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NATIONAL PROGRAMME ON TECHNOLOGY
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(NPTEL)

Discrete Mathematics

Module-01
Set theory
Lecture-03
The principle of inclusion and exclusion

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Today we will discuss the principle of inclusion and exclusion.

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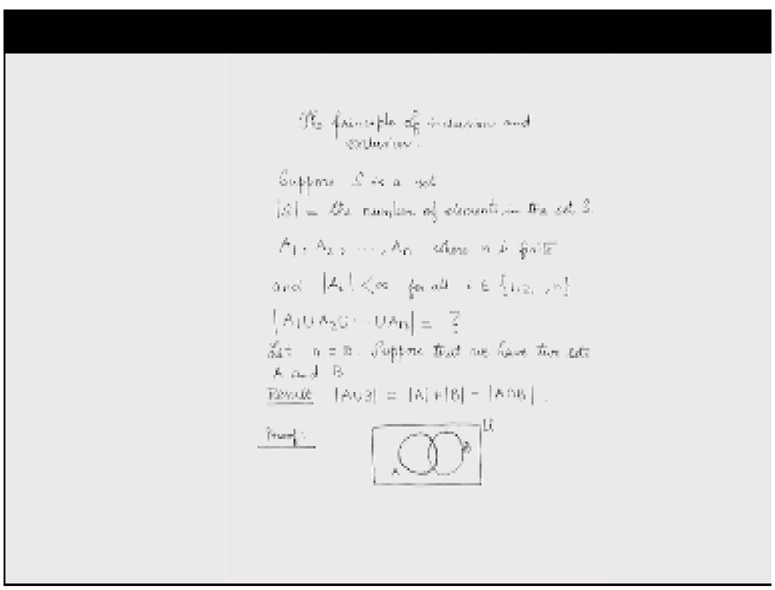

Now the principle of inclusion and exclusion is a principle by using which we count the number of elements in the ∪ of several sets suppose s is a set by the symbol modulus of s we will mean the number of elements in that set now what we are interested in is if we have several sets like even a two and so on up to a n where n is finite and the size or cardinality of the set AI which we did not by let us say modulus of AI $<\infty$ for all I belonging to the set 1 to dot n what is the size or cardinality or simply the number of elements in the ∪ even ∪ a 2.

And so on ∪ n now the principle of inclusion and exclusion lets us compute this precisely to begin with we start with n = 2and instead of writing a1 and a2 we will consider the sets a and B now we have this result number of elements in a ∪ B is equal to number of elements in a plus the number of elements in B minus the number of elements in A ∩ B the question is how do we prove this result to check the proof we will look at the Venn diagram consisting of these two sets.

So suppose this is my Universal set u and inside this Universal set we have two sets a and B now in general we do not have reasons to expect that A and B are disjoint what we will prove are of course true when A and B are disjoint but let us take the general situation where there may be ∩s finite ∩ of A and B now I denote the set a by this circle and set B by the other circle now we see that there is a region which is inside a but not in B and this region is precisely a ∩ complement of B.

Then there is another region which is both in A and B so this region is A ∩ B and the other region somewhat symmetric is the region where which is in B but not in a and this is a complement B so a be complement B complement is this region A ∩ B is the region shared by both A and B and A ∩ B is the region which is in B but not in a now it is not difficult to see that these regions are disjoint from each other.

So the number of elements in the region which is the ∪ of these three regions is the sum of the number of elements in each of them the reason is as I have already said that these regions are disjoint on the other hand we also note that if we take the union of these three regions then we get the set A ∪ B thus we can say that the cardinality of A ∪ B is equal to the cardinality of A ∩ B complement plus the cardinality of A ∩ B plus the cardinality of a complement ∩ B now we move further on now suppose you give me the set a and tell me to split it up into two disjoint portions I can of course split it up in this way A ∩ B complement union A ∩ B.

So we have already seen that this is this portion and the portion A ∩ B now these two sets are disjoint and we know that the union of two disjoint sets will have exactly the number of elements equal to the sum of the number of elements in individual sets therefore we will have cardinality of a equal to cardinality of A ∩ B complement plus the cardinality of A ∩ B.

But we see that the equation one contains an element cardinality of A ∩ B complement and equation two contains the same element what we can think of doing is replace the cardinality of A ∩ B complement in equation bar or equation 1 by an expression that we derived from equation two but we will do that after the next step because in the same way I can prove that cardinality of B equal to cardinality of a complement B plus cardinality of A ∩ B.

Let us call it 3 now from 3 we see that a complement   B can b here placed for here in wall by cardinality of B minus cardinality of A∩ B if we do that we have A ∪ B equal to cardinality of a minus cardinality of A ∩ B plus cardinality of A ∩ B plus cardinality of B minus cardinality of A ∩ B some cardinalities of A ∩ Bare getting cancelled.

And therefore we will get cardinality of a ∩ B equal to cardinality of a plus cardinality of B minus cardinality of A ∩ B thus we have got the result that we wrote down over here we can solve some problems related to this counting principle.

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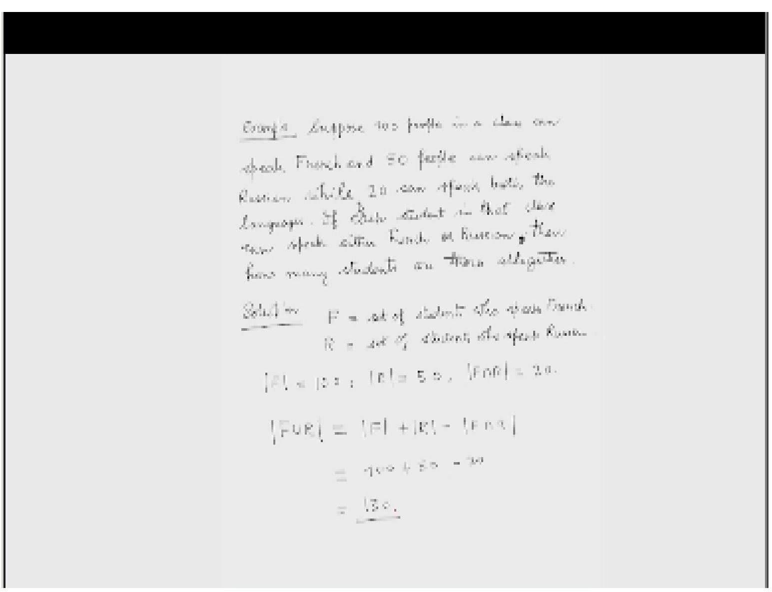

Let me write down one problem suppose hundred people in a class can speak French and 50 people again speak rush on while why 20can speak both the languages if each student in that class fix either French or Russian if each student in that class can speak either French or Russian.

Then how many students are there all together now let us denote by F the set of students set of students who speak French and by our set of students who speak Russian now it is clear that the number of students who speak French is equal to hundred the number of students who speak Russian is equal to fifty.

And number of students who are in both the groups is equal to twenty and we have been told that any student in the class speaks either French or Russian so the total number of students in the class is the number of students in the set F∪ R or the cardinality of the set F∪ R which is given by cardinality of F plus cardinality of our minus cardinality of F ∩ R and therefore we are going to get 100 +50- 20 which is 130.

Now let us look at another example which I will leave as an exercise from a group of 10 doctor show many ways a committee of five can be formed. So that at least one of doctor A and doctor B will be included what I claim is that this problem also can be attempted by the principle of inclusion and exclusion I leave it as an exercise next we move to principle of inclusion and exclusion for three sets here we will take the three sets.

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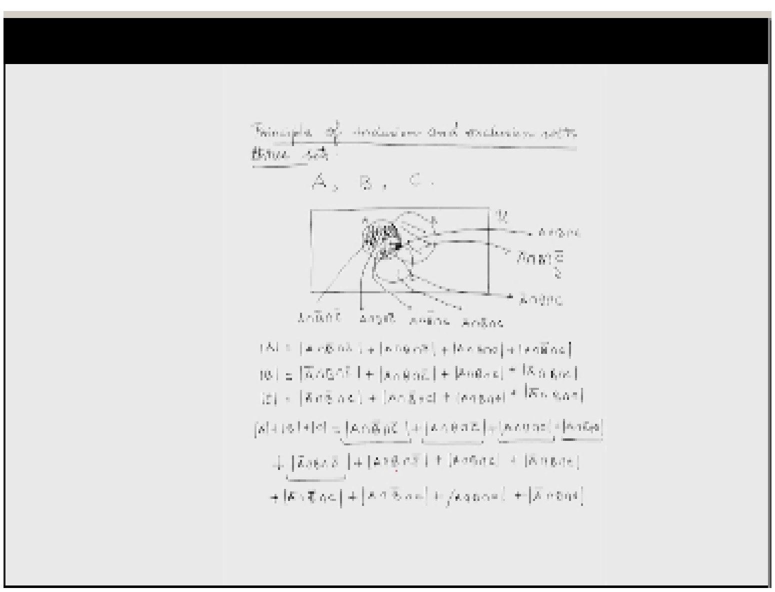

To be A B and C we will draw a Venn diagram showing the general situation so I have a here a then B and then C now this region is A ∩ B ∩ C complement this region is a ∩ B ∩ C complement this region is A ∩ C or rather let me write a ∩ B compliment ∩C this region is A ∩ B complement ∩ C this region is a complement ∩B ∩C and lastly this one is A ∩ C complement let me write B first so I will write over here B, B and not the complement so let me remove this portion so this is A complement ∩ B ∩ C complement.

Now therefore we can write down the cardinalities of ABC cardinality of a is equal to cardinality of A ∩ B complement ∩ C complement + A ∩ B ∩ C complement plus a ∩ b ∩ c+ A ∩ B ∩ C incidentally I have left out one set over here this said this is a ∩ B ∩ C cardinality of B in a similar way is a complement B ∩ B ∩C complement plus a ∩ B ∩ C complement plus a ∩ b ∩ c plus A complement ∩ b ∩ c cardinality of C is a complement ∩ B complement ∩ C plus a complexion B complement ∩ C plus a ∩ b ∩ c plus a complement ∩ B.

And if I sum all of them then I am going to get we have these expression now what we will see is that A ∩ B complement ∩C complement which is this portion then A ∩ B ∩ C complement which is this portion A ∩ B ∩ C which is this portion then A ∩ B complement ∩ C which is this portion this is hole of a and then we have a complement BC complement this is this portion and then we have we have a complement ∩ B ∩ C which is this portion.

And lastly we have a complement ∩ B complement ∩ C which is this portion all of them together gives me the cardinality of A ∪ B ∪ C so I can write cardinality of a ∪ B ∪C which is sum of these terms plus a ∩ B ∩ C complement plus a ∩ b ∩ c plus a ∩ B complement ∩ C plus a ∩ b ∩ c plus a complement ∩ b ∩ c as five elements now if we consider these two terms we will see that we get A ∩ B because this is A ∩ B ∩ C complement and A ∩ B ∩ C so together we get A ∩ B.

So let me write down this is cardinality of A ∪ B ∪ C plus A ∩ B plus this one and this one together gives me A ∩ C and lastly I have a complement ∩ B ∩ C therefore I have an expression like this so A ∪ B ∪ C is equal to cardinality of a plus cardinality of B plus cardinality of C minus cardinality of a ∩ B minus cardinality of A ∩ C minus cardinality of a complement ∩ B ∩ C complement.

Now if we consider the set B ∩C see then we see that B ∩ C is a ∩ B ∩ C cardinality plus a complement ∩ B complement a section C if I replace this above I get a ∪ B ∪ C cardinality is equal to cardinality of a plus cardinality of B plus cardinality of C minus cardinality of a ∩ B minus cardinality of a ∩ C minus cardinality of B ∩ C plus cardinality of a ∩ B ∩ C which is the final result once we come to the this point we wonder that what should be the general term we have seen for two sets.

A and B what is the cardinality of A ∪ B in terms of cardinalities of A B and A ∩B we have seen for three sets ABC just now an expression of cardinality of A∪ B ∪ C in terms of cardinality of a ABC and the mutual ∩s what about the general expression in general we can have a finite number of sets each with finite cardinality and we are interested in the cardinality of their ∪ now let me write the general statement.

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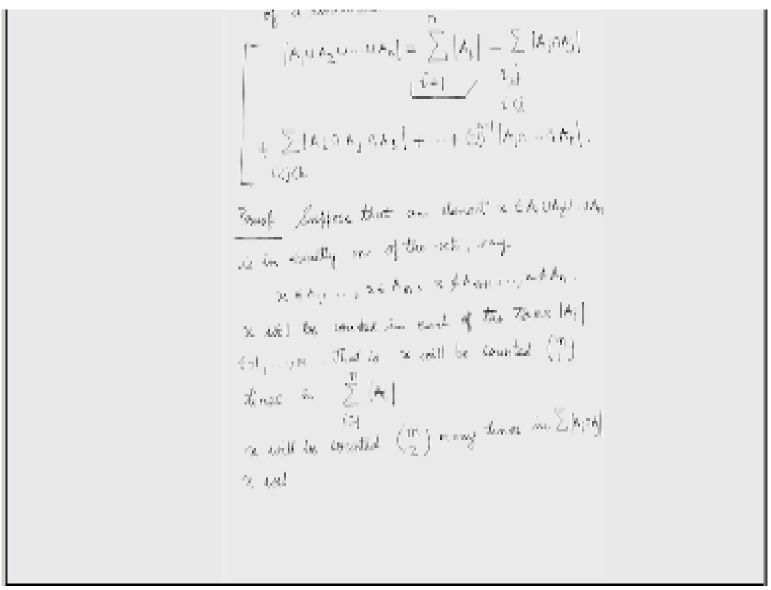

First I write the header as the general statement of that principal of inclusion and exclusion if a I I = 1 to dot n are finite subsets of a universal set you then cardinality of a in ∪ a 2 ∪ soon up to ∪ a n is equal to summation I equal to 1 to N cardinality of AI minus σ I comma J where I is less than J cardinality of AI ∩ AJ + σ I less than J less than K cardinality of AI AJ okay that is cardinality of AI ∩ AJ ∩ a K and it will go on like this to the final term 1^{n-1} even ∩ up to ∩ a n now this is the general form the question is that how do we prove it.

Now I will give a short argument of the proof let me write down proof suppose that and element X belongs to a 1 ∪ a 2∪ up to a n is in exactly M of the sets say X belongs to a 1 up to X belongs to a M and X does not belong to a M + 1 up to X does not belong to a n so I start by considering an element which in which is in exactly M sets and without loss of generality I assume that X is in the first M subsets and strictly not in the other ones now the question is that how many times X will be counted.

So X if you look at this expression then X will be counted in many ways in the first sum X we will be counted in each of the terms a I I =1 up to M that is X will be counted choose one times in I = 1 to n σ AI similarly X will be counted M choose too many times in σ AI ∩ AJ X we will be counted choose three many times in σ AI ∩ AJ ∩ a K and soon so if we keep on increasing the number of sets then we can count the number of times X will be counted they will find that we have a series.

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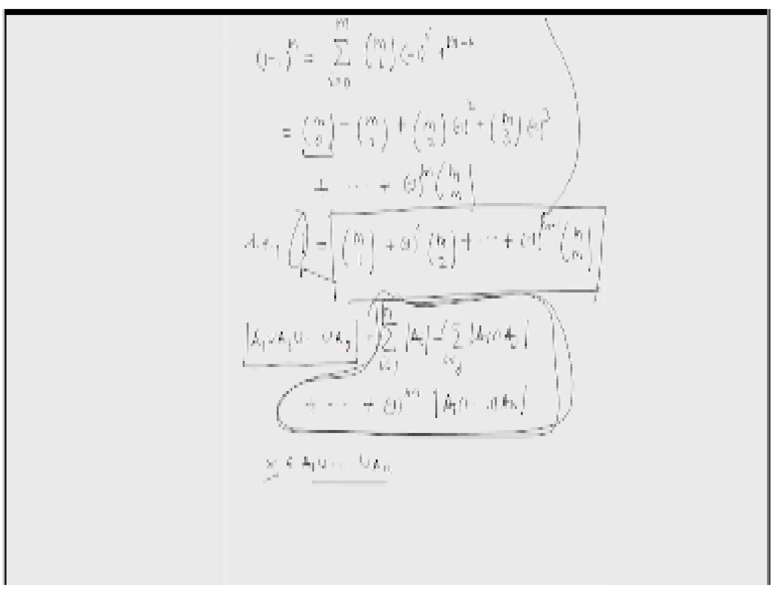

So the X is counted in this way so X so X is counted choose one many times in the first some and then in the second some it is counted choose too many times but we subtract the second from the first then we add up the next one and we proceed in this way to go up to - 1 $^{m-1}$ to them now we see so this is the total number of times any X belonging to a 1 ∪ up to a n will be counted so why did that it is in M many subsets of the considered subsets.

So now we see the binomial expansion of1 -1 raised to the power M which is equal to σ I = 0 to M M choose I- 1 [I] 1raised to the power M -I which is equal to M 2 0 - M choose 1 +M choose 2 -1^2 + M choose $3-1^3$ plus and so on up to minus1 raised to the power M aim to them and remembering that m to 0 is 1 and transposing we will have 1 =M choose 1 minus or just write plus minus1 [1] M choose 2Plus and so on +- 1 raised to the power M -1 aim to them and so if we see that this sum appears over here and which is equal to 1 this means that when I consider that expression let us recall that expression again in this expression if I take any X belonging to a 1 ∪ a n and see that number of times it is counted here and appropriately add and subtract those numbers depending on the signs that adds up always to 1.

So any X here is counted once when I consider this total number of counts therefore this total count is exactly this and this is the general principle of inclusion and exclusion I will stop here today and in the next lecture we will work out certain problems on the principle of inclusion and exclusion in the general form thank you.

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