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NATIONAL PROGRAMME ON TECHNOLOGY
ENHANCED LEARNING
(NPTEL)

Discrete Mathematics

Module-06
Relations

Lecture-06
Closure of a relation (2)

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In this lecture we will be discussing on the transitive closure of a relation.

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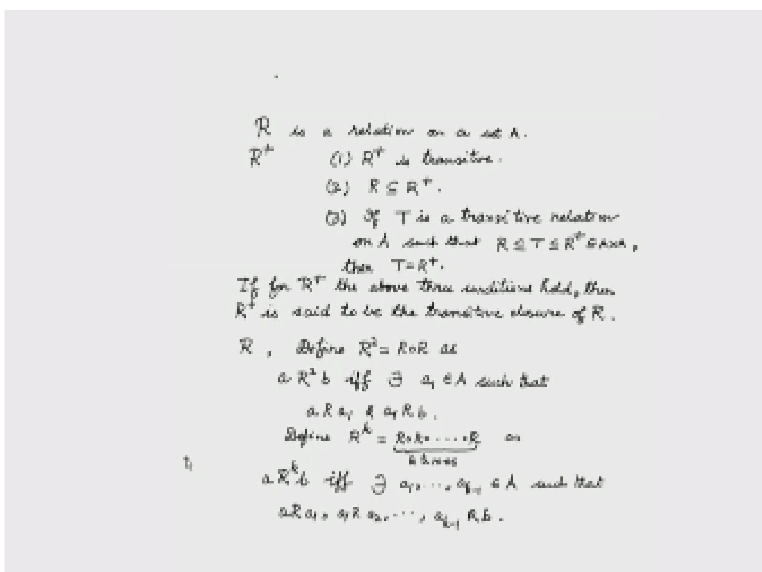

Suppose R is a relation on a on a set a then we call the transitive closure of our another relation which we denote by r + such that r is transitive our contents sorry r + is transitive r is contained in r + + r + is the smallest transitive relation containing R. So there are three points to be remembered r + must be transitive second point to remember is R is a subset of r plus and the third point to remember is that if T is a transitive relation on a such that our is a subset of T which in turn is a subset of r + then t = R + if for R +the above three conditions hold then r + is said to be the transitive closure of R.

Now our problem here is to find out a way of computing r+ from r we start by checking the powers of our that is are composed by itself, so we consider our and then we define R ^{2} which is also written as our composition are as a r ^{2} B if and only if there exists a 1 belonging to a such that a related to a one and A 1 related to b, in this way we can extend the this idea to R ^{K} so we define r ^{k} which is essentially our composition and so on our composition and composition are and this whole thing is k times alright as a R^{kb} if and only if there exists a sequence of elements of a A1 up to AK + one all inside a such that a R a 1 a 1 or a 2 so on up to a k + 1 are a here it is b.

So the last element here is AK + 1 Rb all right, now the result that we are going to prove here is our plus that is a transitive closure.

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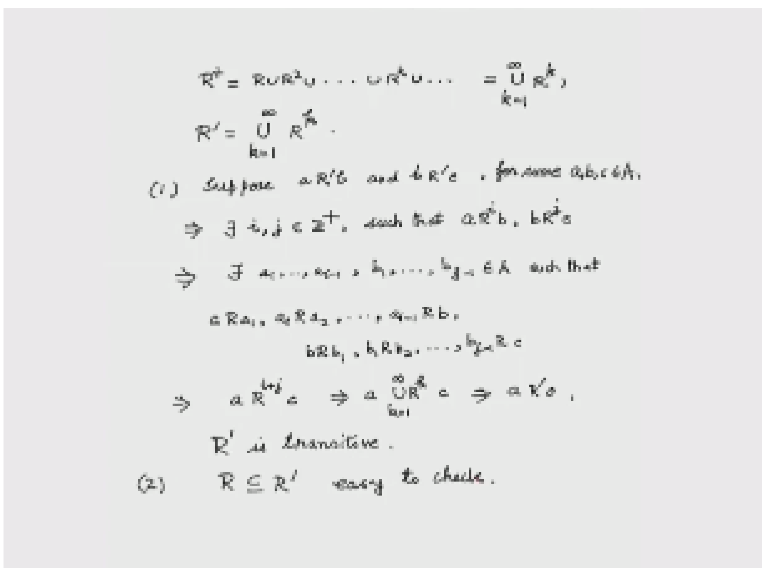

R + that is a transitive closure of R is same as R ∪ r ^{2} ∪ and so on up to RK, but we do not stop here we keep on going, so I just keep on taking powers of our and add in the ∪ and the totality that we will we get is our plus that is what we claim here so I can in a compact notation write this is equal to k = 1 to infinity R ^{K}. Now the question is that where is the proof and that is exactly that we are going to do now we will write for the time being R' as the ∪ k equal to 1 to infinity R^k and we will prove that R' is indeed the transitive closure of R.

So to do that first of all we have to prove that R' is a transitive relation 1 suppose a R' B & B R' C for some A B C belonging to capital A this means that there exists I, J belonging to the set of positive integers such that a R^i and b R^j this is because R' is ∪ of our all are raised to the power case and therefore if a is R' B then of course there is some element I for which a is R^{ib} and similarly for be B and C.

Now by definition of the power of relations what we have here is that there exists a one up to a i + 1 and B 1 up to be j + 1 all belonging to a such that all right we have a chain starting from a related to a one even related to a two and we proceed in this way to a i - 1 related to b but what happens here that the chain does not stop here we can pick up from B which is related to B 1 and B 1 related to be 2 and so on and ultimately we come to bj + 1 related to C, and therefore if we combine this whole chain then we will get a related to R^{i+j} c but this means that a ∪ k equal to 1 to infinity R^k C which in turn means that a r - c thus at least we have proved that R' is transitive.

The second point that we have to prove is somewhat easy because we have to prove that R is a subset of R' and that is true because after all R' is ∪ of our and other powers of our therefore it is easy to check that R is a subset of R'. So I will write here that it is easy to check.

Next we move on to proof probably the most difficult part of the proof that is R' is the smallest transitive relation containing R.

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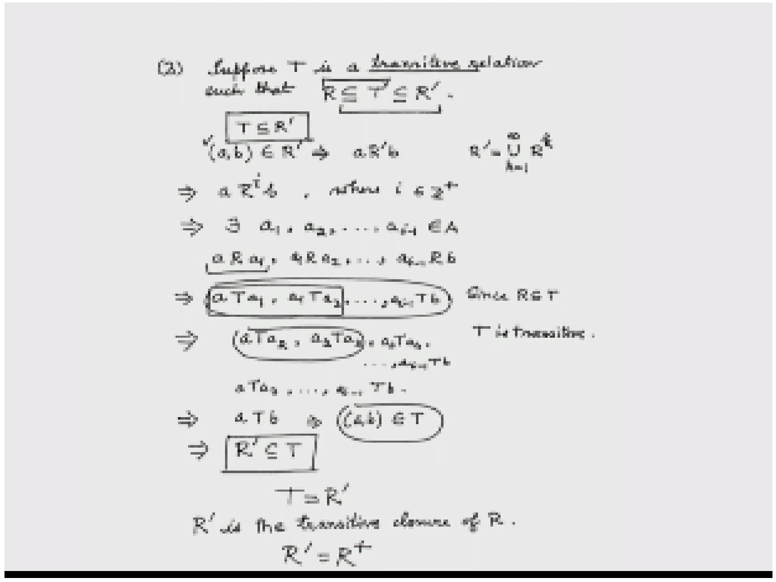

We write it as the 0.3 now let us think that how to prove this fact, so I would like to prove that R' is the smallest transitive relation containing art that means I have to prove that there is no relation containing r which is transitive and properly contained in R'. So let us suppose that we have a relation which is which let us denote by T and which is sandwiched in between our and up R'.

So here suppose T is a transitive relation such that R is a subset of T which in turn is a subset of R' what we will prove is that if such a thing happens then t is forced to be equal to R' and that proves that there can be no proper subset of R' which contains our and transitive at the same time. Now to do this we have to prove that T = R' and that that means a set theoretical et is a subset of R' and R' is also a subset of T, now this part of the chain already tells us that T is a subset of r prime so there is nothing to prove.

So we have to prove the other way round that R' is a subset of T to do that we have to start with an element of R', now suppose we have an element of R' we denoted by A B well technically we can write A B belongs to R' which essentially means that a is R' be right and this means remembering that R' is nothing but K starting from 1 to infinity R^K all right, so since a B belongs to R' that therefore a B has to be in some R ^{I} for some positive I so therefore I can write a R^I B where I belongs to z^+ that is positive integer and since a is R^b we will have i + 1 elements from a such that we can build a chain of relations as we have seen before.

So this implies that there exists a $_1$ a $_2$ so on up to a $_i$ + 1 all belonging to a such that a R a_1 a_1 r a_2 we proceed like this and then at end we have a i + 1 R B, now in the next step we realize that by our assumption we have are a subset of T since our is a subset of T all right since our is a subset of T if we have two elements related through R then they are also related to T. So therefore we can just change it to a T a_1 a_1 ta_2 and we proceed like this and ultimately we will have a i + 1 t b now we know something more about T we know that t is transitive, so T is transitive YT is transitive because we have assumed it to be so.

So since t is transitive what we realize is that we can kind of collapse this chain here to just a t b, why? Because if you consider these two points in the chain since t is transitive we can write a t a 2 the next element will be a 2 t a3 and if well there is a next element to this then it will be a 3 t a for and ultimately we will arrive at a i + 1 t b but then I can collapse it again we can combine these two to write a t a3 and proceed and ultimately get AI + 1 T b this is all because t is transitive and therefore at the end we will end up with ATB, but what does it mean?

This means that the pair A, B is an element of T, and therefore we now check the whole chain of arguments we started with assuming that the ordered pair a B belongs to R' and we end up by deriving that the ordered pair a B belongs to t this means that R' is a subset of T now we check that we have already observed that T is a subset of R' and that is by the definition and we have derived that R' is a subset of T all by using the property that T contains R T is transitive and T is inside R'. Therefore we can write that T = R' and this proves that R' is the transitive closure of R.

Now as we have started by writing transitive closure of relation by our to the Power Plus therefore we can write in symbols that are prime is equal to R^+ so that is to say again the same thing that R' is the transitive closure of R.

Now we will see that this whole thing becomes a simpler if our underlying set is a finite set we need to simplify this a whole scenario because as we have seen that okay if you give me a relation R then r + is a ∪ of a infinite sequence of elements.

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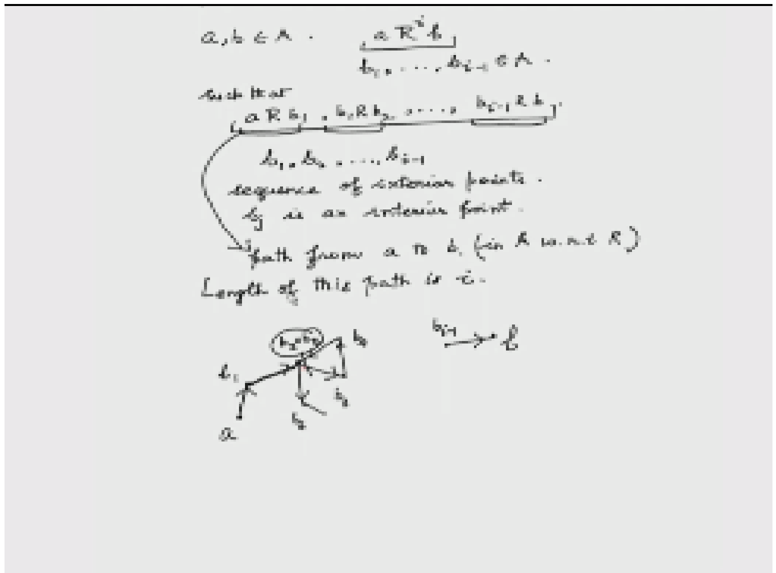

Namely r $\cup$ r^2 $\cup$ and so on some maybe r to the power k and so on which we are writing as $\cup$ k from 1 to infinity AR ^{k} well okay theoretically we have proved this, but it is it is not necessarily true that we will be able to complete the computation in general because we have a we have to compute $\cup$ of infinitely many elements therefore we need something simpler and we would like to have something simpler for at least finite cases and indeed we have a much simpler result when the set a contains only n element which we will write as a_1 a_2 up to a n okay.

So by putting a within two vertical lines we denote the number of elements of a and in this case it is n, what we will show first is that suppose two elements in a let us denote them by A and B are related by some r to the power I of B then we will as we have seen before have a chain of relations, so to say connecting A to B. So we will have elements like a $_1$ up to a I will rather change the definite change the notation over here because I am writing the elements of a as a $_1$ a $_2$ and all these things.

So what I will be doing instead is that I will say that suppose they are we when we have got a R^{ib} then we have some b $_1$… Bi – 1 belonging to A such that a Rb1 b1 Rb2 up to bi - 1 r be here we have to remember few things that this bi is has nothing to do with the ordering ai and I may be much larger than n, so and another thing that we have to not over here that I have not told that be eyes are distinct they may repeat. So I have essentially a sequence of elements b_1 b_2 up to b $_i$ + 1 where there may be repetitions and we may call it the sequence of internal the points or the sequence of interior points.

So I will be calling them sequence of interior points alright, now a an element from the sequence, let us say bj will be coy will be called interior point, and this whole chain starting from a and one after another a sequence of alternatively are and some bi this whole chain is called a path from A to B path from A to B in we can call it in a with respect to the relation R, so I can write it I will just say it is a path from A to B if we assume that we know a and we know are now there is a there is an important parameter associated to this path which is called the length of this path is simply I, why I? Because we see that this one we have got one then two and like this.

So we will have I number of places where we are using the relation if we have i - 1 interior points please note again that this interior points need not be distinct if we think in terms of digraph this is very intuitive what we have here essentially a .a and a .B in a or in the set of vertices when we are looking at this whole setup as digraphs then we have a and B and then we have some let us say be one and a relation a related to b1 means that we have a directed path then we again have something else be too then we have B_3 then we have let us say be four but this is where what I am coming to that this these interior points need not be distinct.

So from before we may go back again to b2 but then this b_2 is also b 5 and from be five we will go to be six and similarly we will proceed till we get to be from be i + 1 and the number of links that we have used is essentially the length of the path. Now what we will notice here that if along the path a vertex or an element or a point of a is repeated like in this portion b2 to be five we can essentially cut this loop out and in the process reduce the length of the path we can do this more systematically like here.

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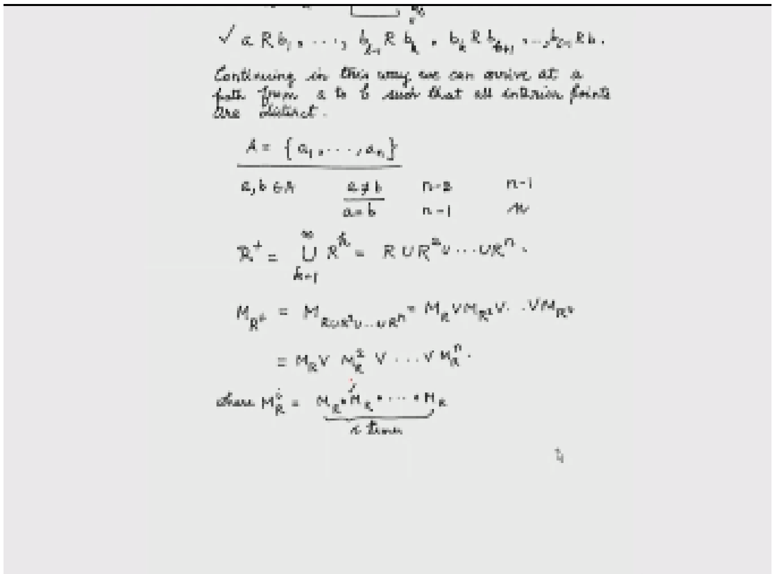

Please see that I am first writing the path this is a and this is a rb1 then a r sorry let me remove this right and then we have b_1 or b_2. Let us suppose we come to some b L + 1 r BL and then BL are be l plus 1 again we proceed then at some point we get another element let us say call it b k + 1 RB k and then we have BK r b k + 1 and we proceed again till we reach the end of the path which is B i + 1 r and the last one is B, we have a path like this and suppose b l = B K L is strictly less than k then as we have seen in the diagram.

But in this case more formally we can write a path from A to B as a starting from b1 and so on till we reach be L + 1 R and here instead of BL I can just write BK the reason is that be l and BK are same and then continue in the same sequence to get BK b k + 1 and so on up to be i + 1 r b right we get this. Now of course this is a path from A to B of length I we have obtained another path from A to B of length strictly less than I because L < K and well what we have done is that we have cut out to two equals in interior points just merge them and then gone on with our path.

Now we can keep on doing this process and at the end we will have a path which does not have any repeated interior points that means a path such that all interior points are distinct, so continuing in this way we can arrive at a path from A to B such that all interior points r distinct. Now let us go back to the set on which we are considering the relation this set has got only n elements a1 up to a n and so if we take any two points in a if they are connected by a path we can we know that we have already seen that we can and we can always do that connect a and B through a path where interior points are distinct but these interior points are going to come from the set itself and nowhere else.

Therefore if a and B are different and both are in a the number of possible interior points that we can get is n – 2, so any path can be reduced to a path containing n - 2 distinct elements of a if a and B are not equal therefore the length of the path will be n = 1 suppose a and B are equal then we are left with n - 1 elements of a and therefore at most we can have a path with n - 1 distinct interior points in fact it will be a loop starting from a point of a and going through the points of capital a and go back to the original point.

So the length of this path will be n, so in this case if we have two elements if they are at all related by some r^{I} they are related by some R^{j} where J is strictly $\leq$ n and $\geq$1 therefore R + which is ∪ k = 1 to infinity are raised to the power K is simply the ∪ of a finite set of relations this, now we can do computation with this suppose we have to find out the matrix corresponding to the relation R +.

Then it will be the matrix corresponding to the relation R ∪ r^{2} ∪ and so on up to R^{n} which in turn is the matrix M_r or the matrix M_r^{2} and so on up to the matrix M_r^{n} which in turn is the matrix M_r or the matrix M_r^{2} the matrix M_r^{n} where this M_r^{i} will mean $M_{r,}$ M_r and so on up to M_r i times where this particular operation is the operation of binary matrices that we have discussed before which corresponds to the composition of the relations this we have covered in previous lecture. So we have this situation, now we can start checking an example yeah.

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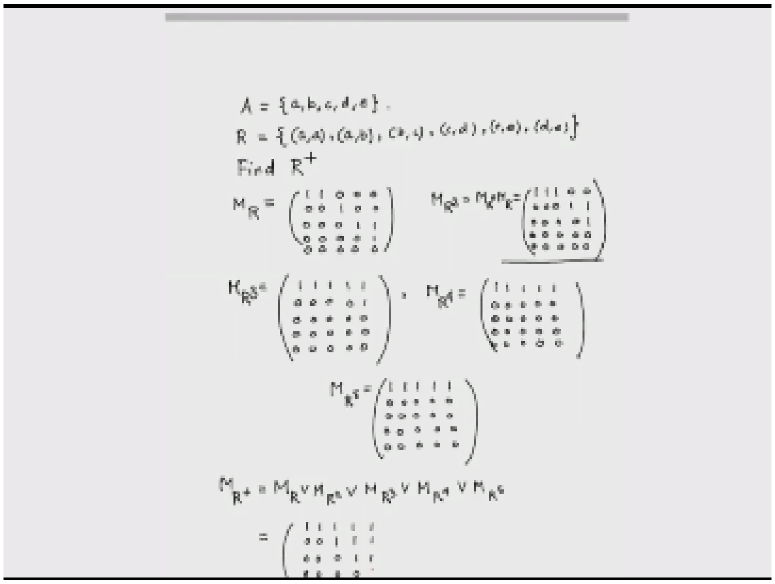

So let us take a set and relation, so now my set a is a b c d and e and the relation r is given by (A, A) (A, B) (B, C) (C,D) (C, E) (D, E) our problem is to find r + in the first step we construct M_r now if we check carefully we will see that M_r is 1 1 0 0 0 0 0 1 0 0 0 0 0 1 1 0 0 0 0 1 and the last row is all 0 like this, this we have to find out by finding the matrix corresponding to our that we have discussed in a previous lecture.

Now we keep on considering the powers of this relation where $M_r{}^2$ is nothing but M_r x M_r with a special rule that we have discussed before and that will give us 1 1 1 0 0 0 0 0 1 1 0 0 0 0 1 and 0 0 0 0 0 along with 0 0 0 0 0, so this so I will request you to check all the calculations and I will tell you one way of doing this when we are trying to compute M_r square which is M_r product M_r what we can do is that we can just take matrix multiplication as such.

And then at each entry we have to check that whether the entry is 0 or nonzero if the entry is 0 keep it as 0 if the entry is nonzero change it to one and then you will get a matrix like this and if you try the same rule with $M_r{}^3$ we will get 1 1 1 1 1 0 0 0 0 1 and rest of the rows all zeros and in exactly the same way multiplying M_r three four times by using the same product we will get M_r^4 which is 11 111 and rest of the rows are zeros alright.

And then lastly we have $M_R{}^5$ which is again same as $M_R{}^4$ alright and then our job will be to take the ∪ of all these relations and in the matrix form it will be M_R + = M_r or $M_r{}^2$ or M_r^3 or M R^4 or M R^5 and this if we check carefully peas 1 1 1 1 1 0 0 1 1 1 then 0 0 0 1 1 0 0 0 0 1 and the last row is 0 0 up to0 all zeros.

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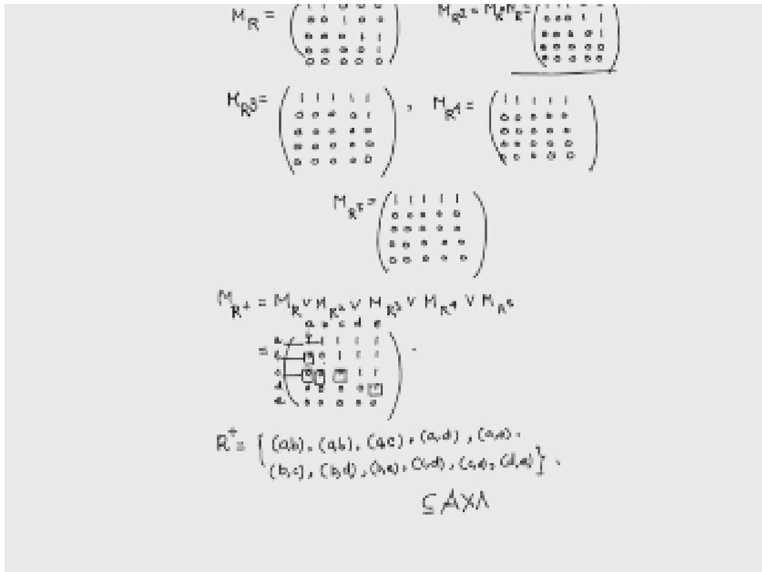

So this is the matrix corresponding to M_r + then we have to construct the relation corresponding to this matrix for that again we will use whatever we have studied before we remember that in all these cases we have not tampered with the ordering of A, therefore these columns are labeled by a b c d e and the rows are labeled by a b c d e therefore we see that the entry corresponding to a a gives one.

So it is in the relation so a is in the relation a B is also in the relation AC is also in the relation ad is in the relation a e is in the relation we come to the second row where we have B we see that b a the corresponding entry is 0 so it is not in the relation BB is not in relation but bc is in the relation so we will write B ,C then we write B , d we write B, E then we come to the third row C a is not in the relation CB is not in the relation CC is not in the relation but it starts from CD CD and CR in the relation so we hide CD CE and lastly we see that DE is in the relation and there is no other element in the relation, so ultimately we have our plus as a set and set it is a subset of a Cartesian product A.

Now the problem with this technique is that we have to do lot of work as we have seen that each time we have to keep on multiplying M_r with whatever we have obtained before what we are doing is probably little less complicated than matrix multiplication but it is ultimately the same in terms of the number of number of elements that we have to compare in the worst possible case, therefore we would like to know whether we can do it in a faster way and indeed there is an algorithm called what shells algorithm which does it in a much more faster and convenient way this algorithm we will study in the next lecture for today this is the end thank you.

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