INDIAN INSTITUTE OF TECHNOLOGY
ROORKEE

NATIONAL PROGRAMME ON TECHNOLOGY
ENHANCED LEARNING
(NPTEL)

Discrete Mathematics

Module-10
Recurrence relations
Lecture-04
Applications of recurrence relation

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Today we will study some applications of recurrence relations; in particular what we will see today is that many counting problems can be solved by using recurrence relations, relatively easily. The basic strategy of handling these problems is that given a problem we will try to build up a recurrence relation on certain numbers and then we will solve that recurrence relation.

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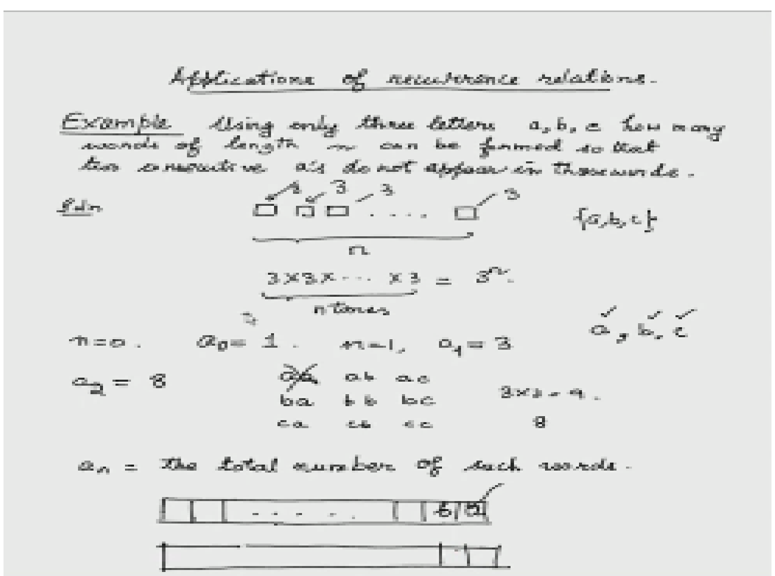

So our topic today is applications of recurrence relations let us look at the first example using only three letters a b c how many words of length n can be formed. So that to consecutive is do not appear in those words no here we have a counting problem, now if somebody asked me that given three letters how many words can be formed of which are of length n then I would have considered n positions like this total number n and then I know that each position can be occupied by three ways.

Since we have three symbols ABC and therefore here we have three, here we have three, here we have three and the last also three so the total number is 3 x 3 x and so on up to 3 again n times which is $=3^n$ but here we have a slightly different condition because here the problem says that I cannot consider the words which have consecutive days. Now let us consider if n =0, if n =0 then that means in the words there is no letter so the total number of such words is let us call it a1 which is =1.

Now when n =1 and I have got three letters then, that sorry here I will write a 0 this is a 0 of course because this is this corresponds to n =0, now we have n =1 for which we have a 1 and a 1 is =3 the reason is that there are 3 letters and I can write a word a another word B and the third word C. So these are three distinct words, so a 1 =3 now if we consider a 2 for n =2 then let us count how many ways we can write a 2.

The number of ways we can build up 2 letter words out of the three symbols is let us start counting we can write a b, a c and then we can write b a b, b b c and lastly ca, cb, cc so there are altogether 3 x 3 9 words. Now what we see here is that there is one word which contains 2 consecutive is so we have to cancel this. Therefore now we have 8 words and a 2 = 8 now like that of like this we can go on but we would like to have a compact relation in the form of a recurrence relation.

Therefore what we do is that we say that suppose a_n is the total number of such words, so a n =the total number of such words. So I have some n positions where letters have been put in now we ask a question, what about the last position? The last position can be occupied by a b or c, now then we ask that suppose the last position is occupied by a, then what will happen if the last position is occupied by a then of course in the position previous to that a cannot appear again.

So either it will b or it will be occupied by c, so now let us count the positions this is the position number one this is position number 2 and so on this is position number n - 2 this is position number n - 1 and this is position number n. Similarly here this is position number 1 this is position number 2 and we move up in this way and then ultimately we arrive at position number a and - 2 position number a and - 1 and lastly position number kn.

Therefore we see that if we have a word in the set of all words which are ending with a and not having 2 consecutive a then ending with a means the position has a then the N - 1 its position of those words will not have a will have either B or C, so we have these 2 configurations and if we look at the segment from n to n - 2 in both these words, then we will see that this segment can be occupied by any word of length n - 2 consisting of the three symbols ABC and not having 2 consecutive case.

Now according to our notation a n - 2 is = the total number of words of length n, such that no consecutive is appear. So we will at least have a n - 2 + a n - too many words in the set of words of length n having no consecutive phase. Now let us move to the next page here.

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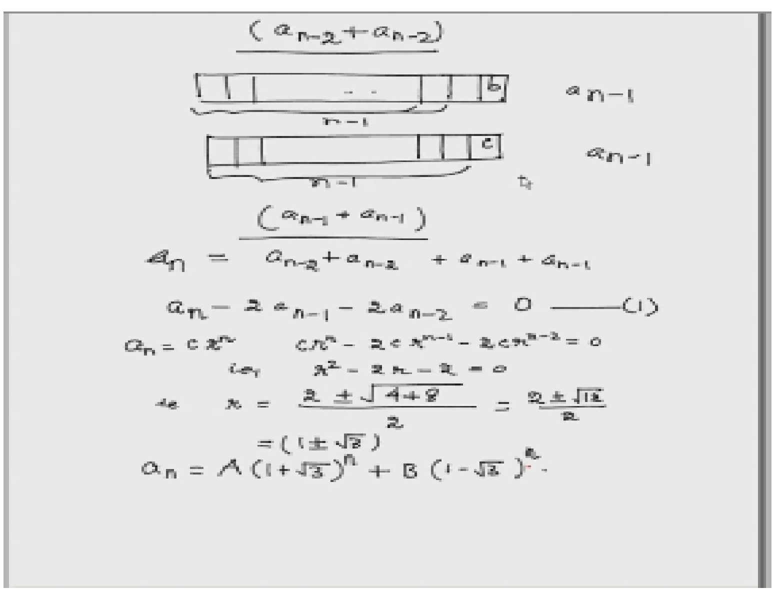

So what we have seen is that when we are considering length and segment ending with a when we are considering length end segment ending with a then the previous position can be either B or C and there are n - too many positions and those positions for each case V and P or C can be occupied by a n - too many words of length n – 2, therefore I have got a n - 2 + a n – 2. Now we look at the other situation when the word of length n is not ending with a then it may end with either B or C.

In either of these cases the initial segment consisting of n - 1 letters can be occupied by a n - 1 and a n - 1 words so total number of words that we have is a n - 1 + a and - 1 therefore we come to the conclusion that if we have words of length n ending with a, then there will be n - 2 + a and - 2 many words of length n ending with a and not having 2 consecutive is anywhere and if the in it in length word is ending with B or C then we will have n + 1 + n + 1 many words and these are the only possible words of length n consisting of three symbols ABC so that no 2 consecutive is appear.

Therefore we can write a recurrence relation in this form a n =a n - 2 + a n - 2 + a n - 1 + a and - 1 in other words we have the recurrence relation a n - 2 times a n - 1 - 2 times a n - 2 which is =0. Once we have done this we need to solve this recurrence relation we note that it is a second order recurrence relation with constant coefficients and of course it is a linear recurrence relation. We consider the solution n =C R^n and substitute it in the recurrence relation which we denote by one to obtain C $_R^n$ - 2 C $_R^n$ - one - 2 C $_R^n$ - 2 which is = 0 that is taking the common factor C and R^n- C x R^n- 2 out from all the terms.

We will get R^2- 2 times R - 2 =0 and solving this we get R =2 + or - 4 + 8 and this gives me 2 + or - √ of 12 by 2 which is 1 + or - √ of 3 therefore we get the general solution as a n =a constant capital a times 1 + √ of 3 raised ^{n}+ B x 1 - √ of 3 $^{n.}$ Now we have to substitute the values of a 0 and a 1 as we have already noted that a 0 is 1 so let us let us go to the next page.

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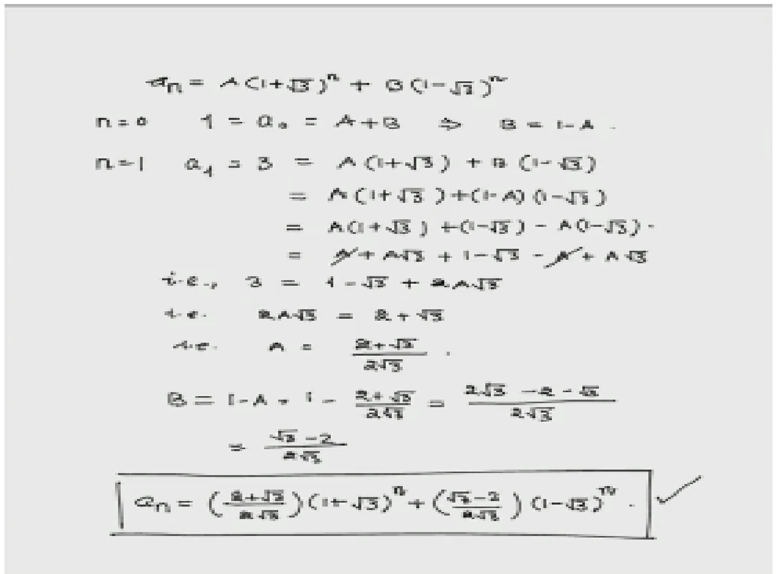

And write the general solution of the recurrence relation a n =a 1 + √ of 3 raised ^{n}+ B 1 - √ of 3 whole raised ^{n} and for n =0 1 =a 0 =a + B this implies that B is =1 – a, if we put n =1 we know that a 1 is =3 which is =a 1 + √ of 3 + b1 - √ 3, if we substitute the value of B in this equation then we will get a 1 + √ of 3 + 1 - a 1 - √ of 3, which gives us a 1 + √ of 3 + 1 - √ of 3 - a 1 - √ of 3 well and proceeding further we have a + a √ of 3 + 1 - √ of 3 - a + a √ of 3.

Now there is some there are some cancellations first of all a will get cancelled and let me write the equation here, which is 3 =1 - √ of 3 + 2 times a √ of 3 that is 2 times a √ of 3 =2 +√3 which gives me a =2 + √ 3 / 2 √ 3 from this we see that be =1 - a which is =1 - 2 + √ of 3 2 √ 3 which is =2 x √ 3 in the denominator and in the numerator 2 √ 3 - 2 - √ 3, which is =2 √ 3 √ 3 – 2. Therefore we have the solution to our recurrence relation which is a n =2 + √ of 3 /2 √ of 3 this is the value of the first constant capital a and this is multiplied by 1 + √ of 3 raised ^{n}+ √ of 3 - 2 /2 √ 3 x 1 - √ of 3 ^{n} this is our final answer, let us look at another problem, now this problem states that.

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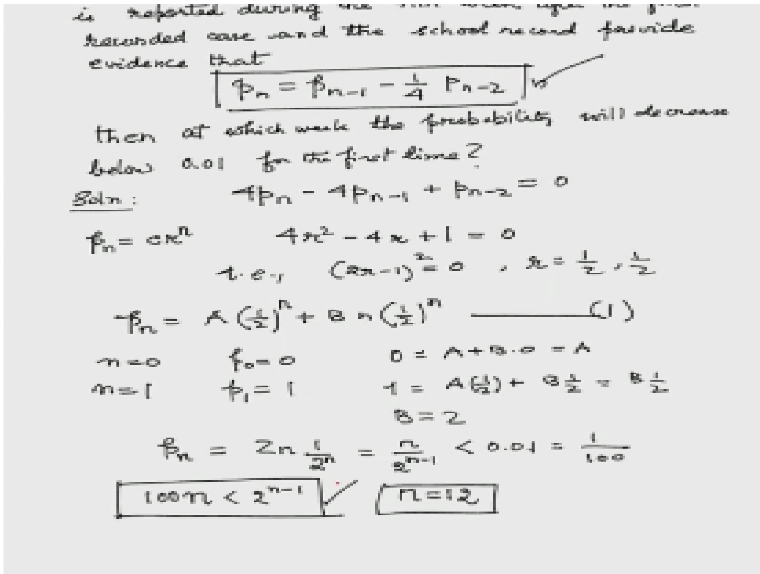

is reported during the nth week after the first recorded case and the school record provide evidence that

$$\boxed{p_n = p_{n-1} - \frac{1}{4} p_{n-2}}$$

then at which week the probability will decrease below 0.01 for the first time?

Soln: $4p_n - 4p_{n-1} + p_{n-2} = 0$

$p_n = cr^n$ $\quad 4r^2 - 4r + 1 = 0$

i.e., $(2r-1)^2 = 0$, $r = \frac{1}{2}, \frac{1}{2}$

$$p_n = A\left(\frac{1}{2}\right)^n + Bn\left(\frac{1}{2}\right)^n \quad \text{——— (1)}$$

$n=0 \quad p_0 = 0 \quad 0 = A + B.0 = A$

$n=1 \quad p_1 = 1 \quad 1 = A\left(\frac{1}{2}\right) + B\frac{1}{2} = B\frac{1}{2}$

$B = 2$

$$p_n = 2n\frac{1}{2^n} = \frac{n}{2^{n-1}} < 0.01 = \frac{1}{100}$$

$\boxed{100n < 2^{n-1}}$ $\quad \boxed{n = 12}$

If a first case of measles if a first case of measles in a certain school system is recorded and pn denotes the probability that at least one case is reported during the n^{th} week after the first recorded case and the school record provide evidence, that P n =P n - 1 - 1 / 4 P n - 2 then at which week the probability will decrease below 0.01 for the first time.

So what we are looking at here is that suppose a school system checks the reports of a certain disease, let us say measles among children and it starts counting from a week when one recorded event has occurred that is some case of measles have come up. Now what the school record says that starting from a week where at least one measles attack has been recorded, if we start counting the number of weeks then the probability that in the n^{th} week there will be at least one reported case is given by this recurrence relation.

Now we have got no control over this record isolation it is from some data source possibly from which the school record is from the records of different years this has been observed let us say and we are asked a question that after how many weeks for the first time this VN is going to go below 0.01, for that we have to solve this recurrence relation in order to do that we first write down the recurrence relation in the form 4 P n - 4p n - 1 + PN - 2 which is =0.

And now considering that Pn is of the form C raised to the power R^{n} we have the characteristic equation as for R^{2}- 4 R + 1 =0 and this leads to 2 R - 1 whole ^{2}=0 and therefore R is 1/2 ½, so 1/2 is a repeated √ of this equation. Now from the discussions of the previous days we have seen that this means that Pn =1/2 raised ^{n}+ B in 1/2 raised ^{n} and now we try to look at the initial conditions.

If we put n =0 that is in the 0 is weak then by definition p0 is 0 because we have not started our observations, now if we start from p1 what we have told in the statement of the problem that we start counting in from the point of observing at least one disease case, therefore P 1 is 1. Now in the equation 1 if we substitute these values we get 0 = A + B x 0 which is =Aso the constant A reduces to 0 and we have one =A times 1/2 + B times ½.

Now A is 0 therefore we have only B times 1/2 therefore we have B is =2, thus we have the probability PN is =2 times n x 1 by 2 ^{n} which is essentially n /2 ^{n}- 1. Now we want this probability to drop below 0.01 for the first time. 0.01 is 1 by 100 therefore we would like this equation to hold that is 100 n < 2 ^{n}- 1. Now if we check this inequality and keep on varying in from 1 onward like 1 2 & 3 4 like that then we will find that the cutoff is n =12.

So when n =12 then for the first time this inequality will be satisfied therefore PN will become < 0.01 01. Now we will look at a case of application of non homogeneous recurrence relations, now we will again discuss one example.
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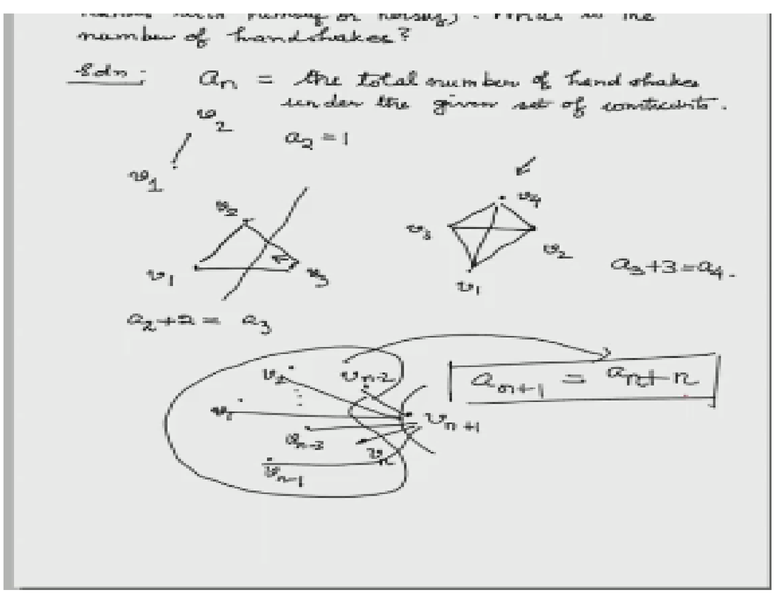

Now let me write down the example first for N ≥ 2 suppose that there are n people at a party and that each of these people shakes hands exactly once with all the other people there, of course of course nobody shakes hands with himself or herself. Now our question is that what is the number of handshakes what is the number of handshakes when we think of this solution we again start denoting the total number of handshakes when n people are present under the given constraints as a n.

So a_n is = the total number of handshakes under the given constraints under the given set of constraints all right. Now I would like to build a recurrence relation now let us try to see what happens if we start with let us say 2 people because we have seen that we do not want n to be to go below 2 because if n is = 1, then of course there is no handshakes. So we start with n =2 so suppose I have got person number one and person number 2 let us call them V 1 and V 2 and I will join them to signify that shaking hand exactly once so we see that a 2 is going to be 1.

Now if we consider that another person is coming now what will happen, so let us draw the situation over here we have got three peoples we have now a party of three and they are shaking hands with the given constraints. Now what we can always do is that we can remove for the time being one person, suppose we have left out V_3 then there is only one handshake so this is the number of handshakes for 2 people so I have got a too many handshakes.

Now when with V 3 comes in then he does not shake hands with himself what he has to shake hand with all the other people exactly wants, so therefore he will shake hand with this person and she can with this person so the total number of handshakes will be a 2 + 2 which is denoted by a 3. Now suppose we know a three so there are three people who shakes hands with each other v1 v2 v3 or C they are shaking hands with each other and suppose a fourth person is coming in and let us call him before.

And so that means now we have a party of four so when v4 is out the remaining people shakes hands in a three many ways and when v3 v4 comes in he has to shake hand with all the people except for himself, he shakes and with v3 he shakes and with v2 and lastly or whatever he shakes and with v1, so he has to shake another three many hands, so total number of handshakes is a 3 + 3 which is =a 4.

Now we can go to the general case that suppose we have got n people, so we have got v1 v2 and so on and lastly we have got let us say we n - 2 V n - 3 and over here V n - 1 and here we have V n. So we have a party of n people and we know that they shake hand in a in many ways, now suppose that we have got one more person V n + 1 and V n + 1. So we have a party of n + 1 people now suppose they shake hands in the designated manner.

Now what we do is that we can choose any of the vertices, now let us let us take that we are taking out V_N + 1 and removing all the edges from V_N + 1 then we have a party of n and of course David they are shaking hands exactly once, with the others are not checking hand with himself or herself. So we have got total of n - 1 handshake and then when V_N - 1 / B_N + 1 is considered then he is bound to shake hands with each of the people in the other group.

So the total number of handshakes will be n so this is increased by N and which is the total number of handshakes with n + 1 people. So we have built up a reconciliation corresponding to the handshakes let us write down this recurrence elation and then let us try to solve it the homogeneous part of this equation is.

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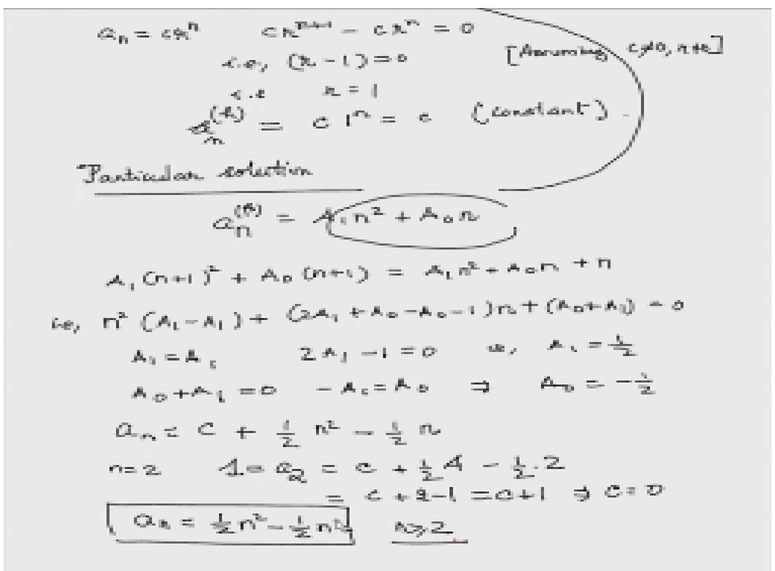

A n + one - a n =0 and we take a trial solution let us call it a n =$C_R{}^n$ then we get $C_R{}^n$+ 1 - $C_R{}^n$ is = 0 and this leads us to R-1 =0 because we can of course assume very reasonably that C 0 =0 and R 0 =0 therefore we have R =1, therefore the homogeneous part written as a n superscript H in bracket is C times 1 ^{n}or C which is a constant. Next we try to find out a particular solution here we choose the trial solution to be in particular = a 1 n ^{2}+ a 0 n which is indeed a rather complicated expression.

And if we substitute this in the recurrence relation then we will get a 1 n + 1 ^{2}+ a 0 n + 1 =a n which is a 1 n ^{2}+ a 0 n + n and if we do the usual manipulations, then we will get n ^{2}a 1 - a 1 + 2 times a 1 + a 0 - a 0 - 1 in + a 0 + a 1 =0 and this should hold for all n. Therefore we have got a 1 =a 1 by equating the coefficients to 0 and when I equate the coefficients of n to 0 I get a 1 - 1 =0 that is a 1 =1/2 and when I equate the constant term to 0, that is a 0 + a 1 =0 then I get a 1 =or - a 1 =a 0 which implies a 0 = - ½.

Therefore we arrive at the general solution a a n =constant + 1/2 n ^{2}- 1/2 in and now if I put the value of n =2, we know that 2 is =sorry one here we have got 1 a 2 is 1. So therefore 1 = a 2 which is =C + ½ 4 - 1/2 x 2 which gives me C + 2 - 1 which is C + 1 this implies C =0. Therefore finally we have the solution as n =1/2 of n ^{2}- 1/2n for N ≥ 2 thus we know. Now that if we have n people in a party and each person shakes hand exactly once with each of the other people present in the party then the total number of handshakes is 1/2n ^{2} - ½ n with this handshaking problem I will end today is lecture thank you.

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