at the point x y is given by d y by d x, and it is given that it equals twice the sum of the coordinate section y. So, we have d y by d x equal to 2 times x plus y. So, this is again a geometric problem where first order differential equation we object we get and then we have let us say another example, falling object. Suppose we have a body that falls from a certain height at time t equal to 0.

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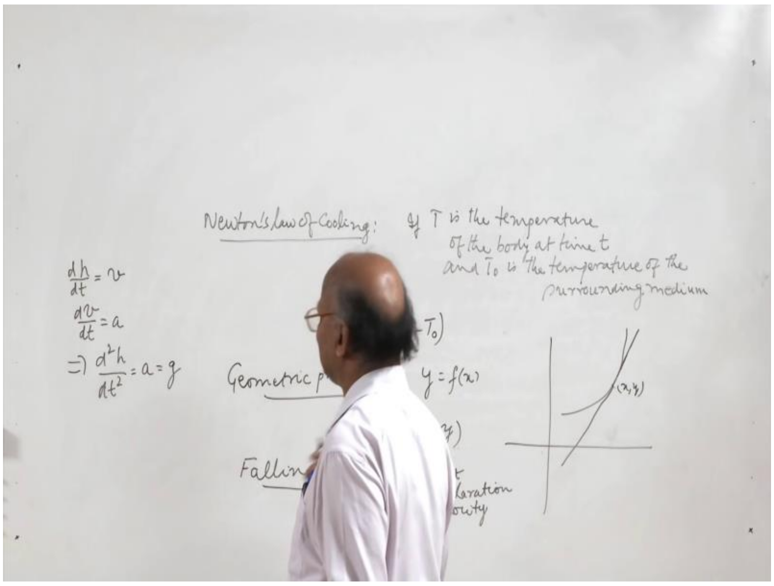

So, at time t let us assume that h is the height of the body, a is the acceleration, v is the velocity; then we shall have h equal to d h by d t equal to v, and d b by d t equal to a. Now from here we can see that this gives you d square h by d t square equal to a. Now in the case of body which we falling from a certain height a is equal to the acceleration due to gravity. So, we can put it as equal to g. So, this gives us a second order differential equation d square h by d t square equal to g. So, there are many such applications which occur in problems in physics then in engineering and sciences and so in. So, for example, in science we can one can talk of population dynamics, or we can have problems where we can study the growth of the bacteria, we can put that model in the form of a differential equation.

So, now we can talk of solution of a second order differential equation. By solution of a second order differential equation we shall we are concentrating on second order differential equation, because the theory that we shall study for second order differential equation can be easily extended to nth order differential equation, with constant coefficients. So, let us concentrate on second order differential equation by solution of a differential equation we mean a function y equal to phi x, which 1 substituted in the differential equation the reduces to it an identity.

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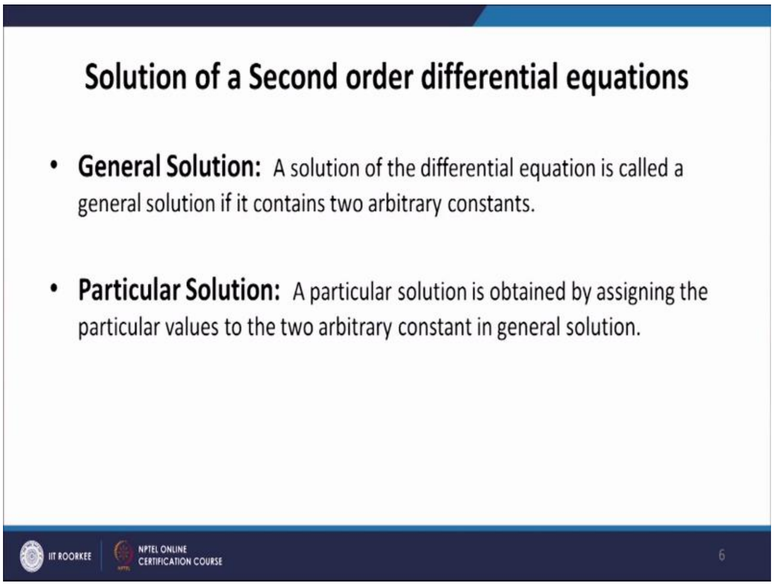

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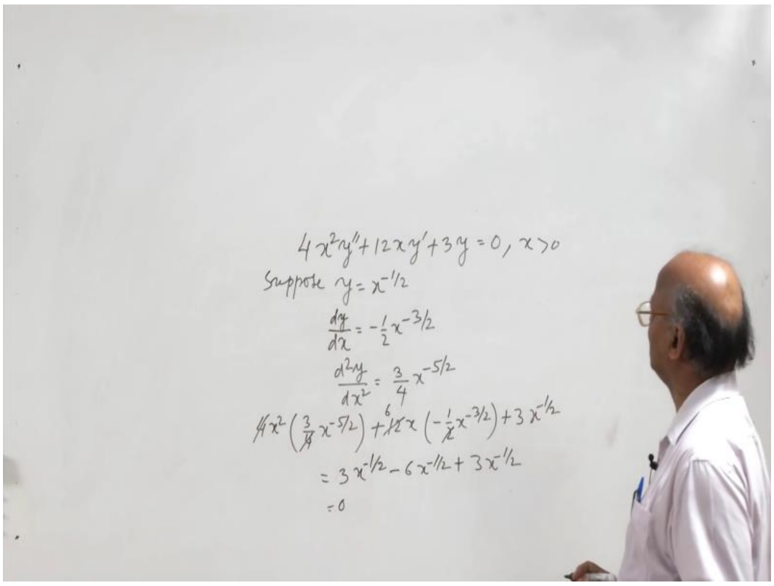

Now, general solution; general solution is the most general form of the solution without any initial conditions, where the initial conditions are not taken into account. If you are solving a second order differential equation, then its general solution contains 2 arbitrary constants. Particular solution is a solution, which is obtained from the general solution by assigning some particular values to the 2 arbitrary constants.

Now for example, let us look at this question. So, we have 4 x square y double dash, plus 12 x by dash, plus 3 y equal to 0. Suppose we have this differential equation, it is a differential equation of second order, now we let us see let us say suppose y equal to. So, be we have a differential equation of second order, and we have to solve it for x greater than 0. Suppose we have y equal to x to the power minus half, we are going to see that y equal to x to the power minus half is a solution of this differential equation. So, to see this let us find its partial derivatives, let us substitute these values of the partial derivatives in the given equation.

So, 4 x square y double dash means 3 by 4, x to the power minus 5 by 2, plus 12 x, then y dash is minus half, x to the power minus 3 by 2 and then we have 3 x to the power minus half; now this is equal to. So, this is 3 times x to the power minus half; then we have, so 6 times x to the power minus half, and then 3 times which is equal to 0. So, y equal to x to the power minus half satisfies this differential equation and therefore, it is the solution of the differential equation, now one may question one may ask is that why where we have used x greater than 0 in this solution.

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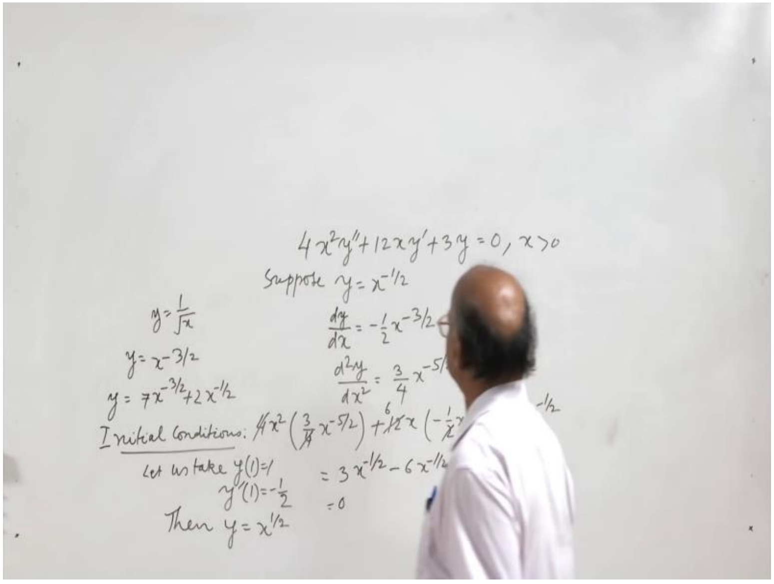

So, let us see the function y equal to x to the power minus half, y equal to x to the power minus half we can write as y equal to 1 over square root x. Now we can see that 1 over square root x it is now square root x is not defined when x is negative, and when you take x equal to 0 again 1 by root x is not defined. So, x greater less than or equal to 0 has to be excluded from the region of solution and therefore, the solution that we have obtained y equal to x to the power minus half is valid for x greater than 0. There is another solution say let us say y equal to x to the power minus 3 by 2, if we take y equal to x to the power minus 3 by 2 we can check that it also satisfies the given differential equation. So, and also you can take any combination linear combination of these solutions, they also satisfy the given differential equation.

Now, the question is one may ask which solution we should have, it depends on the choice of the requirement of the particular person. So, for that if we want per one particular solution, then we need to put some conditions on the differential equation, we call the conditions that are given at a certain point say x naught as the initial conditions. So, initial conditions are conditions that are put on this solution, so that we can obtain a particular solution which we are looking for. So, for example, let us say suppose we have in this differential given differential equation, let us put the conditions let us take y 1 equal to 1, and y dash 1 equal to minus half. Then we can see that y equal to x to the power half satisfies y 1 equal to 1, and also y dash 1 equal to minus half. So, y equal to x to the power minus half is the only solution of the given differential equation satisfying the initial conditions.

Now, the number of initial conditions that will be required for a given differential equation depends on the order of the differential equation. So, here the order of the differential equation is 2. So, we need only 2 initial conditions; for nth order differential equation we shall need n initial conditions.

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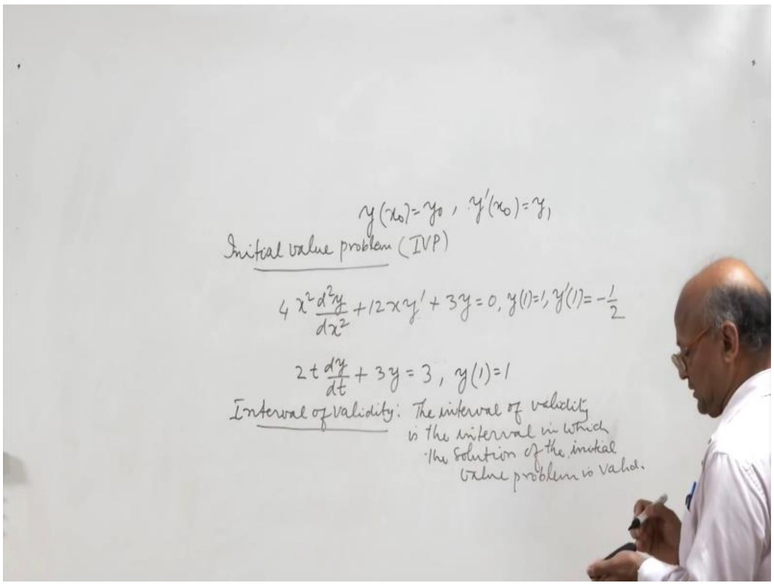

Now why do we call these conditions are initial conditions, because the point y at suppose we are given y at x naught equal to y naught, and y dash x naught equal to y 1, then these conditions are given at the point x naught generally in particular applications this x naught is nothing, but the time variable t, and so the conditions are given at time t equal to 0, that is why we call them as initial conditions.

Now, let us talk of an initial value problem. So, an initial value problem is a differential equation second order differential equation with the appropriate initial conditions. So, for example, we can have x square, d square y over d x square like we have written just now 4 x square d square y over d x square, plus 12 x by dash, plus 3 y equal to 0 where we have y 1 equal to 1, y dash 1 equal to minus half. So, this is a second order differential equation and we are given 2 initial conditions at the point x equal to 1, it is an initial value problem. Now we can suppose we take a second order first order differential equation 2 d y, 2 t d y by d t plus 3 y equal to 3 and put a condition on this let us say y 1 equal to 1, then it will be a again a initial value problem.

Now, interval of validity; the solution will be valid in a certain interval, in the in the case of an initial value problem the interval in which the solution is valid is called as the interval of validity of the initial value problem. So, the initial value problem the interval of validity is the interval in which the solution of the problem initial value problem is valid. Initial value problem in short we also denote by IVP.

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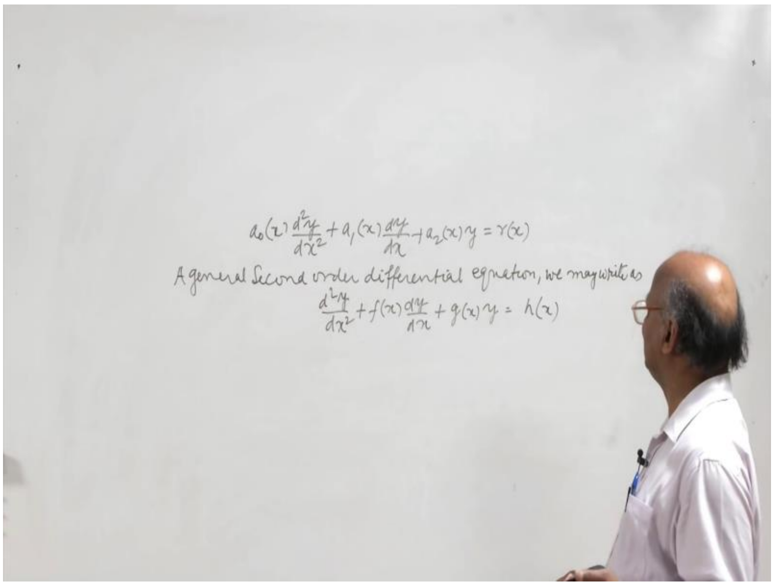

Now when we find this general solution of a differential equation, second order differential equation let us say second differential equation general second order differential equation we may take. So, a naught x, the general second order a differential equation we shall be first concentrating on because it is not easy to solve by general second order differential equation with variable coefficients. So, first we shall be solving second order differential equation with constant coefficients. So, if let us consider second order differential equation a general second order differential equation with constant coefficients; with differential equation; let us say a general second order differential equation we may write as.

So, this is general second order linear differential equation, we shall be considering a particular case of this later on, where we shall discuss second order differential equation with constant coefficients. So, they are these f x and g x here are called coefficients of the equation. So, in those second order differential equation with constant coefficients we shall be taking f x and g x as constants. Now before that when we consider this second order differential equation with the coefficients f x and g x, the general form of the solution involves 2 arbitrary constants. So, it is the matter of great interest whether what are the conditions under which the solutions of the differential equation suppose y 1 and y 2 are 2 solution of this differential equation, then when they will be independent.

The linearly linear dependence independence plays a important role in the solution of the second order differential equation, and they are the Wronskian also plays an important plays an important part by finding the Wronskian of the solution y 1 and y 2, one can see whether they are linearly dependent or independent. So, this aspect we shall be discussing in our next lecture.

Thank you.