INDIAN INSTITUTE OF TECHNOLOGY
ROORKEE

NATIONAL PROGRAMME ON TECHNOLOGY
ENHANCED LEARNING
(NPTEL)

Discrete Mathematics

Module-09
Discrete numeric functions
Lecture-02
Generating functions

With
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In today's lecture we will be discussing generating functions.

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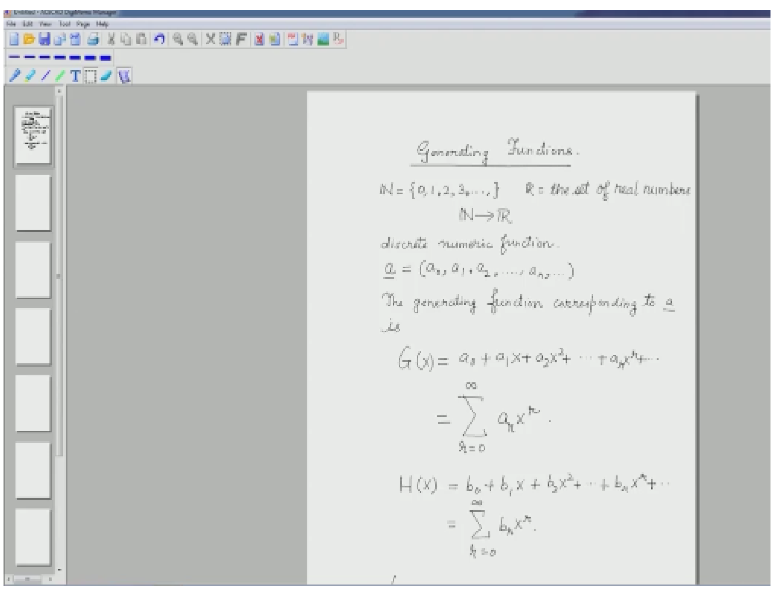

We have already discussed in our previous lecture the idea of a discrete numeric function we recall that if we have the set of natural numbers 0, 1, 2, 3 and so on and the set of real numbers then any function from N → R is called a discrete numeric function now we have also seen that a discrete numeric function is written in a somewhat different form than a than any usual function that is we write it in form of a sequence we write in print we write a bold letter a otherwise we may write a and put an underline on a and this is a sequence a_0, a_1, a_2 .. a_r and so on.

It is understood that this a represents the concerned discrete numeric function and a_0 is its value at the point 0 a_1 is its value at the point 1 a_2 is the value at the point 2 and so on given a discrete numeric function like this we can write a power series type of expression which is called the generating function corresponding to the discrete numeric function the generating function corresponding to a is $G(x) = a_0 + a_1$, $a_1x + a_2\ x^2$ + and so on up to ar x^r + and so on in a compact notation we can write this as Σ r = 0 to ∞ a sub r x^r.

We will see that given generating functions we can add and multiply by scalar and also we can mutually multiply the generating functions so suppose H(x) is another generating function corresponding to a discrete numeric function b which is the function having the values b_0 b_1 and so on so we get a generating function of this type or in our compact notation r from 0 to ∞ br x^r

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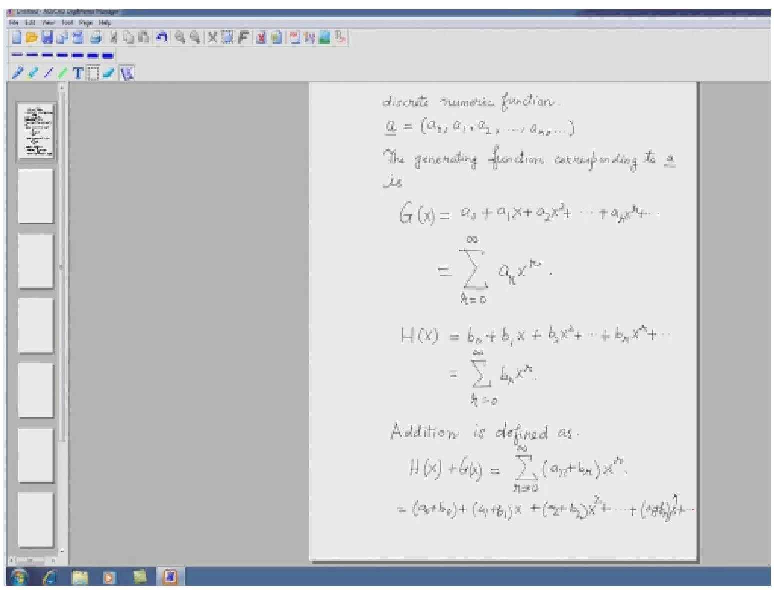

Then addition is defined as H(x) + G(x) = Σ r = 0 to ∞ ar + br x^r in other words this is $a_0 + b_0 + a_1 + b_1\ x + a_2 + b_2x^2$ moving onward to the general term ar + br x^r + and so on now if we remember the discussions on the previous lecture we showed that generating functions can be added.

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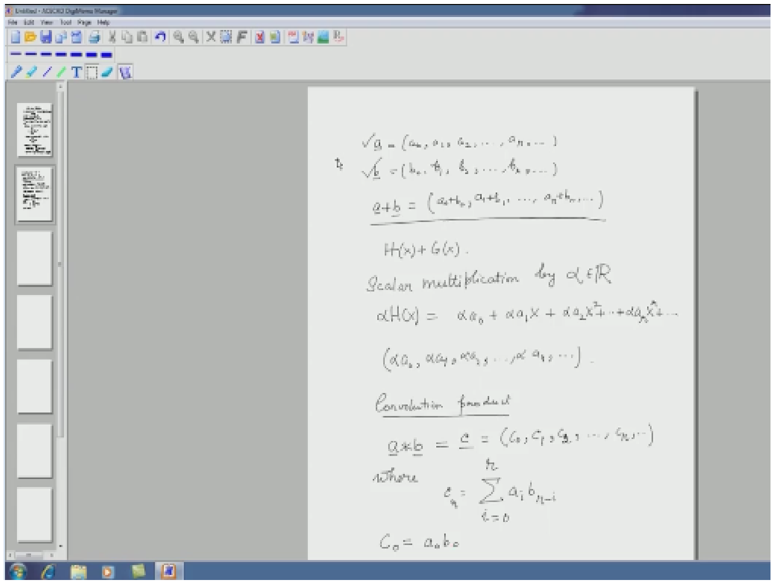

So if we have the generating function a_0 a_1 a_2 a_r and so on and b has b_0, b_1, b_2 and so on the general term Bbr and so on we have seen that the sum of the of these two discrete numeric functions is $a_0 + b_0$ $a_1 + b_1$ $a_r + b_r$ and so on and from what we have seen just now these discrete numeric functions generating function is H(x) + G(x) in the similar way we can define scalar multiplication by α which is an element of r if we take a generating function H(X) then α times H(x) is simply α a_0 + α times a_1 x + α times a_2x^2 and the general term α is sub r and x^r and so on.

And so we see that this is of course this corresponds to the discreet numeric function α a_0 α a_1 α a_2 and α a_r and so on thus we see that the operations on generating functions correspond to the operations on discrete numeric functions parallely now we define another operation or discrete numeric function which is called convolution products now if we take as before the discrete numeric function a and the discreet numeric function b then the convolution of a and b is defined as a discrete numeric function given by a* b.
Let us say it is = c where the elements of c are c_1 c_0 c_1 c_2 and so on up to c_r where c_r is a_i b r – i, i running from 0 to r for example we see that if r = 0 c_0 is a_0 b_0

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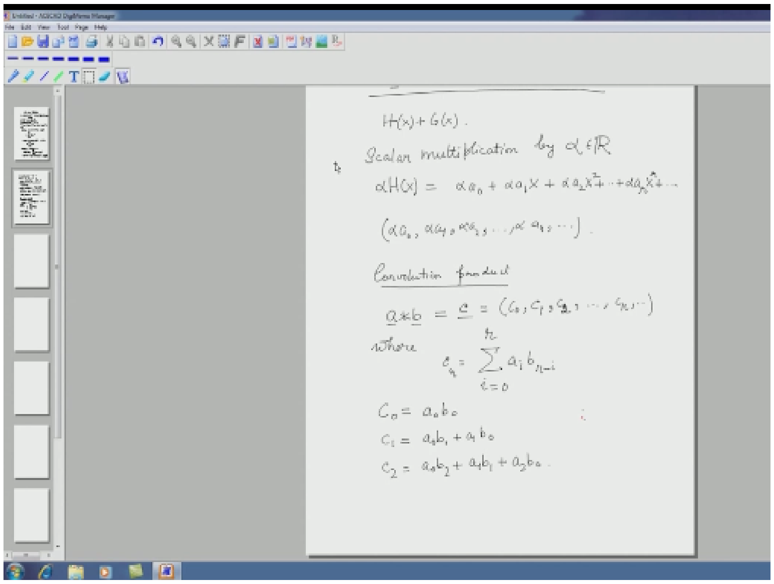

c_1 is a_0 b_1 a_1 b_0 c_2 is a_0 b_2 + a_1 b_1 + a_2 b_0 and so on now we ask that what is the corresponding operation on generating functions of these discrete numeric functions now as we see that this operation is nothing but the product of the two generating functions so we take H(x) and G(x) as the generating functions of b and a.

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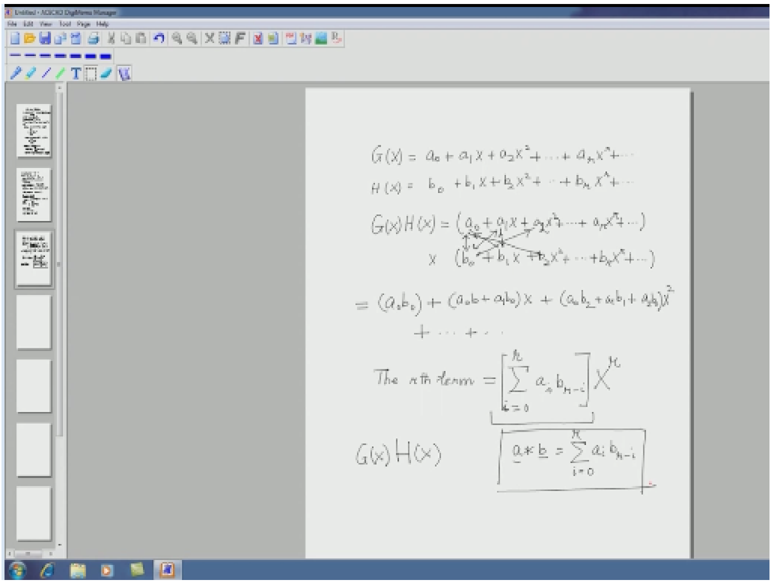

So let us write afresh suppose G(x) is = $a_0 + a_1x + a_2\ x^2 + a_r\ x^r$ and H(x) = $b_0 + b_1\ x + b_2x^2$ + and so on $b_r\ x^r$ and so on if we take the product G(x) into H(x)G(x) into H(x)then this is product of these two sequences $a_0\ a_1\ xa_2x^2 + a_r\ x^r$ + and so on multiplied by $b_0 + b_1x + b_2x^2$ and soon $b_r\ \ x^r$ + and so on now this is = see the first term is by combining a_0 with b_0 so I have got $a_0\ b_0$ the second term is the indeterminate x along with the coefficient a_0 b1 + $a_1\ b_0$ it goes like this so I have $a_0\ b_1 + a_1\ b_0x$ then the third term is $a_2\ b_2$ sorry $a_0\ b_2$ then a1 b1 + $a_2\ b_0$ all right.

And so on and we have to find the expression for the rth term if you look closely we will find that the expression of the rth term is the correct term Σ $a_i\ b_r$- i, i running from 0 to r thus we see that the of course just to be more specific this will be the rth term is has x^r we take so the coefficient of the rth term is Σ $a_i\ b_i$ r- i, i runs from 0 to r however we see that the coefficient of x^r to the power r in the product G(x) times H(x) is the convolution product of a and b.

Now this product has several interesting applications the first application is very useful in writing generating functions in compact closed forms now we take two generating functions.

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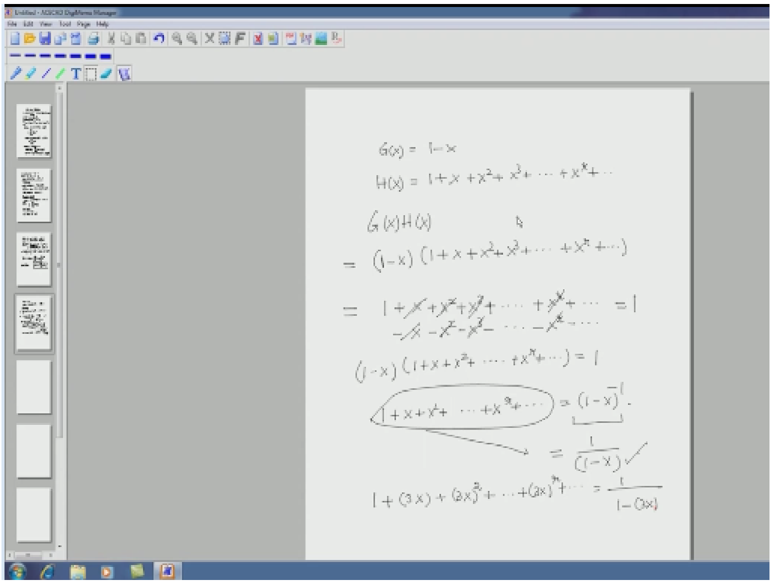

1 is given by G(x) is = 1 - x the other one is given by H(x)is = 1 + x + x^{2} + x^{3} and so on up to x^{r} and then onward now if we multiply G(x) and H(x) then we get 1 - x multiplied by 1 + x + x^{2} + x^{3}d + and so on up to x^{r}+ and so on so if I now multiply the first term 1 to all the terms in the second factor I will get 1 + 1 1 into x x + x^{2} + x^{3} + + and so on up to x^{r} and onward and then the second term of the first factor - x is multiplied to the all the terms in the second factor and we obtain - x – x^{2} - x^{3} and so on – x^{r} and so on.

So we see that there are many cancellations in fact all the terms will cancel except 1 therefore we will have 1 - x into 1 + x + x^{2} + x^{r}+ and so on = 1 and therefore we can write 1 + x + x^{2} + and so on up to x^{r} and so on as 1 – x^{-1} because this infinite series 1 + x + x^{2} + x^{r} when multiplied by 1 - x and in fact we can check that the order does not matter it may be 1 - x and this series or the series into 1 - x in both the cases we get 1 therefore this series is the inverse of 1 - x and therefore can be written as 1 – x^{-1} and we can even write it as 1 - 1 / 1 – x.

Here we see that we are not putting any numerical values in the place of x so these infinite series that we obtain as generating functions of discrete numeric functions are purely symbolic or formal so there is no concept of convergence that we get in real analysis so there is nothing called divergence nothing called convergence it is only important that these series should be well defined that is if you give me an r no matter how large I should be able to compute the coefficient corresponding to up to r in finite time and that is all what I need.

Therefore I can have an agreement of writing 1 + x + x^2 and so on as simply 1 / 1 - x we see here that x is a pure symbol so x can be replaced by anything else and for example if I took 3x instead of x then I would have got x + 3x + $(3x)^2$ + and so on up to $3x^r$ and so on this is = 1 - 3x we can check that by multiplying according to the rule of multiplication that we have defined so if we now process the left-hand side of the equation we see that 1 - 1 / 1 - 1 / 1 - 3x = 1 + 3 times x + 3^2 times x^2 + and so on up to 3^r times x^r and so on.

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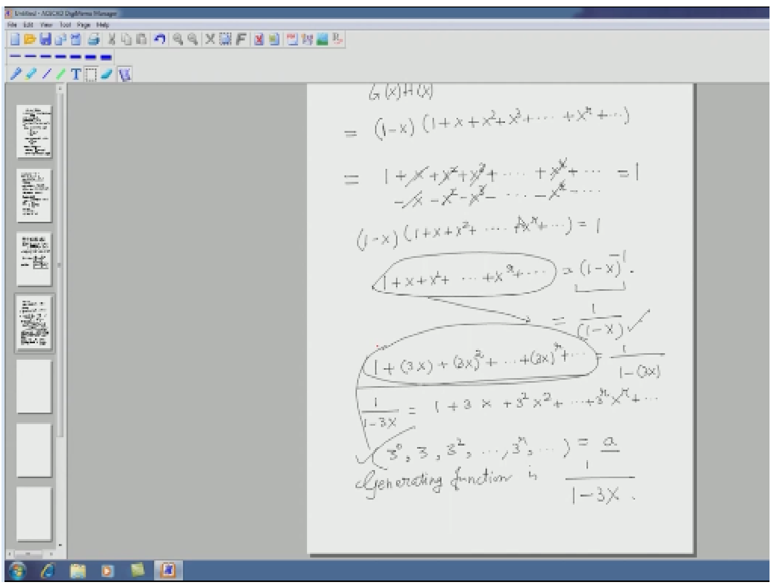

Therefore we see that the generating function corresponding to the discrete numeric function 3^0 3, 3^2 and so on 3^r and so on let us call this a the generating function corresponding to this a is 1 / 1 - 3x thus we have seen how to get a generating function of a discrete numeric function in it is closed form suppose we are given this discrete numeric function we then could have written this expression as generating function and by using the rules that we have derived already we could have come up to this.

We can now move on to another example of finding discreet numeric function corresponding to generating functions so let us look at this generating function.

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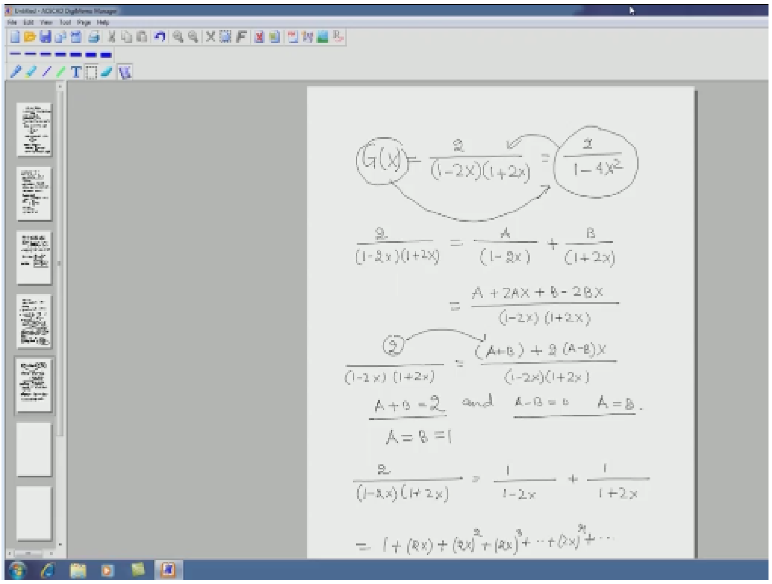

Let us call it G(x) which is = 2 / 1 - 2x 1 + 2x which is of course = 2 / 1 - $4x^2$ so it is possible that somebody tells me that look there is a discrete name like farad there is a generating function G(x)which looks like this find out the discrete numeric function corresponding to this G(x) for that we will first factorize the denominator and write in this form and then decompose it in terms of partial fractions let us do that 1 - 2x 1 + 2x if we decompose in terms of partial fractions then it will be like this a / 1 - 2x + b / 1 + 2x and we have to find the values of a and b.

So we write 1 - 2x 1 + 2x and in the numerator it is a + 2A x + B - 2B x this is = now since this is an identity therefore we must have A + B = 0 and A - B = 0, A - B = 0 means they = B I am sorry this is not 0 but it is two because we must remember that this expression should be identically = 1 - 2x 1 + 2x so therefore 2 is = A + B and there is no x in the left hand side so the coefficient of x should be 0 therefore I have got A - B = 0 and A + B = 2 so from A - B = 0 we have got A = B and therefore from this equation it is clear that A = B = 1 thus we have the decomposition of 2 / 1 - 2x 1 + 2x as 1 / 1 - 2x + 1 by 1 + 2x.

Now by using our previous rule we can expand each of the terms of the right hand side into power series which will correspond to the generating function so it will be 1 + 2x + $(2x)^2$ + $(2x)^3$ + so on $2x^r$ and so on.

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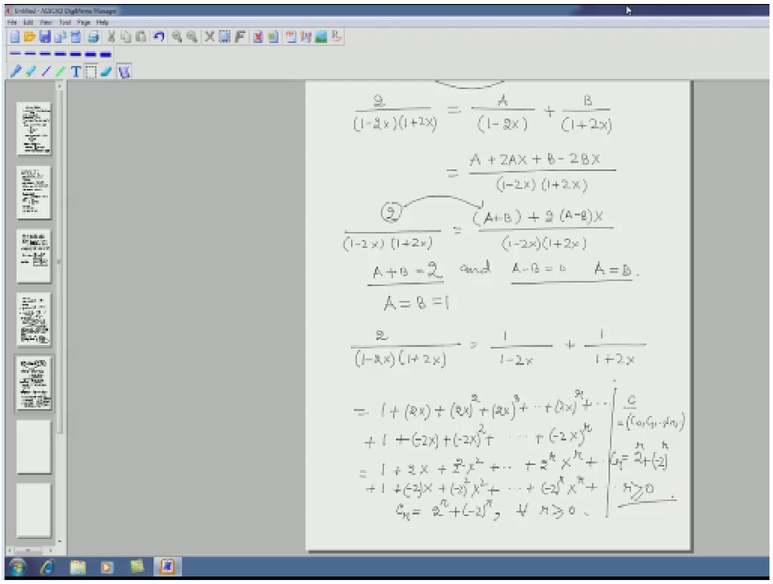

And the other term will give me 1 + -2x + - $(2x)^2$ and so on up and the other term is - $2x^r$ and we can process it in this way we can write this is 1 + 2x + 2^2 x^2 and so on up to 2^r x^r and onward and 1 + - 2(x) + - 2^2x ^{2} + and so on + - 2^r x^r and so on so the coefficient of the rth term for r ≥ 0 is some cr = 2^r + - 2^r for all r ≥ 0 thus the corresponding discrete numeric function is i write here some c which is = c_0 c_1 c_r says that c_r is 2^r + - 2^r for r ≥ 0.

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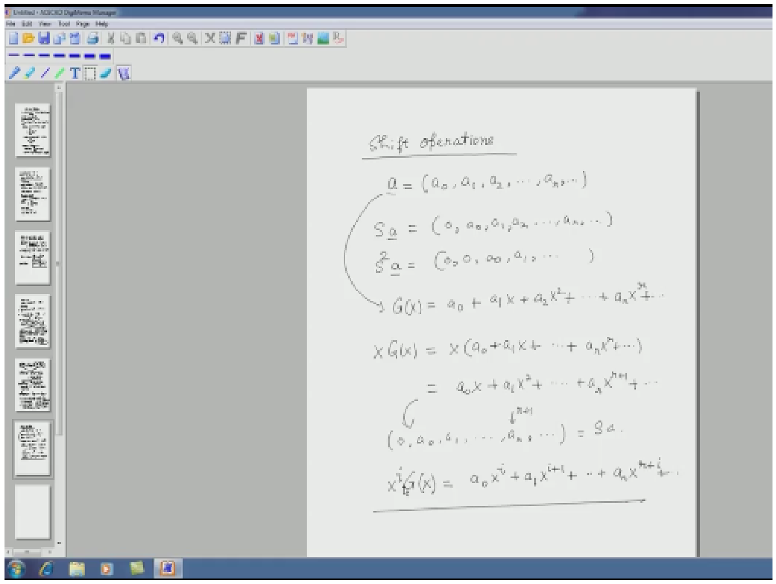

Now we will start discussing some more operations on generating functions we have seen in case of discrete numeric functions that we have shift operations now if we have a discrete numeric function a given by a_0 a_1 a_2 and so on a_r like this then one right shift denoted by Sa is the function 0 a_0 a_1 a_2 a_r and so on if we apply s^2 to a then we have to write shifts which is 0, 0 a_0 a_1 and so on now in case of generating functions it will be just multiplication by x so consider the generating function corresponding to a $G(x) = a_0 + a_1 x + a_2 x^2 + a_r x^r$ and so on.

Now we apply x on G(x) that is we multiply x to G(x) then we will get x times $a_0 + a_1x +$ up to a_r x^r and so on this is $= a_0x + a_1x^2$ and so on then I have got a are x^r so see now the coefficient a^r becomes the coefficient corresponding to x^r+ 1 therefore the corresponding discrete numeric function will be 0 a_0 a_1 and so on a_r and so on where this is in the r+1th position thus this is nothing but Sa similarly we get a^2a if we multiply by x^2.

And in general if you want to find the discrete numeric function corresponding to S^a by using generating function then we must multiply G(x)/ x^i. So this is i G(x) which will give us a_0 x^i + a_1 x^i + 1 + and so on a_r x^r+ i and so on now we can also talk about the left shift operation and we can in fact derive it starting from applying S^{-1} to the discrete numeric function a so a is a_0 a_1 a_2 a_r and so on.

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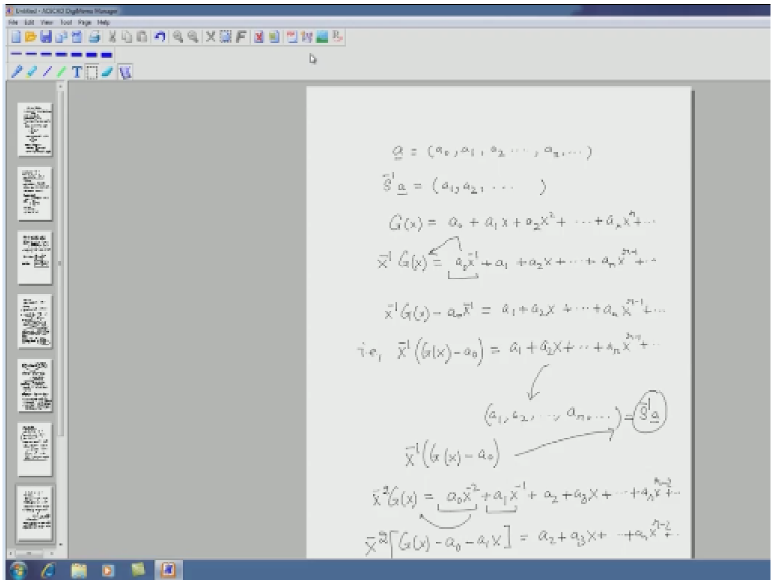

So I know that s inverse a or I should I should not say S inverse but probably I should say S^{-1} it corresponds to one left shaped it is a_1 a_2 and so on so the generating function G(x) is given by a_0 + a_1x + a_2x^2 and so on up to a_r x^r+ and so on if I multiply by x^{-1} G(x) then I get a_0 x^{-1} + a_1 + a_2x + and so on we have a_r x^r- 1 + and so on and then we may like to transpose this first term to the right hand side to obtain x - 1 G(x) - a_0 x^{-1} = a_1 + a_2x and so on up to a_r x^r- 1 and onward.

And the right hand side becomes now it is clear that the discrete numeric function corresponding to the right hand side of this identity is a_1 a_2 and so on in the r - 1 at position we have a_r and move onward which is = S^{-1} a therefore we see that x $^{-1}$G(x) - a_0 is the discrete numeric function corresponds sorry is a generating function corresponding to S^{-1} of a now what happens if you multiply G(x) by x^{-2} we check that this is x^{-2} into G(x) is a_0 x^{-2} + a_1 x^{-1}+ a_2 x^0+ a_3x and so on up to a_r x^r- 2 and we move on.

Therefore we will see that transposing the first two terms of the right hand side to the left hand side and taking the common factor x^{-2} out we get x^{-2} G(x) - a_0 - a_1x is = a_2 + a^3 x + and so on a are x^r- 1, 2 and so on which is of course the generating function corresponding to the discreet numeric function.

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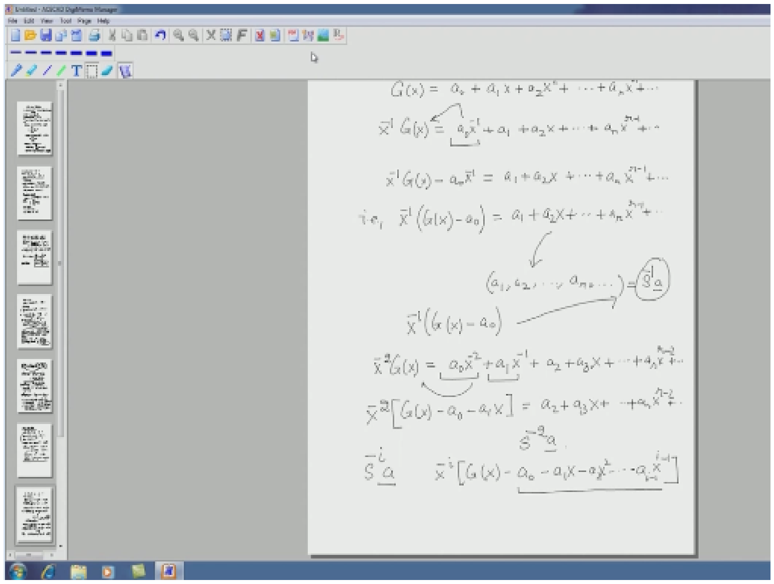

S^{-2} a now we have a rule the rule is that the generating function corresponding to S^{i} a is x^{-i} G(x)- a_0 - a_1x - a_2x^2 - and so on up to –a i – 1x ^{i}-1 and it ends here so we have a finite series that has to be subtracted from G(x) and then we multiply by x^{-i} and we obtain the generating function of s S^{i}a this is all about generating functions for today as a last topic.

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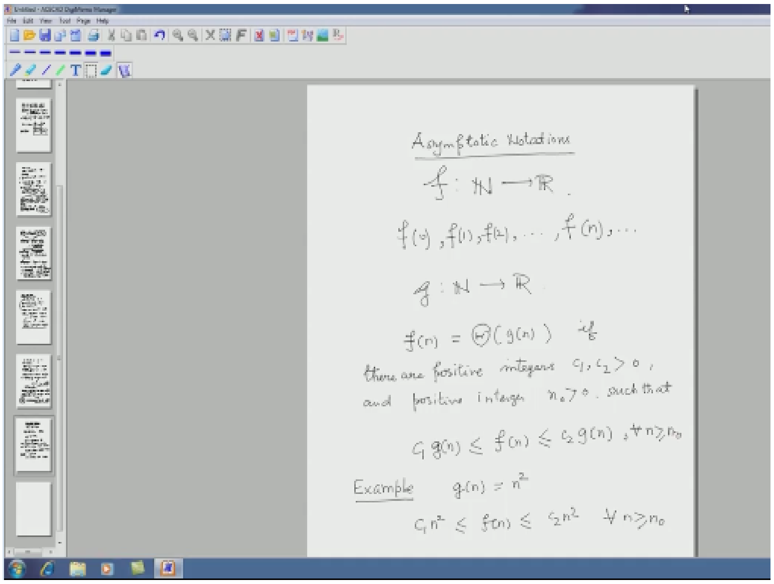

In the discussion of discrete numeric functions and the generating functions corresponding to these discrete numeric functions we discuss the asymptotic notations here for convenience we will denote discrete numeric functions as usual functions so we denote a function f from $N \rightarrow R$ which is a discrete numeric function and we are interested in knowing the growth of f so f takes values f_0 f_1 f_2 so on in general.

Let us say it takes a value f(n) and it goes on we would like to know about how it grows so we compare f to certain other functions let us say G which is also a discrete numeric function we say that $f(n) = \Theta(g(n))$ if there are positive integers c_1 and c_2 and positive integer n_0 such that c_1 g(n) $\leq$ f(n) $\leq$ c_2 g(n) for example if g(n) is simply n^2 then this will mean that there exists positive integer c_1 c_2 and n_0 such that $c_1 n^2 \leq f(n) \leq c_2 n^2$ for all $n \geq n_0$.

Here also we have to say that for all $n \geq n$ 0 so in language it means that if f is Θ n ^{2} then I should be able to find two constants which must not vary c_1 and c_2 and a constant n0 all positive such that for all $n \geq$ n0 the value of f is bounded above by c_2 n^2 and bounded below by c_1 n^2 there are other notations involving this idea.

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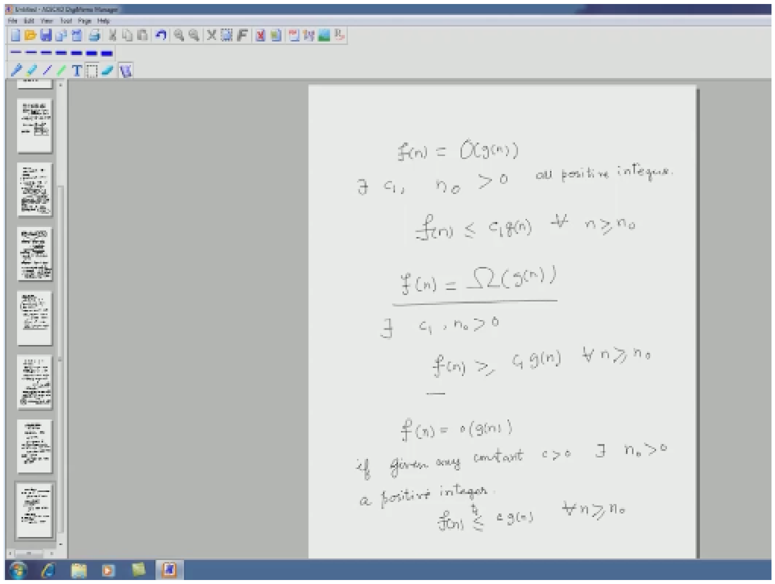

We say that f(n) is = O of g(n) if there exists c_1 c_2 and n0 > 0 all positive integers such that if n is ≤ I am sorry we do not need two constants over here we just need c_1 and n_0 so it is only c_1 and n_0 says that f(n) is c_1 n ≤ c_1 n for all n ≥ n 0 and we say f(n) = Ω g(n) just one correction in the previous statement we have to say that this is = c_1 g(n) alright and f(n) is Ω g(n) if there is there exists c_1 and n_0 > 0 such that f(n) is ≥ c_1 g(n) for all n ≥ n_0 and finally we say that f(n) is o(g(n)).

If there if given any constant given any constant c > 0 there exists n_0 > 0 a positive integer such that if n is ≤ c times g(n) for all n ≥ n_0.

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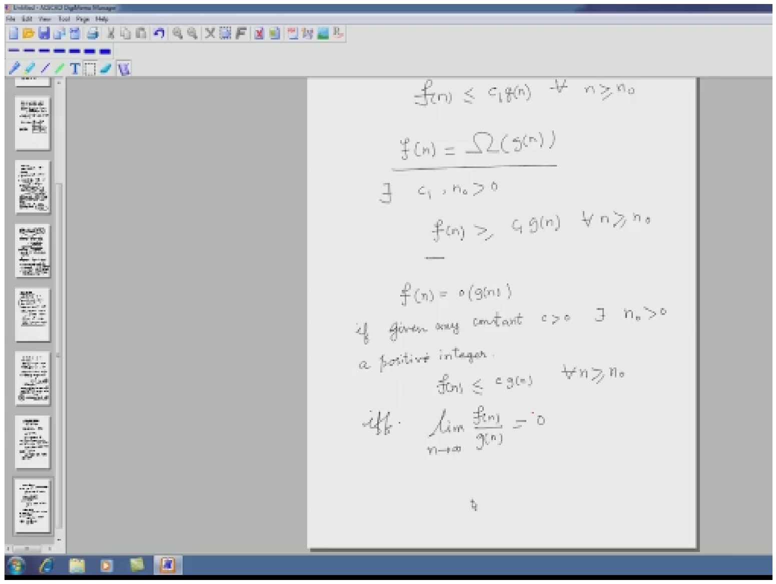

It can be shown that this happens if and only if limit n tending to ∞ f(n) / g(n) is = 0 these are the 4 asymptotic notations which are used very commonly to discuss the growth of discrete numeric functions.

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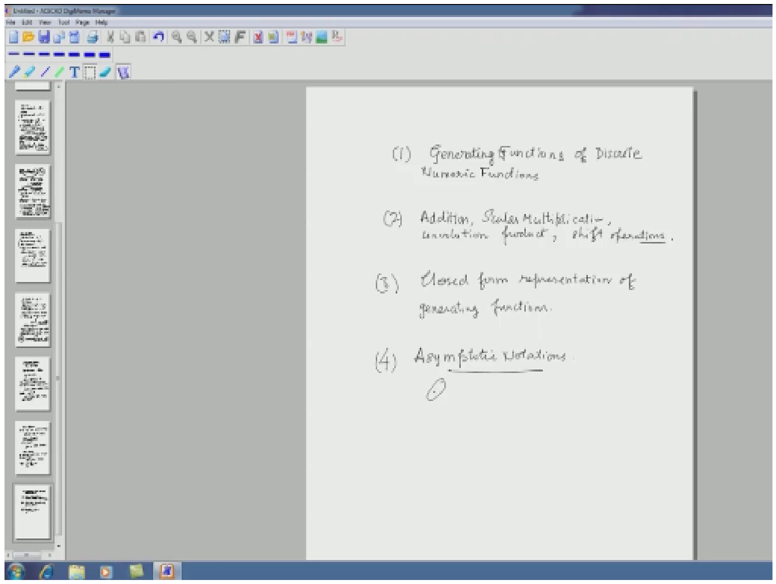

So in this lecture we have discussed generating functions of discrete numeric functions we have also discussed several manipulative techniques of generating functions like addition scalar multiplication convolution product then shift operations on generating functions apart from this we have learned how to write a generating function in a closed form.

So closed form representation we have also seen in case we are given a closed form representation of a generating function how to get the corresponding discrete numeric function and finally we have discussed the necessary notations used to study growth of discrete numeric functions these are called asymptotic notations among these asymptotic notations we have discussed the most important 4 asymptotic notations is Θ then we have Ω and lastly o this is all for today thank you.

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