INDIAN INSTITUTE OF TECHNOLOGY
ROORKEE

NATIONAL PROGRAMME ON TECHNOLOGY
ENHANCED LEARNING
(NPTEL)

Discrete Mathematics

Module-10
Recurrence relations
Lecture- 01

Introduction to recurrence relations

With
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In today's lecture we will be studying recurrence relations.

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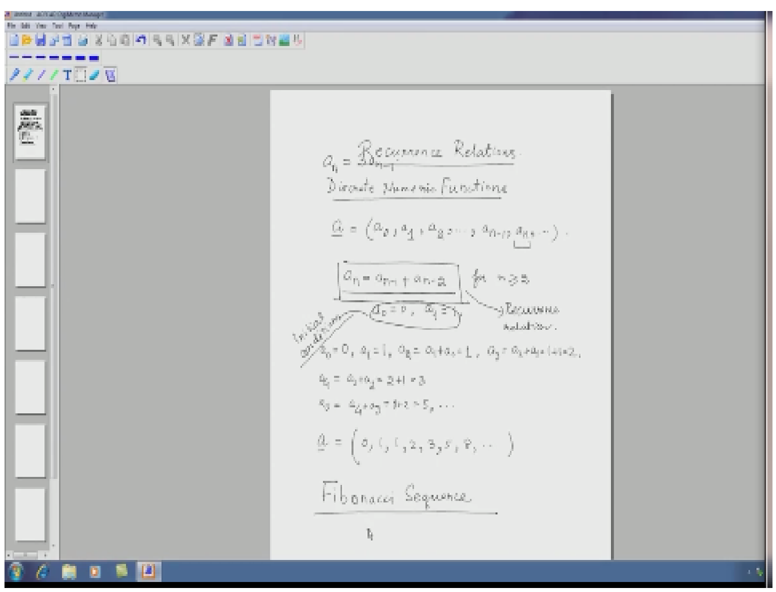

Our starting point is discrete numeric functions which we have discussed in last two lectures suppose a is a discrete numeric function which is written as a = a sequence of real numbers which we denote in general by a0 a1 a2 and so on, then n -1 an and Han world now sometimes what happens is that the entry an of the discrete numeric function can be related in certain ways to the previous entries for example let us look at this relation n equal to an - 1 + an - 2 for n > = 2 and a0 = 0.

And a1 = 1 now we start from a 0 and of course it is given that a0 is 0 then we come to the next entry that is a1 which is also given as a1 is 1 and then a2 according to my rule over here he is a 1 + a0 which is equal to 1 again after that a 3 which is equal to a2 + a1 which is 1 + 1 = 2 then we have a 4 which is equal to a3 + a2 which is 2 + 1 that is 3 then a 5 = a4 + a3 this is 3 + 2 = 5 and we have to proceed in this way thus we have a sequence whose first few terms are 0 1 1 then 2 then 3 then 5 and then of course 8 and so on.

Now this is an example how a discrete numeric function can be generated recursively and we get very interesting sequence sequences, for instance the sequence that we have just seen is the famous Fibonacci sequence and the relationship that we get here is called a recurrence relation and the conditions, that we write down as a0 = 0 and a 1 = 1 are called initial conditions now we can have many more examples of recurrence relations for example we could have written an = 3 – 1 - 2 times an -2.

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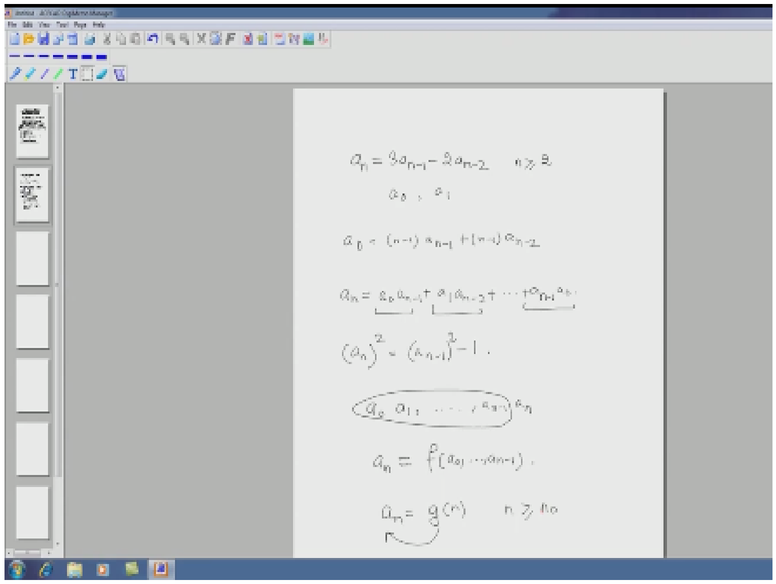

So I write an = 3 times an - 1 - 2 times an - 2 and I start from n < = 2 and for a0 and a1 I have to choose some initial conditions like this we can have recurrence relations like an = n - 1 an 1 + n - 1 into an - 2 or we could have had an = a0 n -1 + a1 an - 2 + and so on, upto some an - 1into a0 where the corresponding discrete numeric function is a0 a1 and all that, so we see that at each term over here we have a product of two entries of the discreet numeric function then we could have had an^2 = an – 1 1^2 – 1.

So we see that there are many different discrete numeric functions, but the essence of sorry there are many different recurrence relations but the essence is that given a discrete numeric function a 0 a1 an - 1 n I try to build up a relationship involving n equal to the previous terms sum F of a0 up to an - 1 our goal in this topic is to find out expressions of an purely in terms of n.

So we would like to find out a function of n explicitly written in terms of n which gives me the values of an for n, let us say greater than or equal to some number n0, now in general this is a difficult problem and there is no general technique of handling this problem if somebody gives us a general recurrence elation, but we can restrict the class of recurrence relations that we consider and build up some strategies or some general techniques of solving those recurrence relations.

So now our job is to find out a special class of recurrence relations and these will be called linear recurrence relations, so I go to the next page.

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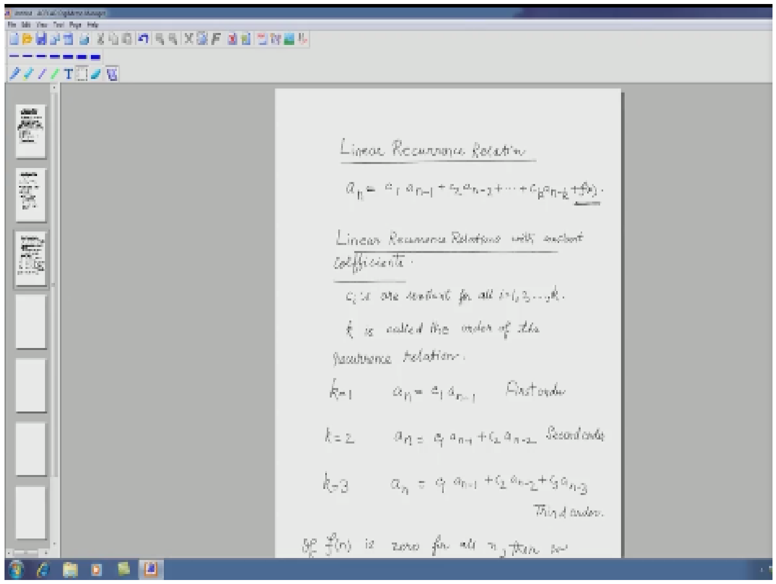

Linear recurrence relation all right so in case of a linear recurrence elation we will write n in terms of some coefficients and the previous values of the discreet numeric function so I have c1 an - 1 + c2 an - 2 and so on up to ck an – k + fn here in general the coefficients ci can be functions of n, but we further reduce this class to a more restricted one called linear recurrence relations with constant coefficients no linear recurrence relations with constant coefficients are those linear recurrence relations for which the CIS are constants.

So in this case C is are constant for all I = 1 2 and so on up to k and now one would like to ask that what is this k, k is of course something less than n and it signifies the situation that we look back up to stuck some steps but we stopped at after a while, so k is called the order of the recurrence relation k is called the order of the recurrence relation for example if k is 1 the recurrence relation will be an = c1 an - 1 if k = 2 then the recurrence relation will be an = c c1 an - 1 + c2 an - 2 if k is equal to 3 then the recurrence relation will be a n equal to c1 an - 1 +c2 an - 2 + c3 an -3.

So the first one will be called first order recurrence relation first order the second one is second order the third one is third order and so on, so we see that when we are looking at the first order linear recurrence relation then we are just looking one step back and an is some constant times the previous entry to an when I am looking at the second order recurrence relation linear recurrence relation let us say with constant coefficients then an is nothing but a constant multiplied to an - 1 + another constant multiplied to an - 2 and similarly for the third or the fourth and so on.

Then we question what about this F_n this F_N can be identically 0 that means it is possible that F_n is 0 for all n, now if F_n is 0 for all n.

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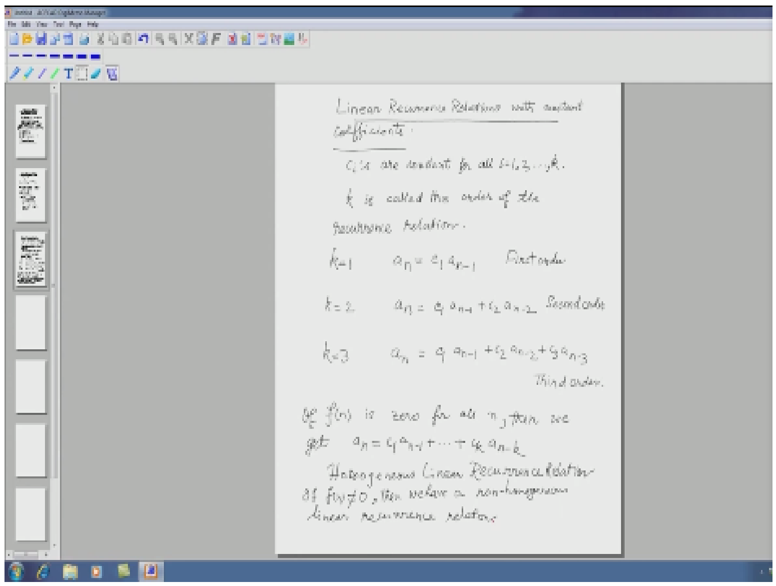

Then we get n = c1 an – 1 up to seek an - k this recurrence relation is said to be linear recurrence relation which is homogeneous or homogeneous linear recurrence relation, now it can be with or without constant coefficients if n is not 0 then we have a non-homogeneous linear recurrence relation, so we have a classification of recurrence relations so we started with general recurrence relations which can be absolutely anything, but we restricted ourselves to linear recurrence relations and a subclass of linear recurrence relations called linear recurrence relations with constant coefficients. And within linear recurrence relations there are two different classes linear recurrence relations.

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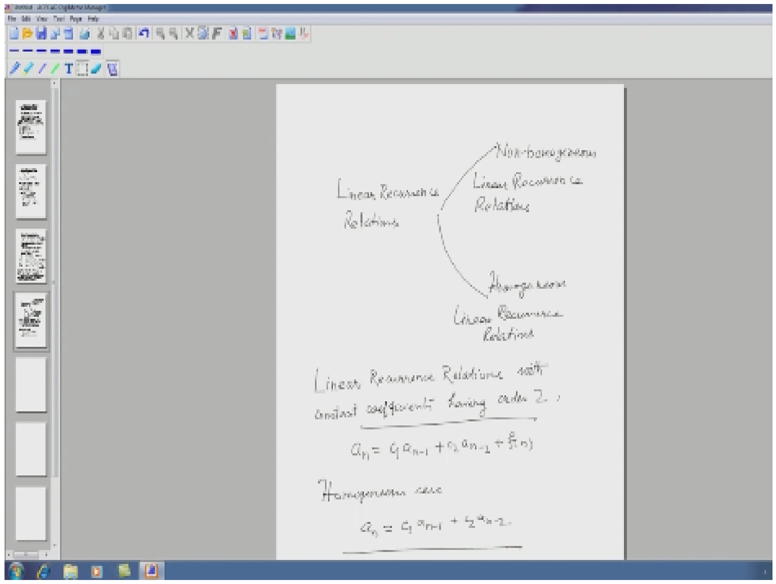

Get split up into non homogeneous linear recurrence relations and homogeneous linear recurrence relations now what we are going to do is to further restrict ourselves, so we will restrict ourselves to $k = 2$, so the class that we are going to consider now is linear recurrence relations with constant coefficients having order to first we will check order to and in the process we will automatically understand what to do with order one, now if we have this then the recurrence relation will be of the type F $an = c1\ n - 1 + c2\ an - 2 + Fn$.

Now I further restrict I make $Fn = 0$ so I take the homogeneous case which gives me $an = c1\ an - 1 + c2\ an – 2$ my aim is to find a way to write in purely as a function of n equal to some Gn and let us see how to do that.

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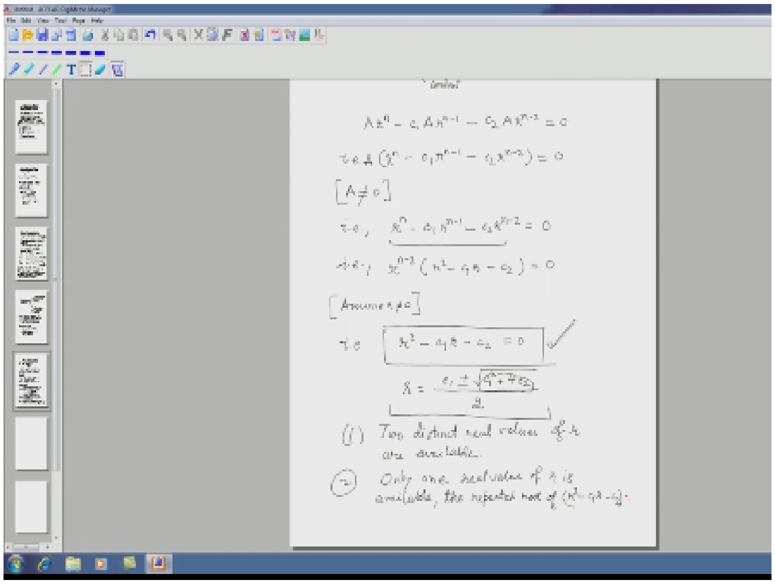

So we have a recurrence relation of this type where c1 and c2 are constants and ai are are values from some discrete numeric function, we have to find that function what we do is that we take a so called trial solution we say that let an is equal to some capital a times R^n I do not know r, but I know that a is a constant and our is also some values that we would like to find out we place this in the equation above to get Ar^n and we transpose the elements of the right hand side to left hand side so I get c1 a^ n – 1 + c2 Ar $^{n-2}$ = 0.

And from this we get R $^{n-c1}$ R $^{n-1}$ + c2 R $^{n-2}$ the whole thing inside the parentheses and into a = 0 it is very reasonable to assume that $a_0 = 0$ because this is a constant a = 0 then we have nothing to do we have the solution a n equal to 0 and of course that is a solution but we can do very little with that that solution, so we write the equation as there is a mistake here it will be minus instead of plus, so I replaced by minus here and this is also - so now I have - and here also this is - so - see - are n - 2 = 0.

Now again we can do something that is we can take the common factor R $^{n-2}$ out from this expression, so I have R $^{n-2}$ = R^{2-c1} R -c2 = 0, now again it is very reasonable to assume that R is = 0 because if R = 0 again we have the 0 solution which is of course a solution in this case but it is useless, so we have R^2 – c1r – c2 = 0, we see that proceeding as above we have arrived at a degree two polynomial equation and we know how to solve that so we can write the solution as R = 1 /2 into c1 +or - root over $c1^2 + 4c_2$ at this point we have the well-known theory of solving quadratic polynomial equations.

In one variable and we know that R has three choices are can have two different values for which this quadratic equation is satisfied or we can get a single value that is c1 / 2 which satisfies this quadratic equation or this expression inside the square root can be a negative 1 and therefore R may admit only two complex roots complex that is the equation will be admitting only two complex solutions, in this lecture we will restrict ourselves to the case where one two distinct real values of are available to only one the real value of our is available which is the repeated root the repeated root of the equation.

Or one can say a polynomial R^2 – R1 c1 R - c2 repeated root so there are two cases that we will be studying in this lecture, now the first case is where I have got two different solutions I call them.

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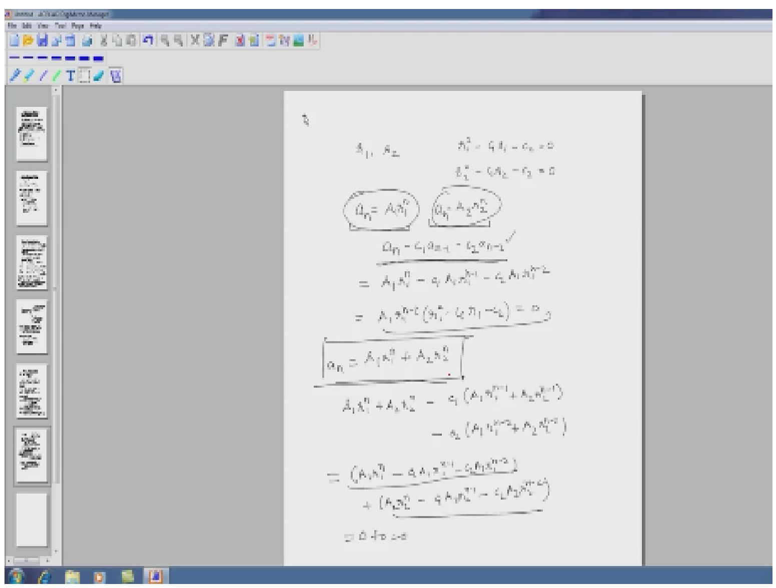

R1 and R2, so I know that $R1^2$ - c1 R1- c2 = 0 I also know that $R2^2$ - c1 R2 + - c2 + - c2 = 0 so what I see is that I if I go back to the if I if I, let us say if I build two different solutions one as R equal to some A times R1 n and R equal to some letter site a1 and letter site a to R2n then I can replace each of them in the recurrence relation which is given by a1 - c1 an - 1 - c2 an - 2 if I replace just I should write this as an as we have taken as a solution it's not our but this is a n so this is a n and this is also n.

What I want to say is that the an has two possibilities so then if I put n as the first possibility of a n in the recurrence relation I will see that a1 $R1^n$ – c1 a1 $R1^{n-1}$ - c2 a1 $R1^{n-2}$ which is equal to a1 $R1^{n-2}$ $R1^2$ - c1 R1 - c2 and this is of course equal to 0, so this of course satisfies the recurrence relation given by this and similarly the other solution also satisfies the recurrence relation, but what is interesting is that if we add these two solutions and get something like this a 1 $R1^n$ + a2 $R2^n$ let us call this n.

Now if I replace this in my in the left hand side of the recurrence relation which is essentially this then I get a1 $R1^n$ + a2 ^{n} – c c1 times a1 $R1^{n-1}$ + a2 R2 $^{n-1}$ - c2 a1 R1 $^{n-}$ + a2 R2 $^{n-2}$ and rearranging the terms I will get the terms that we have already got that is a1 $R1^{n-n}$ c1 a1 R1 $^{n-1}$ c2 a 1 $R1^{n-2}$ + a2 R2 $^{n-c1}$ a2 R2 $^{n-1}$ c2 a2 R2 $^{n-2}$, now this is of course 0 first term is 0 we have already seen over here second term is also 0 that we can check because after all this is a solution, so 0 + 0 = 0 thus we see that we can construct essentially an infinite number of solution by taking linear combinations.

Of R1 ^{n} and R^2 I am essentially free to take any constant values for a1 and a2 thus we have obtained several solutions of the linear recurrence relation, we started with but what is most surprising is that advanced theory of recurrence relations tells us that for the reconciliation under consideration these are essentially all the solutions of that one therefore the job that remains for us is to put the initial conditions and obtain the values of the constants which will give us the particular recurrence relations.

That we are looking at we will check the first recurrence relation that we build up which is corresponding.
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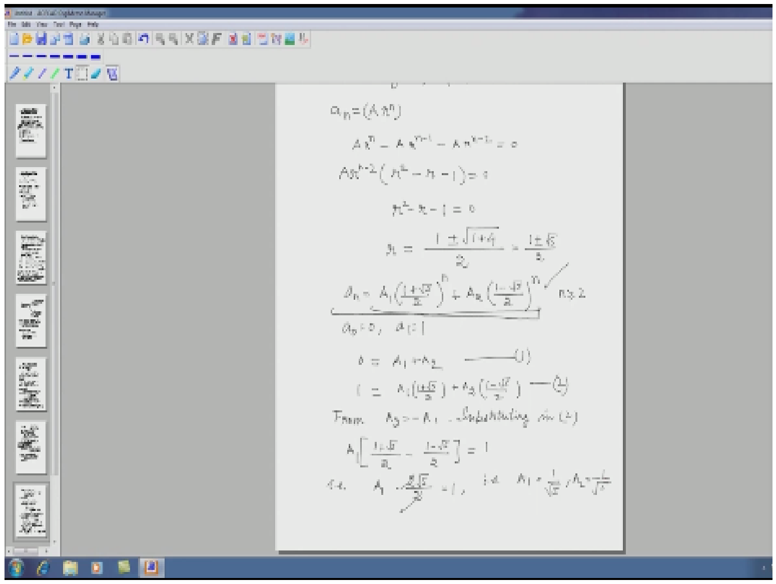

To Fibonacci sequence that if our recurrence relation is an - an -1 - an - 2 equal to 0 for all n > = 2 along with initial condition a0 = 0 and a1 = 1 now let us take the trial solution as an = a constant times are to the power n and if I put this expression in the right hands left hand side I will get a R^n - a R^{n-1} - a R^{n-2} = 0 and then if I take the common factors of all the terms then I get a into R^{n-2} multiplied by R^2 - R - 1 = 0 and we know that this means that I have to solve the quadratic equation given by R^2 - R - 1 = 0.

And we can of course do that by writing the general form and which is R = 1/2 of 1 + or - 1 + 4 which is equal to 1 + or - root 5 / 2 therefore the general equation for R is in general solution of this recurrence relation is of the form n equal to some constant a1 times 1 + root 5 / 2 raised to the power n + a2 1 - √ 5 / 2 ^{n} this is for n greater than or equal to 2 and for here zero, I know it is zero and for a 1 it is 1 now what I do here is that I try to fit the expression that I have got over here from n = 0 onward.

So I force a0 = 0 and force the this expression in the right hand side to get a1 + a2 and then 1 equal to a 1 times 1 +√5 / 2 + a2 times 1 - √ 5 / 2 that is all and so I have generated two equations suppose I am able to solve them and get the values of a1 a2 then I can plug in the values of a1 a2 in this equation to get a solution for the Fibonacci sequence reconciliation, so what I do here is that from 1 we have a2 = - of a1 substituting in 2 we have a1 = 1 + √ 5 divided by 2 - 1 - √ 5 / 2.

Which should be it should be equal to 1 and this is same as a1 times 2 / 2 √ 5 equal to 1 2 gets canceled, so I have a1 = 1 √ 5 and a2 – 1/ √ 5 therefore we have arrived at the solution of the recurrence relation that we are considering here.

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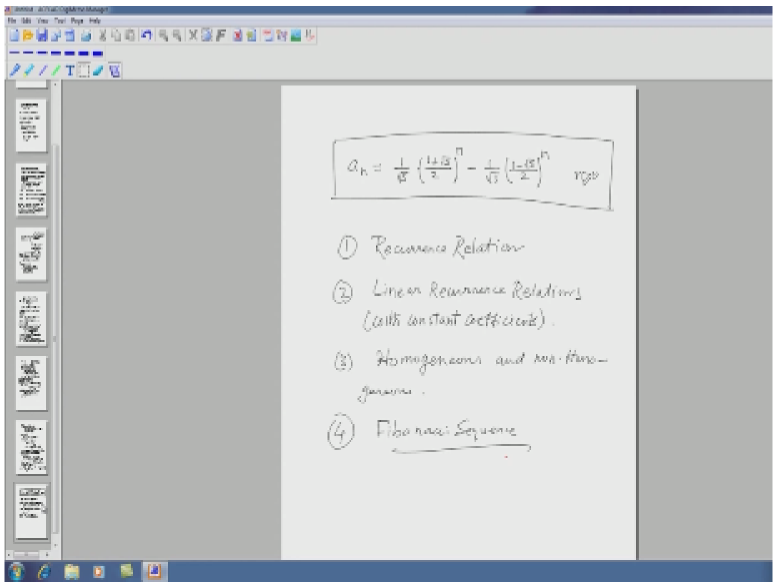

We will say a n equal to 1 / √ 5 into 1 + √ 5 by 2 raised to the power n minus 1 by root 5 1 - √ 5 / 2^n for all n > = 0 thus in today's lecture we have introduced the idea of recurrence relation then we have introduced linear recurrence relations and restricted that to the class of constant coefficients, we have introduced the classification homogeneous and non homogeneous recurrence relations linear recurrence relations and lastly, we have considered a homogeneous linear recurrence relation with constant coefficients which corresponds to Fibonacci sequence and completely solved that relation this is all for today's lecture thank you.

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