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NATIONAL PROGRAMME ON TECHNOLOGY
ENHANCED LEARNING
(NPTEL)

Discrete Mathematics

Module-06
Relations
Lecture-05
Closure of a relation (1)

With

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In today's lecture we will be discussing on closures of a relation now suppose I have a set A.

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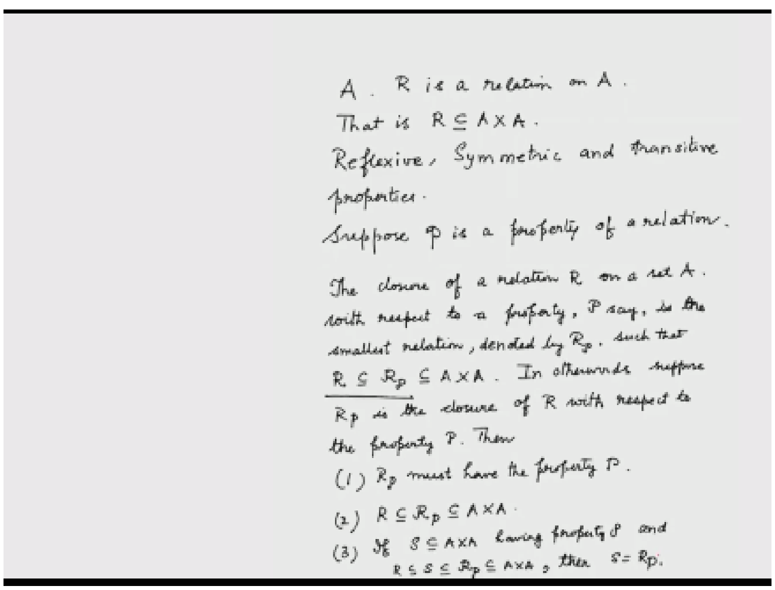

And a relation are on the set A that is R is a subset of A x A now we have already seen that relations satisfy certain properties like reflexivity, symmetry, transitivity, anti symmetry and so on so, we will be particularly discussing on three properties reflexive symmetric and transitive so reflexive symmetric and transitive properties however to start with we will just take a general property let us say P that is let us denote the property by P suppose P is a property of a relation in particular P can be as i said reflexive symmetric or transitive property.

Now a generalization are may or may not have P so that is our starting point, so suppose we are considering a relation we which does not have a problem which does not have the property P then we may ask that I would like to extend the relation to a relation and this extension should be minimal such that the extended relation has the property P, for example if the property is reflexivity or let us sell reflexive property and suppose R is my relation which is not reflexive I would like to extend it extend R to a relation let us say R sub B such that that R sub P is reflexive and it is a smallest reflexive relation containing are now in general let us let us give a definition with respect to the general property RP and a general relation R.

The closure of a relation R on a set A the closure of a relation R on A set A with respect to a property P say is the smallest relation denoted by R_P such that r is a subset of R_P which in turn of course is a subset of a cross a, now here there are certain issues that needs clarification first of all the point which is more or less clear that what we mean by a relation containing another relation because after all we have defined relation to be a subset of a cross a on which it is defined and then if I say that the relation R_P is contained in our that means R is a subset of RP as a set.

But a more A more critical point is that this R_P has to be minimal with respect to the property the question is what do we mean by that we are coming to it shortly, but let us write again the same thing more explicitly so suppose R_P is the closure of R with respect to P then what are the properties that are P must have in other words suppose R_P is the closure of R with respect to the property p then one the first thing is that R_P must have the property P then second our must be a subset of R_P which in turn is a subset of a cross a and third.

If we consider a relation s with property p containing our then and contained in R_P then s must be equal to R P if s is a subset of a cross a having property P and R is a subset of S which in turn is a subset of RP which in turn is a subset of A x A of course then S must be equal to R_P and the third point specifically say what we mean by the minimal extension it means that we have a relation R here and then we have A x A and R_P is somewhere in between such that R is a subset of RP + RP is a subset of A x A.

It is to be remembered that R_P satisfies the property P now suppose we have some S which also satisfies the property P and which is a superset of R as I have drawn here and subset of R_P then this will force S to be equal to R_P, so that means that we have our the relation R over here and R_P on the top of it and which satisfies P, but there is no and there is no relation in between R and R P satisfying P and properly contained in R_P so R_P in that sense is the minimal extension of our having property P now we will start looking at closures of specific relations that is closures of relations with specific properties the first property and probably the easiest property that we have in hand is reflexivity.

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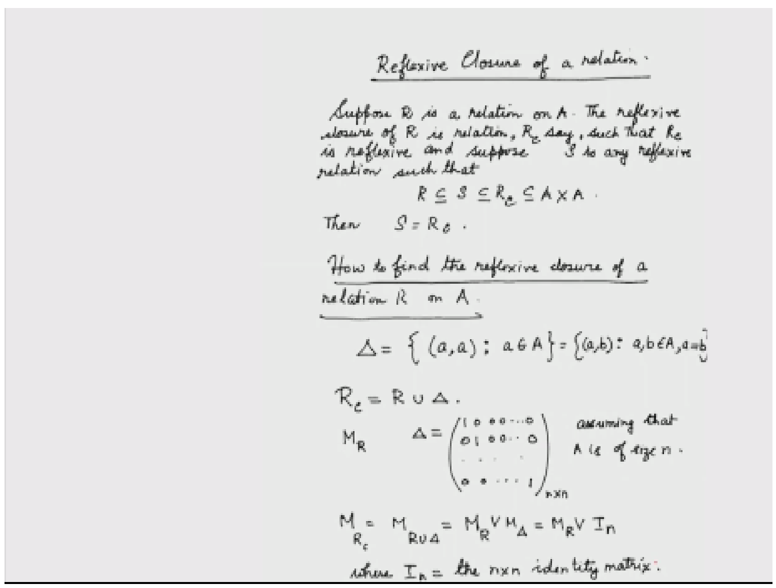

So we consider now reflexive closure of a relation suppose R is a relation on A the reflexive closure of R is a relation R_C say such that R_C is reflexive and suppose S is any reflexive relation such that R is a subset of S which in turn is a subset of R_C which in turn is a subset of A x E_A then S equal to R_C thus we see that this is exact translation of the general case that we have discussed just some time back considering a general property, now the property is no more general this is the reflexivity property and we have said what we mean by reflexive closure of a relation.

Now we ask the question how to find how to find the reflexive closure of relation the reflexive closure of a relation are on A, so let us be very specific that we have said suppose A is the relay is the set on which relation R is defined and we would like to know the reflexive closure and of course if R is reflexive the reflex reflexive closure of R is R itself, but we do not know that therefore we define a relation which we denote by capital Δ and which is essentially the Equality relation.

So capital Δ consists of all points a,a such that a belongs to a or in other words i can write it contains all pairs a, b such that a,b belongs to a and a equal to B alright, so this is the Equality relation now what we do is that we simply augment if at all necessary this a, a type of elements to our and we call that that is R_C that is a reflexive closure of a relay of the reflexive closure of R and i do not think i need to explain anything more because it is very straightforward R_C is equal to our union capital Δ the Equality relation.

Now suppose we are given are in terms of a matrix that is instead of R we are given the matrix corresponding to r that is M_R and then it is very direct that the Equality relation is nothing, but the identity matrix all right so only the diagonal elements will be one and rest will be zeros so we are assuming here assuming that A is of size n all right so Δ is this thing the identity matrix and M_R of course is a is a matrix the matrix corresponding to R and if you want to know the matrix corresponding to R_C.

So that is M_{RC} we can just write this is equal to M_R U Δ because we have already told that RC is R U Δ and we know that from our previous lecture that this is equal to M_R or MΔ and MΔ essentially is the identity matrix therefore this is M_R or In where In is equal to the n/n identity matrix alright, so this is easy but well this somehow captures a basic idea of closure that we just add new, new elements to the relation original relation R to maximally extend it to its closure with respect to certain property.

And we have also seen here that if we write in terms of matrices sometimes it is possible to write the whole computation very neatly because we just know, now that we have to just take the or of the identity matrix to the original matrix of the relation and the matrix that we get is a matrix corresponding to the closure, now let us look at an example.

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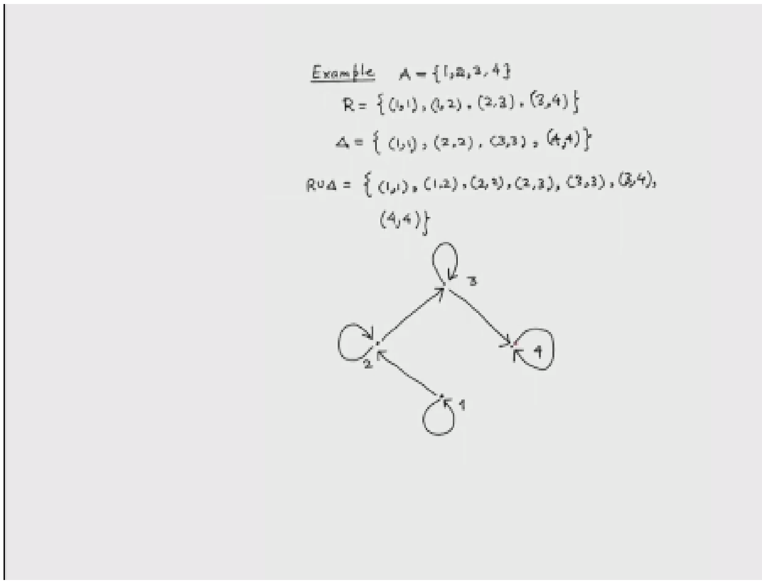

Right now let us consider A to be the set 1 2 3 and 4 and we consider a relation R equal to say 1,1 then 1,2 2,2 not me not linking this one is the link these are I am doing it, but now it will so I have to press here it was not happening actually when I press there but it was not happening but anyway alright, so this is 3,4 okay now suppose we want to find out the reflexive closure of R well then we have to write down the Equality relation which is very straightforward because it is 1,1 2,2 then 3,3 and then 4,4 4, 4 all right now we take RUΔ this gives us 1,1 and then 1,2 and then 2,2 then 2,3 then 3,3 3,4.

And lastly we have 4,4 of course this is the reflexive closure if we now look at the graph corresponding to this relation then we will see that we have another way of looking into the reflexive closure, so let us look at the graph corresponding to R defined on A so we have got 4 vertices we label them by 1 2 3 and 4 and here we notice that 1,1 is there 1,1 means there is a self loop from one to one and then we have a B we have an edge from one to two and then we have an edge from 2 to 3.

And then we have 3 to 4 this is the original relation given by the given by our and finding out it is reflexive closure is just putting self loops at each vertex that makes each of the vertex related to itself and where there is already a self loop, that is in this case one we do not have to do anything so this is the reflexive closure of the relation and a relation R and the graph corresponding to it.

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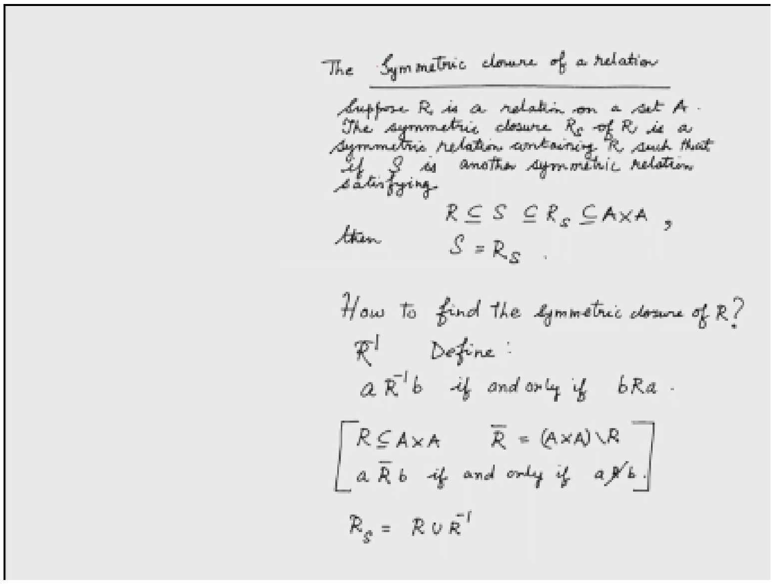

Alright again we start with R suppose R is a relation on a set A the symmetric closure R_S of R is a symmetric relation containing R is a symmetric relation containing are such that if S is another symmetric relation satisfying our subset of S subset of R_s subset of A x A then S = R_S, now we come to the question of how to find the symmetric closure of R in order to do that we will first start by defining another relation corresponding to R which is called the inverse of our we denote it by our inverse and define as a R^{-1} b if and only if bRa now at this point we must not confuse our inverse with compliments of R.

Since R is a subset of A x A there is a set theoretic complement of our which we usually denote by R over line which is essentially A x A – R, now when we translate it in the language of relations this will mean that a R complement be if and only, if a is not related to be this is R compliments but we are not here defining compliment of R we are defining our inverse where we say that A is related to B if B a by our inverse if B is related to A/R, now what we claim over here is that the symmetric closure of a relation R that that we are denoting at R sub A is nothing but R U R inverse we have to see why it is true.

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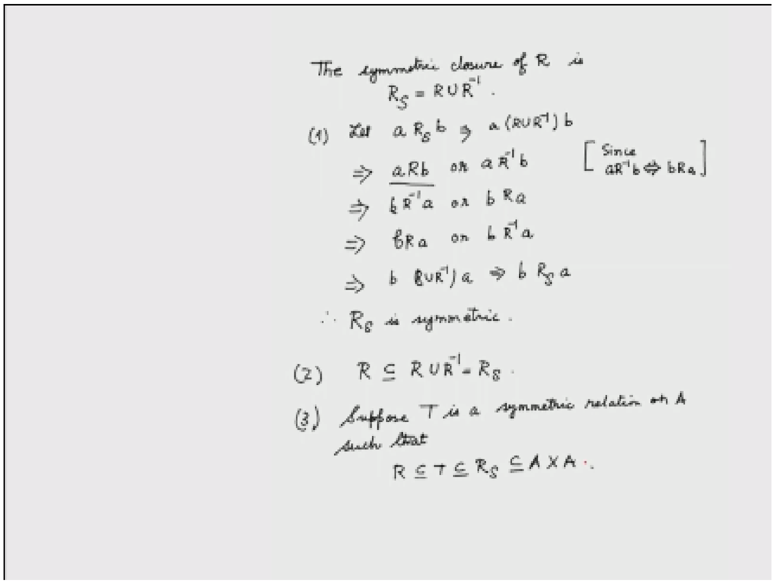

So first we have to show that R sub S is symmetric for that let A R sub S be now this implies that A R U R → B which implies that A are B A R → B which in turn implies that be R → A well that is the definition of R → we have said that a R B if and only if a RB R → A, so since here we have got a R_B here therefore i can write B R → A since A R → B, if and only if B are a now R be our a therefore we see that this is B area or B R → A, but that means that B is R U R which in turn means be R_S.

A because R_S is RUR inverse therefore we see that A R sub S B → B R sub S A but that is the property that R_S has to have if it is symmetric and, so R_S is symmetric the next property that we have to show is that R is a subset of Rs, but that is extremely straightforward over here because RS U of R & R inverse and therefore we can write that R is a subset of R U R inverse which is equal to R_S, now we come to the third property which is the minimality so now let us suppose that we have a relation let us call it P which is symmetric and which is sandwiched between R and T.

So suppose T is a symmetric relation on a such that R is a subset of T which in turn is a subset of R_S which of course is a subset of A x A, now let us start let us start from let us let us try to prove that t is equal to R_S we already know that d is a subset of R_S clearly t is a subset of R_S we have to prove the other way round.

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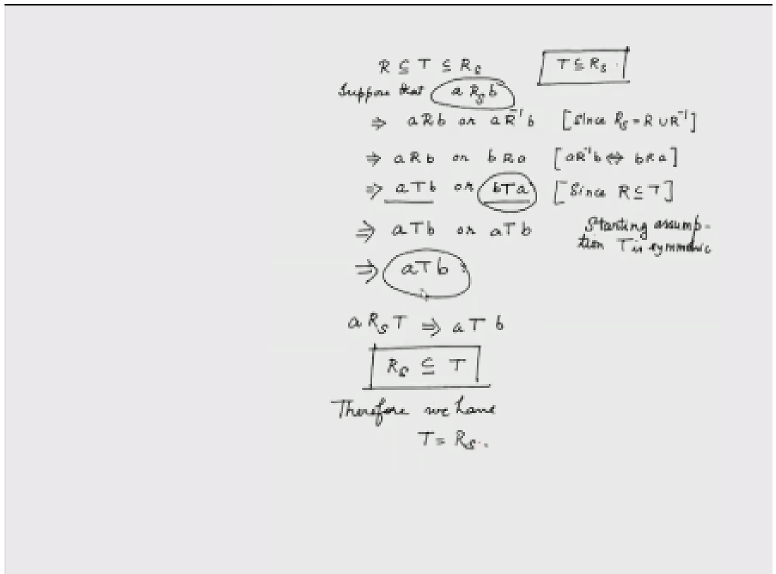

So we start it in a fresh page right, so let us write again so we have are a subset of T which in turn is a subset of Rs of course T is a subset of Rs it is already known now i will start from R_S side, so suppose that A is related to B by R_S now this implies that A is related to B by R or A is related to B by R → this is because R_S is a subset of our union or inverse now this implies that a is related to B that is all right or B is related to A okay, because that is by the definition of R → we know that A R → B both the → B.

Area so therefore we can write that a SB or b S a why since R is a subset of T, so just let me correct this is not S but this is T so instead of S there I must write P this is T because since R is a subset of TA RB means A TB be are a means B T a therefore we have come to a scenario where A is related to B or B is related to A so therefore through R therefore since R is a subset of T I can say that a is related to B by T because it is related to be by R and R is a subset of T and or B is related to a/T.

Now we started with the assumption that T is symmetric starting assumption is that T is symmetric okay, so therefore a TB or Tb here this thing P_{TA} is aTB because T is symmetric and therefore we have the same thing we have a Tb or Tb therefore this → that a Tb, now this means if we now notice from the beginning that is this to the end we have proved that a R_S sub st → Tb that R_S is a subset of T, but already we knew that T is a subset of R_S now we have got R_S is a subset of T therefore we have T is equal to RS.

So this is what we wanted to show to prove the minimality of R_S and this is what we have shown now we will consider the matrix corresponding to R_S, so in general we will consider matrix corresponding to a so what we want to consider now is how to call how to construct the matrix corresponding to R_S, so first we have to know how to construct the matrix corresponding to R → now if you see that a matrix M_R.

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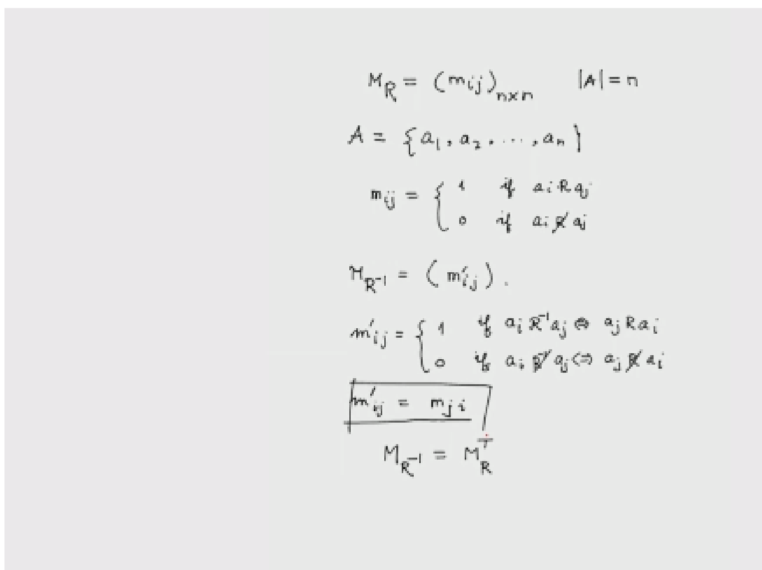

Which corresponds to the relation are defined on a is given by m_{ij} n x n where the number of elements in A is n on which are is defined and further to specify the matrix we need to write the elements of a in F in some order which we fix afterwards, so suppose when we write in that order the elements of a is a one up to a n then m_{ij} = 1 if a_I is related to a_j is 0 if a I is not related to a_j now suppose we consider the matrix corresponding to M_R inverse, now suppose we denote this matrix x m_{ij} bar sorry m_{ij}' now m_{ij} ' is defined in this way mij ' = 1 if ai is our inverse aj that implies aj R_{ai} and 0.

If AI is not R → aj which implies both ways if aj is not in not related to ai that means that M prime sub ij = Mji because when aj is related to ai then M sub ji is equal to 1 well then M prime sub ij is one therefore this relation holds and when j is aj is not related to a I then m_{ji} 0 and same as M prime sub ij therefore this is same thus it is clear from this that M sub R → that is the matrix corresponding to R inverse is equal to the ^{T} of the matrix corresponding to R because in the new matrix the ij is switched that means rows and columns are switched therefore we have this and now since we know that R_S is equal to R U R.

Inverse the matrix corresponding to RS is the matrix corresponding to R U R inverse which is equal to the matrix corresponding to R or the matrix corresponding to R^{-1} which in turn is equal to the matrix corresponding to R or the matrix corresponding to R^T this gives a particularly straightforward method to construct the matrix corresponding to the symmetric closure of any relation and then of course from that we can write the relation or the digraph corresponding to the relation very quickly.

Next we come to the question of finding the transitive closure of a relation the transitive closure of a relation is a relation which is transitive and which it means no transitive relation between itself and the relation under consideration.

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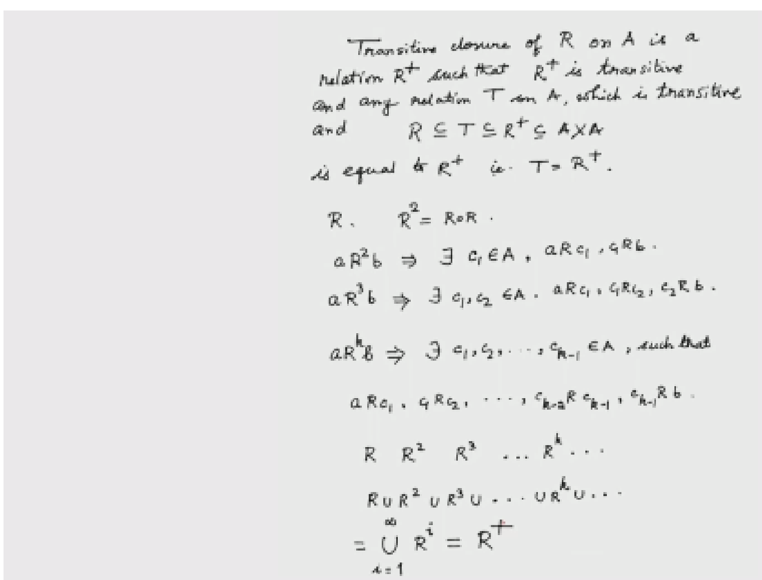

So transitive closure of R on A is a relation we usually define transitive closure as R superscript + we I will read it as R + is a relation R + such that R + is transitive and any relation T on A which is transitive and R subset t subset R+ of course all subset of a Cartesian product a is equal to R + that is T is equal to R + so again we have the problem of finding out the transitive closure of a relation to do that we have to recall few things that we have studied in previous lectures, so if we have a relation R.

Then we can take we can compose this relation R2 with itself several times for example by R^2 we mean the relation R composition are now, when we say that a that an element A is R^2 B this means that there exists an intermediate element C_1 let us say in the set a on which R is defined such that A R C_1 and C_1 R be now suppose we raise R^3then AR QB will mean that there are elements c1 and c2 belonging to a such that A are c1 c1 RC2 and c2 are be if we go forward like this then.

We can define the general case that is let us say R^K B all right this means that there exists c1 c2 so on up to c k - 1 belonging to a such that A R c_1, c1 Rc2, so R c k – 2 rc k - 1 then see k - 1 R be so we see that we can have sequence of powers of are defined in this way that is our r^2 r^3 so on R^K and so on, we can construct a relation by taking the union of all these relations, so we consider the relation that we get by taking R U R R^2 U R^3 union moving in this way R^K U so on in a compact way.

We can write this as I = 1 to ∞ R^i and what can be proved is that this is same as R + that is the transitive closure of R we will stop here in today's lecture and we will continue discussions on closure of relations particularly closures particularly the closure of transitive relations in the next lecture thank you.

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