INDIAN INSTITUTE OF TECHNOLOGY ROORKEE

NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)

Discrete Mathematics

Module-10
Recurrence relations
Lecture-02
Second order recurrence relation with constant coefficients (1)

With
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In today's lecture we will learn how to solve second-order linear homogeneous recurrence relations with constant coefficients.

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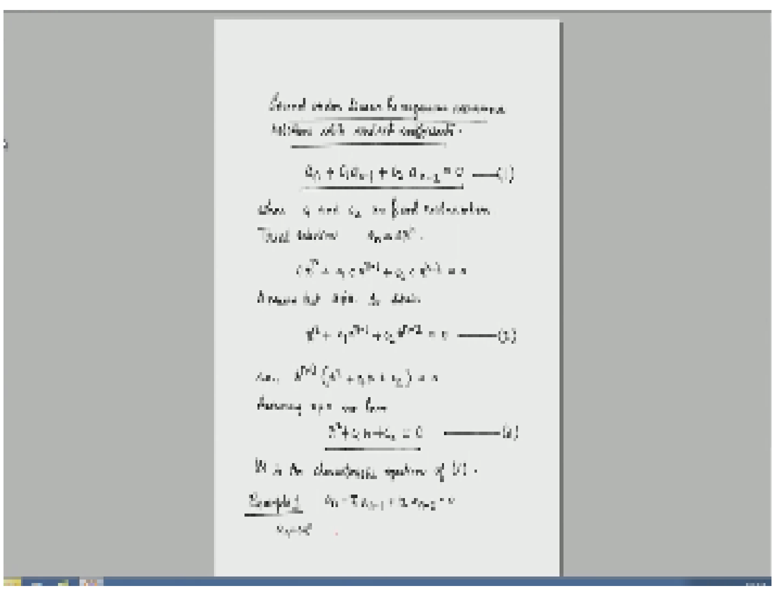

Now as we have seen before that a second-order linear homogeneous recurrence relation with constant coefficients will have the form a n +c 1 a n - 1 + c 2 a n – 2 equated to 0 where c 1 and c 2 are fixed real numbers let us name this equation as one now the solution technique consists of starting with a trial solution of the form a constant c times r^N and substituting this in 1 we get the constant times r^{n+c1} into the constant times r^{n-1} + c2 the constant times r^{n-2} which is equal to 0.

And in the first step we note that a reasonable assumption is c = 0 so assume that c = 0 to obtain our raise to the power n + c1 times r^{n-1}+ c2times r^{n-2}=0 and we name this equation 2 we can further simplify this again with the assumption that r =0 to get this $r^{n-2} = r^2 + c\ 1r + c\ 2$ into 1 = 0 assuming are not equal to zero we have $r^2 + c\ 1\ r + c\ 2=0$ let us call this equation number 3 the equation 3 is called the characteristic equation of the recurrence relation 1usually when we are solving the recurrence relations of this type we can directly write the characteristic equation.

And proceed ahead but in this lecture each time we will start with the trial solution obtain the characteristic equation and then solve the characteristic equation to obtain the solutions of the recurrence relation this way over and over again we will get the practice of solving recurrence relations from, from the basic definitions now the first observation that we have over here is that the recurrence relation has been transformed where to an algebraic equation thus in a way the solution of one is transformed to the problem of solving three which is an algebraic equation.

And for which we have established methods of solution now we start with an example no was before we consider a = cr^n and substituting in let us call this for we have $cr^{n-7}\ c^{r\ n-1} + 12\ c^{r\ n-2}$ =0 that is $^{r\ n-c}7\ ^{r\ n-1} + 12^{r\ n-2}$ =0 call it 5 that is $r^2 - 7\ r + 12=0$ call it six and of course here we are assuming that r=0 so our problem at hand is to solve the algebraic equation.

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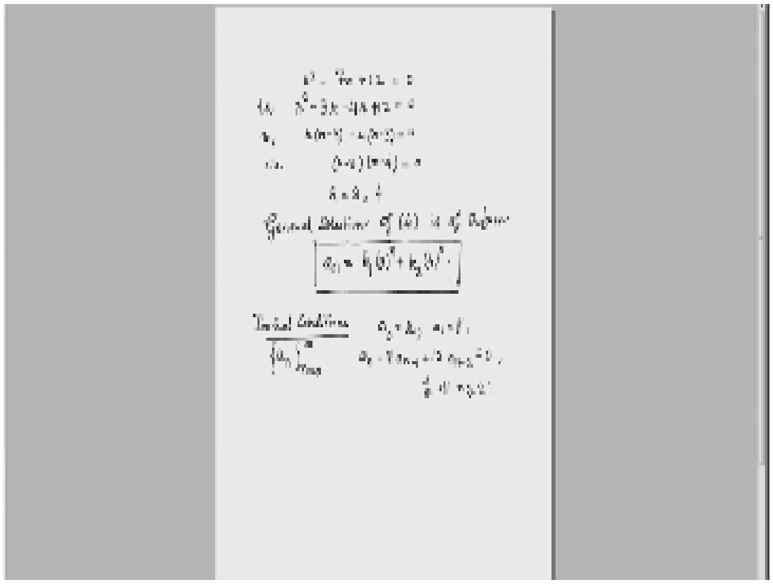

R^2- 7 r + 12 = 0 r ^{2} - 7 r+ 12 =0 we factorize the left-hand side first we can write r^2- 3r- 4r + 12 =0 that is our r - 3 - 4r - 3 = 0 that is r - 3 . r - 4 = 0 therefore the solution is r= 3 or r=4 now when we get two distinct real solutions of the characteristic equation of a recurrence relation then we write the general solution in the form general solution of four is of the form a an equal to some constant K 1 times 3^n + another constant K 2 times 4^n
Now there is Theory dealing with linear recurrence relations which tells us that if we get the √ s of the characteristic equation and if the constant times init power of one √ + another constant times in it power of the other √ we cannot go into details of the theory right now but we can accept that for the time being in order to solve the recurrence relation our for a particular case we have to do something more very often a recurrence relation comes with the so called initial conditions in this case.

Let us assume that we have an initial condition a 0= 2and a 1 = 5 that is somebody tells me that we have a discrete numeric function given by a n where n varies from 0 to ∞ and a n - 7 a n- 1 + 12 a n - 2 = 0for all N > or = 2 which is a problem that we are solving right now the initial conditions 0 = 2 and a 1 = 5 has to be given along with the reconciliation if you want a particular solution.

Now we take the general solution of their conciliation under consideration and then put the value N = 0 for n = 0 we have a 0 = 2 = K 1 + K 2 therefore we know that K 2 = 2 - K 1for n =1 we have a 1 = 5 = 3 times k1 + 4times k2 we can always put the value of k2 in terms of k1 and obtain 3 k1 + 2- K 1 = 3 K 1 + 8- 4 K 1 = - k1+ 8 well this is essentially k1 =8 - 5 which means three therefore we have obtained the value of k1 =3 and K 2 = 2 - k1 and this gives me - 1 thus the solution of the recurrence relation is a n = 3times 3 $^{n - 4 n}$ therefore it is 3 $^{n+ 1}$ - 4 ^{n} next we question that what happens in case we do not get two distinct real √ s.

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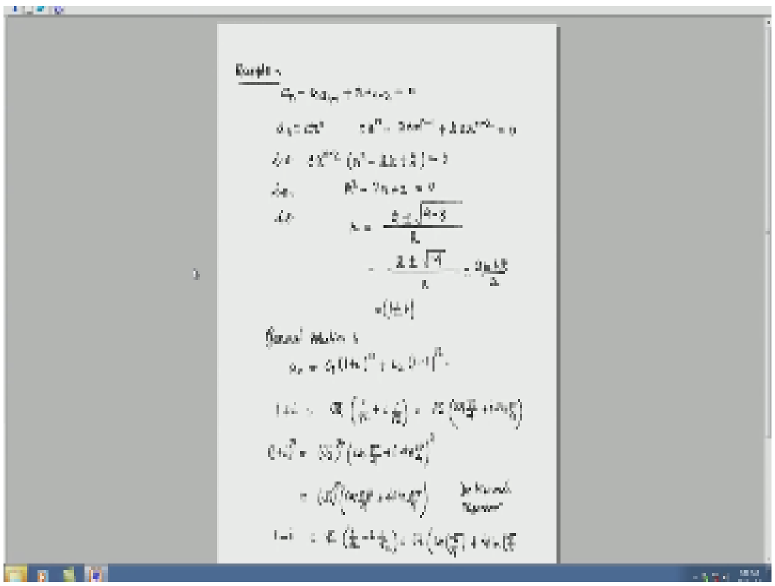

Now let us look at a at a recurrence relation of the form a n - 2 times a n - 1 + 2times a n - 2 =0 this is example to now again we take an equal to C times r ^{n} to obtain the characteristic equation first we get this one cr $^{n-2}$ times cr $^{n - 1 + 2}$ times cr $^{n - 2}$ equated to 0 so we get C times R $^{n- 2}$ = r^2 - 2 times r + 2 = 0 which leads us to the characteristic equation R^2- twice R + 2 = 0 now we solve it so this means that R = 2+ - 4 - 8 / 2=2 2 + - -of 4 √ sign and this is 2+ or - I times 2 / 2 which is 1 + or – I.

Therefore the general solution is n = C 1times 1 + I $^{n+ C 2}$ times 1 - I ^{n} but at this point we will process 1 + I $^{n +1}$ - I ^{n} but we see that 1+ I=√ 2 x 1/ √ 2 + I times 1/ √ 2 = √ 2 cos π / 4 + I times sine π/ 4 within parenthesis therefore 1 + I ^{n} is √ 2 ^{n} and cos π /4 + I sine π/ 4 ^{N} and at this point we will use de movers theorem to write 1 by √ 2 √ 2 s $^{n \cos n}$ π /4 + I sine n π by 4 similarly 1 plot - I is √ 21/ √ 2 - I 1/ √ 2 which is equal to √ 2 cos - π /4 + I sine- bye-bye for and one - I ^{n} will be √ 2 $^{n \cos}$ – p in / 4 + I sine - pi n /4 = √ 2 ^{n}cos n π / 4 - I sine n π / 4 now we will be replacing this and these expressions in place of 1+ I ^{N} and 1- I ^{n}

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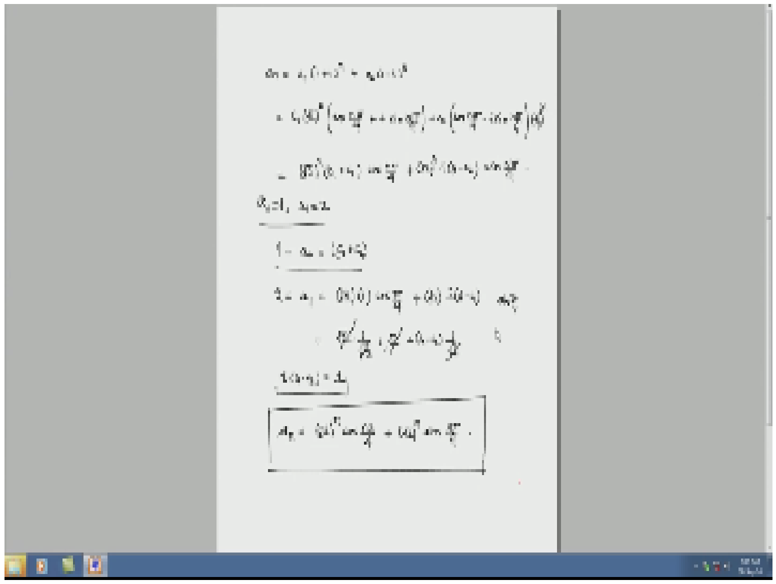

Let us go to the next page now let us recall that we have an expression in the form a n = c 1 1 + I $^{n +}$ c2 a 1 - i ^{n} where we have worked out the expressions for 1 + i ^{N} and 1 - I ^{n} which we substitute now to get C 1 2 ^{n} times cos n π 4 + I sine n π / 4 + C 2 cos n π / 4- sine I sine n π /4 and we have to multiply by 2 $^{\surd\ 2\ n}$ therefore we get route $^{2\ n}$ c1 + c2 cos n π / 4 + √ 2 ^{n} I times C 1 - C c2 into sine n π / 4.

Now suppose somebody tells us that there is there are initial conditions in the form a 0 = 1 and a 1 = 2 then we will substitute these values in the general solution to obtain 1= a 0 = c1 + c2 we have to note here that if we look, look at the expression of a n √ 2 ^{n} if n is 0 is 1 the next factor is c1+ c2 which survives and then cause 0 π / 4 is cos 0 which is 1 + again√ 2 ^{n} is 1 I xC 1 - C 2 will be there but sine 0is 0 therefore the second term does not appear in this expression.

And therefore we get that C 1 + C 2 = 1now if I put n = 1 then we will get- = a 1 which is equal to √ 2x 1 because we already know that C 1+ C 2 = 1 x cos π / 4+ again we have a √ 2 then I times C 1 - C 2 and sine π /4 which we know is √ 2 in fact let me write sine π / 4 here sine π /4 so in the next step we write √ 2 x1 / √ 2+ √ 2x I C 1 - C 2/ √ 2 so √ 2 will cancel both expressions so I get I times C 1 - C 2 = 1 and please see here that we really do not need C 1 C 2 independently separately we just need I times C 1 - C 2 and C 1 + C 2 and both are ones therefore we have n = √ 2 ^{n} cos n π /4 + √ 2 ^{n} sine n π /4.

Thus we see that ultimately the solution that we get after putting the initial conditions is real the discrete numeric function a n for all values of n will take real values although in between we got some intermediate solutions which are complex numbers now the natural question that occurs here is that given a polynomial of degree two an equation whose degree is two.

That is a quadratic equation in single variable we know that there are three possible cases the it may have two distinct real solutions or two distinct complex solutions and there is a third case that is the cases when the solutions are repeated or it is sometimes called the√ s are repeated now we will now look at that case.

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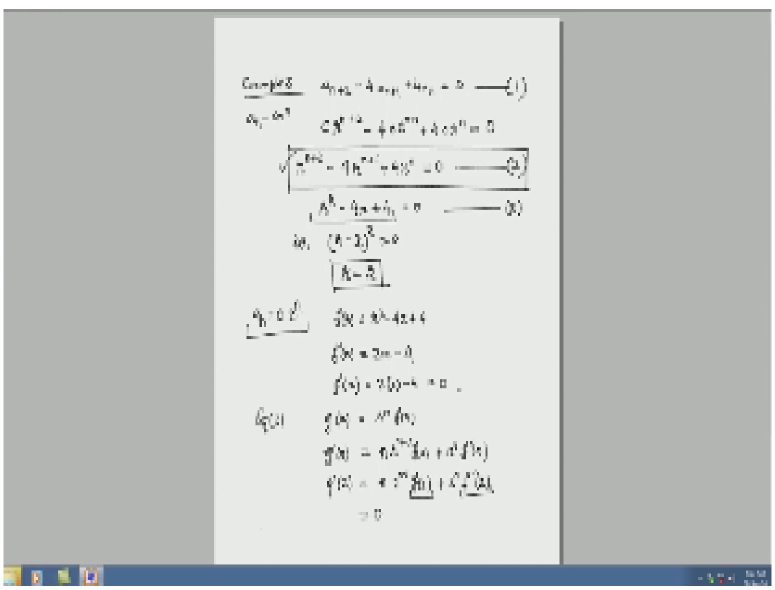

In example three consider the recurrence relation an + 2 - 4 a n + 1 + 4 n = 0 now if we apply the same technique put a n = c times r ^{n} then we get here c times r $^{n+1}$ n + 2 let me so we get C times r $^{n+2-4}$ into c times r $^{n+1+4}$ into c times r ^{n} =0 now first we take out C to get an equation of the form r $^{n+2-4}$ times r $^{n+1+4}$ times r ^{n} =0 this is important therefore let us number the equations we again start from 1.

So on this page the recurrence relation will be referred to as the equation 1 and $r^{n+2} - 4 r^{n+1+4} r^{n}$ will be referred to as equation 2 and then assuming that our is not equal to zero we get r square - 4r + 4=0 let us call it 3 which is the characteristic equation of 1it is obvious that three is same as R- 2^2=0 therefore we have only one √ repeated twice which is R =2 therefore we have one solution as a n equal to some constant times 2^{n} .

But at this point it is very important the theory tells us that we have to get so-called another linearly independent solution of the equation one now roughly speaking when we say we need another linearly independent solution we mean that the discrete numeric function that we are talking about is not a constant multiple of the discrete numeric function that we have obtained here that is C times r^{n} because all the constant multiples have been already taken into account in fact.

The theory says that the general solution of a second-order linear homogeneous equation will be of the form of C one times one solution + C two times another solution which is not a multiple of the first one now from the characteristic equation we get no other solution that is why we consider the equation two now since equation 3 has a repeated √ we know that if we take the derivative of the left-hand side of the equation 3 which is a quadratic polynomial that repeated√ will also be satisfied that derivative will be also satisfied by that.

Repeated √ r2 - 4 r+ 4 the derivative of fr that is f dash r is 2 times r - 4now f dash 2 is 2 x 2 - 4 which is equal to 0 this is true in general now we have equation two over here which is essentially some g r = r^{n} x fr now if we take the derivative of gr that is g-r this is by the chain rule the sorry the product rule we get n into r^{n}- 1 fr + r^{n} if – r now if we put two instead of r then we have n into 2^{n} – 1x F 2 + 2^{n} xF -2.

Now we know that if the f2 is 0 we have already proved that and we have already also seen that F dash 2 = 0 therefore this is equal to 0 now we take the derivative again of 2 in the form given as in the left hand side of the equation 2 if we do that then we get n + 2r n + 2 - 1- 4 times n + 1^{n+4} x n r^{n-1} = 0 and I know that this is satisfied for r= 2 now since we know that R = 0 I multiplied by R both sides to get n + 2 are n + 2 - 4 n + 1are 2 power n + 1 + 4 n R^{n} which is equal to 0.

Now here we observe a startling fact we see that we have already proved that this equation is satisfied for r= 2 therefore we can safely write that this means n+ 2 2 ^{n}+ 2 - 4 n+ 1 2 to the power n + 1 + 4 n2 ^{n} = 0 that means that if I had taken a n as n r ^{n} as a discrete numeric function that discrete numeric function will satisfy the linear recurrence relation given by one let us match it because we have here n + 2 2 $^{n+2}$ which is a n + 2 if a n = n r ^{n} so I have a n + 2 then -4 a n + 1 + 4 a n = 0.

So by considering the fact that if a quadratic equation has repeated √ s then that repeated √ is going to satisfy the equation obtained by taking the derivatives of the quadratic equation and then by using that and the equation2 over here we have obtained a another solution for the original record isolation.

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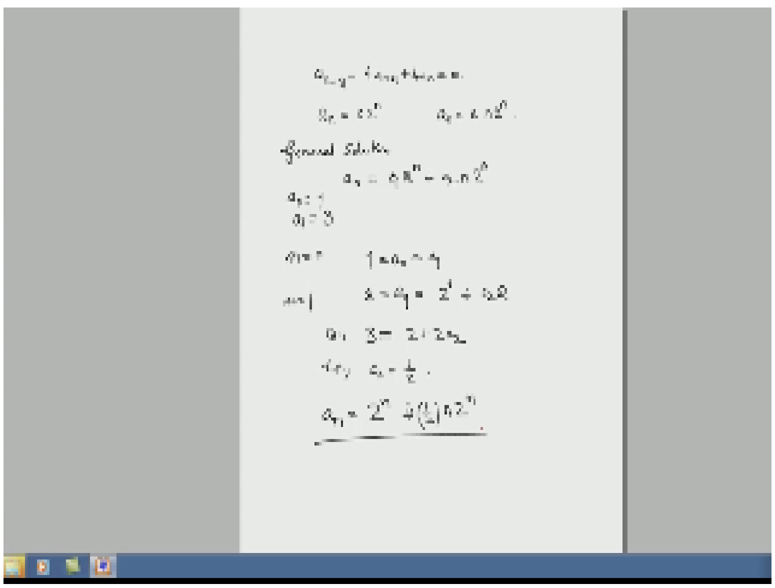

Now let us look at these two solutions let us recall again the recurrence elation that we are considering a n +2 - 4 a n + 1 + 4 a n = 0 and the recurrence this is a recurrence relation and the solutions are a n equal to a constant times 2 ^{n} another solution is n equal to a constant times n 2 ^{n} now general solution will be of the forma n equal to some C 1 times 2 $^{n+C\,2}$ times n ^{n} now initial conditions are given by a 1=3.

Let us see and a zero equal to let us take a 0=1 then if I put n =0 then 1= a 0= C 1 and 2 ^{0} is 1 and the second term is going to be zero because n = 0 so we get C 1=1 and if I put n = 1then we get 3 = a 1 = C 1is 1 so 2 $^{1+C\,2}$ into 1x 2 ^{1} therefore we get 3+ 3 =3 =2 + 2 times C 2which means C

2=1/2 thus the solution is a n = 2^n +1/2 in 2^n thus in this lecture we have discussed recurrence relation which are second-order linear homogeneous with constant coefficients.

And we have seen that a general method of solution exists and it leads to solving quadratic equations and when we want to solve quadratic equations in one variable there are three cases to be considered one when the √ s are distinct real two√ s are complex and three repeated√ s we have seen how to handle these three cases in the context of second-order linear homogeneous equations with constant coefficients this is all for today thank you.

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