INDIAN INSTITUTE OF TECHNOLOGY
ROORKEE

NATIONAL PROGRAMME ON TECHNOLOGY
ENHANCED LEARNING
(NPTEL)

Discrete Mathematics

Module-02
Logic
Lecture-3
Methods of proof of an implication

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In this lecture we will discuss methods of proof of an implication.

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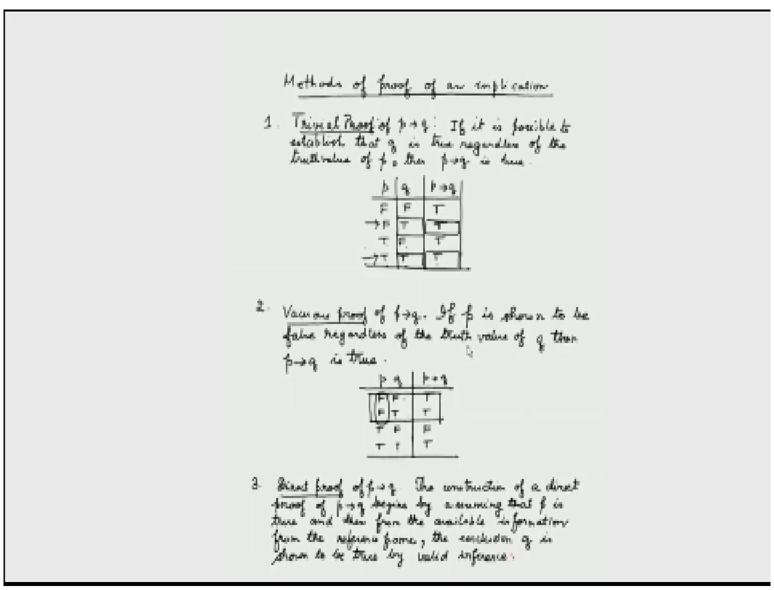

So far we have used truth tables to specify propositional functions including the basic functions involving a single logical connectives. We have also encountered tautologies which are foundations upon which valid inference are based. Now we shall apply valid inference patterns to validate nine common methods for proving implications. These will be required; these will be referred to as methods of proof.

So the first method of proof is trivial proof of P→Q. If it is possible to establish that Q is true regardless of the truth value of T, then P→Q is true. Now this happens because if we see the truth table of P→Q, we see that when P is F and Q is F that is P and Q both are false, then P→Q is true, when P is false and Q is true then P→Q is true, when P is true and Q is false then P→Q is true, and when P is true and Q is true, then P→Q is true.

This means that if so happens that we know that Q is always true, then P→Q is true regardless of what happens to P. This is of course trivial and hence it is called the trivial proof. The second method of proof is vacuous proof. If it is possible to establish that Q is true no, I will reward this. So here if P is shown to be false regardless of the truth value of Q, then also P→Q is true. If P is shown to be false regardless of the truth value of Q, then P→Q is true.

We see this if we again look at the truth table of P→Q; we see that this portion of the truth table says that P→Q is true if P is fixed to F that is P is false whatever be the values of Q. Next we move on to direct proof of P→Q, the construction of a direct proof of P→Q begins by assuming that P is true and then from the available information from the reference frame the conclusion q is show to be true by valid inference.

So direct proof is indeed direct we have to prove p → q so we will assume that the propositional p is true and then use the valid inferences and the information available in the reference frame and to step by step reduce that q is true if q is true when ever p is true then the proposition p → q is true.

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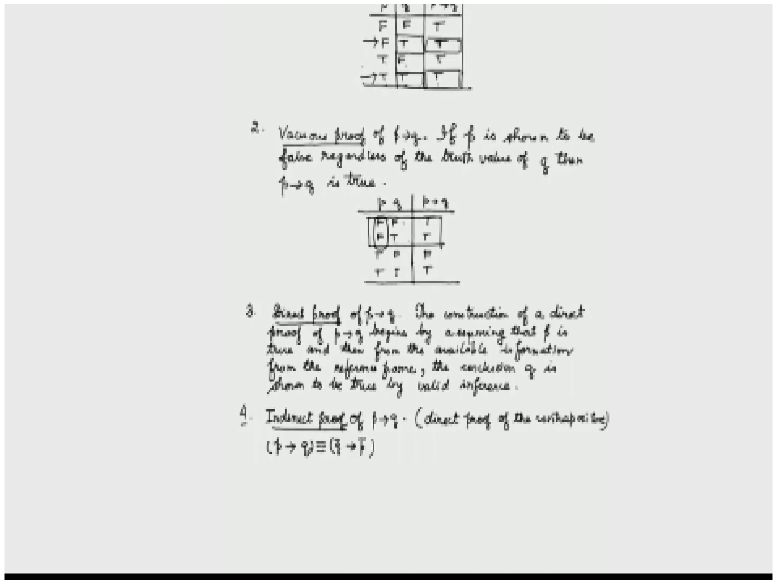

The 4th method of proof is indirect proof of p → q this is basically a direct proof of the counter positive we recall that if we have proposition p → q then counter positive of that proposition is not of q → not of p and p → q and not of → q → not of p are equivalent statements therefore if we can prove that negation of q → negation of p then we have proved p → q we need to work out this proof in 2 steps I am coming to that indirect proof .

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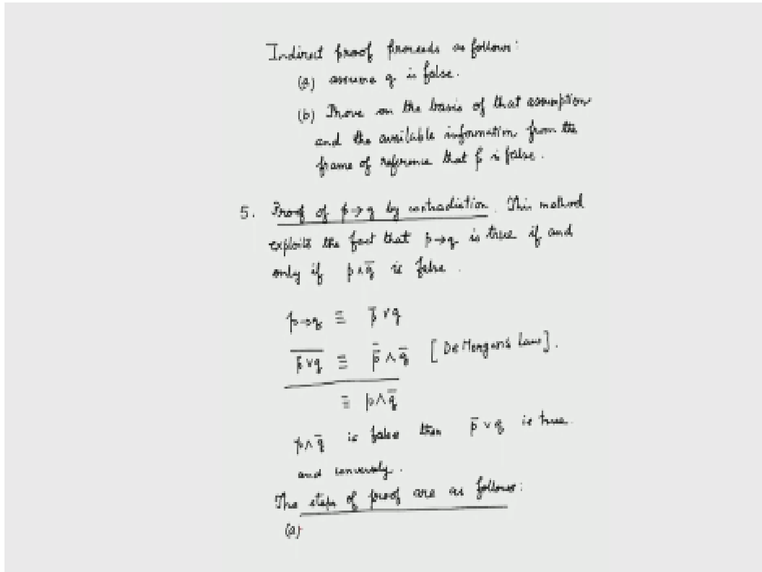

Proceeds as follows a) assume q is false b) prove on the basics of that assumption and the available information from the frame of reference that p is false 5, proof of p $\to$ q by contradiction this method exploiters the fact that p $\to$ q is true if and only if p and negation of q is false now we have seen that p $\to$ q is equivalent to the statement negation of p or q if we take negation of p or q and negation of that then this is equivalent to negation of negation of p and negation of q this is by de Morgan's law.

And which in turn is equivalent to p and negation of q therefore if we can prove that p and negation of q is false then the negation of that which comes from here that is p negation of p or q is true and conversely the proof by contradiction is a very power full tool and to implement such a proof we have to go step wise now I write the steps of the proof by contradiction A assume that p and q negation of q is false assume that this is true not false in the contrary start by assuming that p and q a complement that is not negation of q is true step b dis convert on the basis of that assumption some conclusion.

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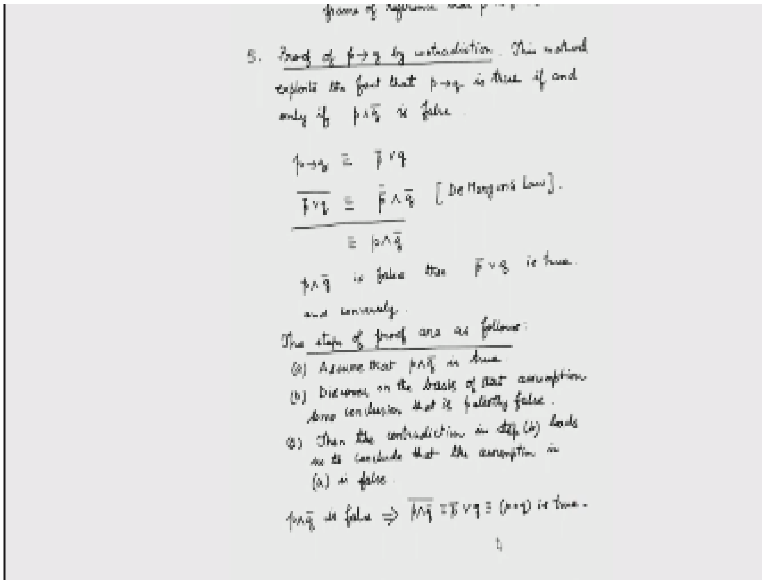

That is patently false, c in the contradiction in the step leaves us to conclude that assumption a is false then the contradiction in step d leads us to conclude that the assumption in step a is false in step a we assumed that p and negation of q is true so that means that p and negation of q is false this will imply that T negation of q negation of that is negation of p or q which is in turn equivalent to T ⇒ q is true this is what is known as the truth contradiction.

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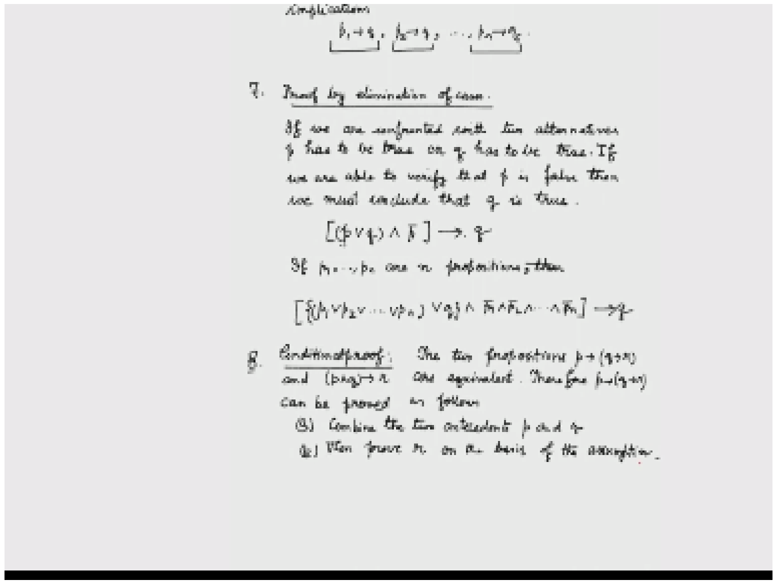

The 6th method is u of p $\Rightarrow$ q by cases now here we are in a situation where p can be split up into core of several prepositions P1 P2 after Pn and we are encountering a preposition like this which is p1 or p2 or and so on up to or tn $\Rightarrow$ q where the left side is essentially p, now what we note that we can establish this fact by proving smaller implications such as p1 $\Rightarrow$ q p2 $\Rightarrow$ q and pn $\Rightarrow$ q if all of them are true then this is also true, the 7th method is prove by elimination of cases if re comforted with two alternatives.

P as to be true or q has to be true if we are able to verify that p is false then we must conclude and if we know that p is not true then that will imply q is true we can extend this for finite number of cases as follows if p1 up to pn are n prepositions then p1 or p2 or q and p1 negation p2 negation and so on pn negation this is q is a tautology.

Next the two prepositions p implies q implying r and p and q implies r are equivalent. Therefore p implies q implies r can be proved as follows combine the two antecedent's p and q b then proves r on the basis of the assumption. Now we have come to the last method of proof that we discuss in this lecture.

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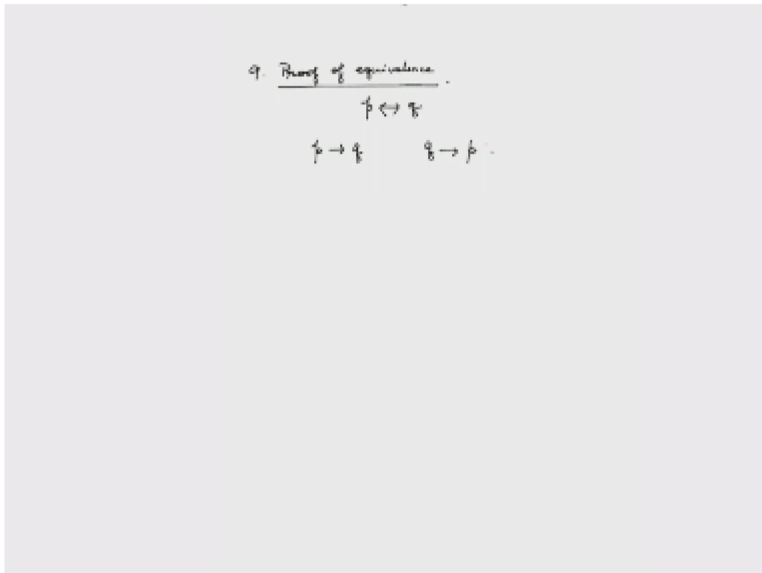

Proof of equivalence now suppose we have a bio conditional if an only if q and we would like to prove it proof of equivalence is that it is enough to prove separately the direct implication p implies is q and its converse q implies p and this is proof of implication. Now that we have discussed some methods of proofs of implications we must also remember that these are not all the possible methods there are other methods as well but these are the methods which are very commonly used and it is a good idea to try to formulate the proves based on these methods.

We will now look at a examples of proves constructed by using proof by contradiction and the proof by contra positive, so first let us check a proof by contradiction.

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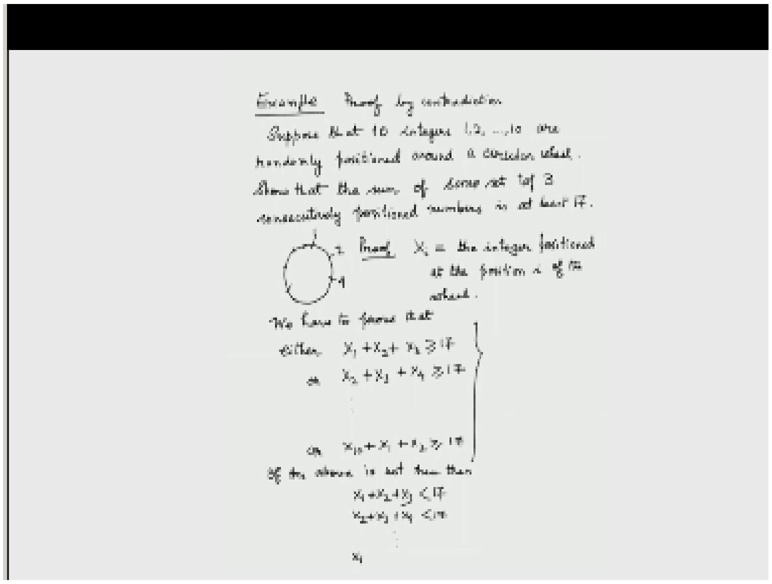

Suppose that ten integers 1 2 up to 10 are randomly positioned around a circle or a circular wheel, so that the sum of some set of three consecutively positioned numbers is at least 17. So I read again now suppose I have got 10 integers 1 2 up to 10 and I have a circular wheel with 10 positions and at each position I am putting some number maybe 1 2 4 and so on like that what I am clamming is that there is three consecutive positions such that if I add up the numbers placed in those three consecutive positions then it will be at least 17 no matter in which way I arrange the numbers between 1 to 10around.

So we start with the proof by defining x_i which is the integer positioned at the position I at the sheet so we have to prove that either $x_1+x_2+x_3>17$ or $x_2+x_3+x_4>17$ and so on or $x_1+x_2>17$ so we want to prove this we have to prove that at least one of this conditions must now we prove by contradiction so we say that let us suppose that it does not hold so if it does not hold then what we hold we have if the above is not true then $x_1+x_2+x_3<17$ please check whether the x quality $x_2+x_3+x_4$ is also >17.

And so on up to $x_{10}+x_1+x_2<17$ and this is end all of the first because if one of them is not true then the sum is greater than or equal to 17 and that means my preposition is correct so suppose this is two now we add up all these equivalent to obtain three times so we are assuming that this is true so that means the three time x_1+and so on up to x_{10} is>16 now we remember that x_1 up to x_{10} are distinct integer between 1 to 10 so we can add them up to obtain 10 and 11 and divided by 2x3 which gives me 165.

And if our assumption is true and once and five is >116 which is off course false so this is a contradiction but all the logical inferences that we have used are valid therefore through a valid logical inferences we are finding that truth is implied if something is false that means that the pen cannot is true and therefore from these we can conclude our assumption is true that is there are consequent three numbers always whose sum is greater than or equal to 17 this is an example of a proof by contradiction as a last topic of today's lecture we will discuss an example of a proof by contra positive.

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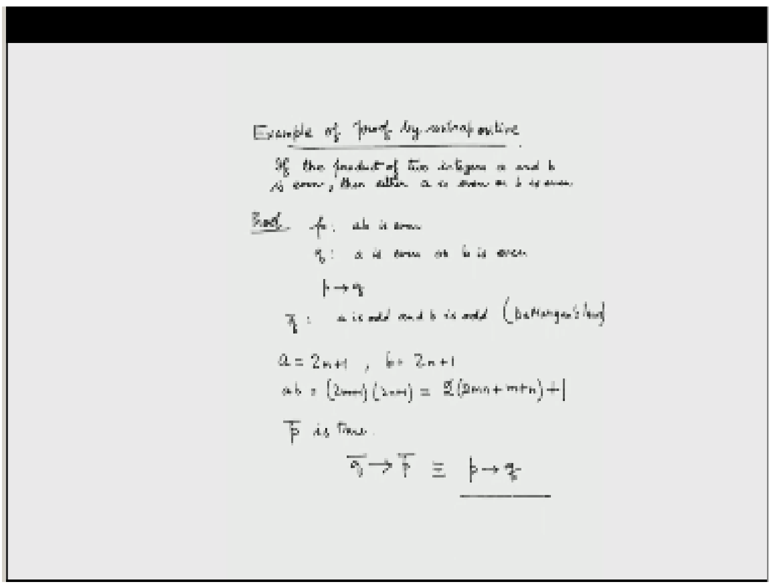

The statement we would like to prove is if the product of two integers A and B is given either A is given or B is even now B denote the n incidence by preposition p that is ab is even and b denotes the consequence a is even or b is even we want to proof that p implies q we start with not of q not of q is a is odd and b is odd this is by Demorgan's laws so we know that a is we are assuming that a is odd and b is odd therefore a can be written as 2m+1.

And b can be written as 2n+1 if we take a product ab and then this is 2n+1.pn+1 which gives me 2 times 2nn+n+n+1 we see that a+b is false sorry axb is false which is the product of ab is odd so therefore the negation of p is true so we have two if negation of q is true whenever negation of 2 is true therefore we have proved the implications negation of q implies negation of p this is two proof by using by representing and this is the end of this lecture thank you.

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