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NATIONAL PROGRAMME ON TECHNOLOGY
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(NPTEL)

Discrete Mathematics

Module-06
Relations
Lecture-04
Matrix of relation

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Today we will be discussing matrix of a relation, what we will see that given a relation from a set a to b or a relation on a set A, we can define a binary matrix corresponding to the relation and this matrix plays an important role in studying properties of relations and operating one relation by the by another relation and so on. So to begin with let us start.

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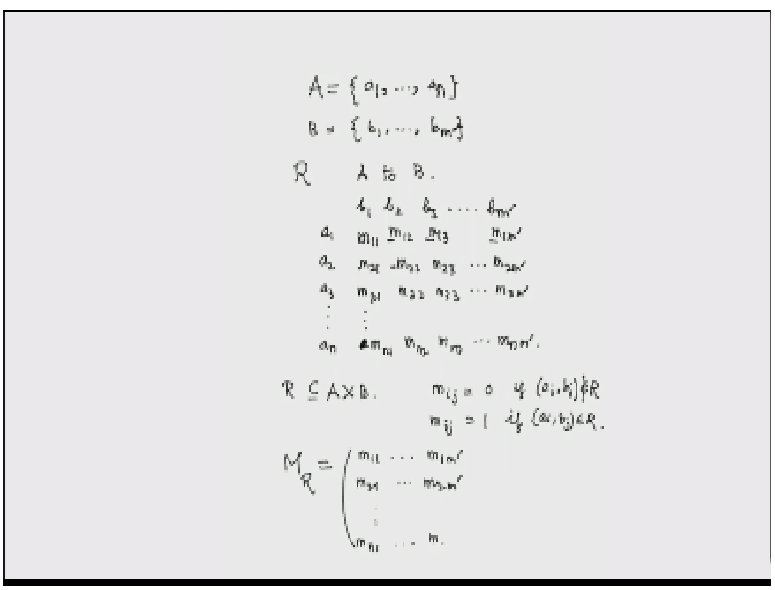

With a set A given by a $_1$ up to a $_n$ and a set B given by b $_1$ up to b$_M$, now suppose R is a relation from A to B and of course n and m are finite positive integers, what we can do is, that we can label the rows and columns of a matrix by the elements of A and B respectively. For example let us write a_1 a_2 a_3 one below another and b_1 b_2 b_3 and so on up to b_m and now we have positions of mattresses indexed by a_1 b_1 a_2 b_2 a_1 b_2 a_1 b_3 a_1 b_m and so on we can fill in these gaps by writing m_{11} m_{12} m_{13} & m_{1}m.

In the next row we write m_{21} m_{22} m_{23} and so on m_{2} m at this point let us just change the notation a little bit and understand that this m and this m are not same, so probably write this as m' so we have got b_1 up to b_m - this is 1 m - 2 m - and so on the next row we have m3 1 m 3 2 m 3 3, so on up to m 3 m ' and then a m n 1, m n2 m n 3 so on up to m in mn '. Now as we have already studied that relation is nothing but a subset of the Cartesian product of A and B.

Therefore what we will do is that if a pair a i_{bj} belongs to the relation we will put one in the corresponding m_{ij} and if it does not belong to the relation then we will put 0. So we will write m_{ij} = 0 if a_I b_J is not in R and we will m_{ij} = 1 if a_I b_J belongs to R and if by using this rule we fill in the fill in the matrix, that we have already written down the m_{IJ} matrix then we will find that it becomes a binary matrix.

So what we will write is that this matrix that we denote by M_R =m 1 1 up to m 1 m' - m 2 1 up to m and so on another in the last column it is m n 1 up to m n m '. Now of course here we will assume that we will know we know the ordering the ordering is fixed the ordering of the set A, in which we have labeled up by using which we have labeled the rows of the matrices rows of the matrix and the ordering of the set B by using which we have labeled the columns of the matrix M sub R.

It is now clear that the matrix M_R is a binary matrix, there is a special case of this which is used very often that is when A and B are same.

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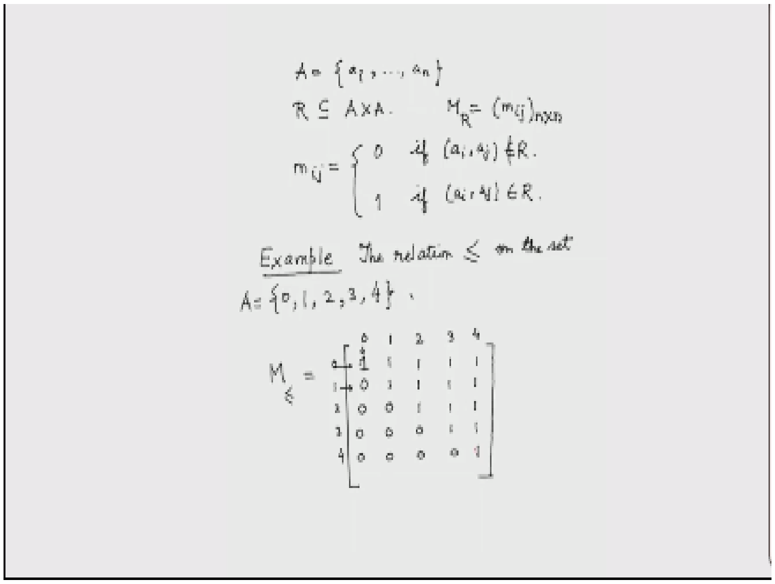

So in this case we have only one set A which is finite consisting of elements a 1 up to a n and we have a relation R well which is a subset of A x A, so this relation is a relation on A and then we write M_R which is the matrix corresponding to the relation R which is given by m_{ij} which is an n by n matrix defined by m_{ij} =0 if a_i a_j is not in R and 1 if a_i a_j is in R. We will now look at some examples related to this idea of a matrix of a relation.

So first of all we check a well-known example that is the relation ≤ to on the set 0 1 2 3 4, so in this case the set A is the set 0 1 2 3 4, we may even write that here. Now we are we are to construct the matrix corresponding to the relation given by ≤ to let us call that matrix M sub ≤ to which = now at this point we are we have to label the rows and columns, let us do that we label the columns by 0 1 2 3 4 and label the rows by 0 1 2 3 & 4.

Of course we have a matrix over here and there are entries that we have to fill in so let us start constructing the matrix, we take 0 & 0 from here and here and we ask the question is 0 ≤ to 0 the answer is yes, therefore we write 1, after that we asked the question is 0 ≤ to 1, the answer is yes therefore we write 1, then we ask the question is 0 ≤ to 2? The answer is yes and therefore we write 1 over here and like this we fill in once in all the entries of the first row because 0 is less than all the elements of the set 0 1 2 3 4.

Then we move to the second row and we start checking we ask the question whether 0 is I am sorry whether 1 is ≤ to 0, which is not true therefore we write 0 over here and then we write whether when then we ask whether 1 is ≤ to 1 and of course we know that 1 is ≤ to 1 therefore we get 1 over here 1 is ≤ to 2, so we get 2 over here 1 over here and similarly this. In the third row the first column entry is 0 because 2 is ≤ 0 2 ≤ 1, so 0 over here and then again 1.

In the similar way we can fill in all the entries of the matrix and we get a matrix as this one right, we will now look at another example where we are given a digraph and we are asked to find out well the corresponding matrix, well that means that essentially a digraph is connected to a relation a relation, is connected to a binary matrix.

Therefore in effect any digraph is related to a binary matrix and if somebody looks at graph theory by not looking at relations then of course there is a graph such related to binary matching from binary matrices and those matrices are called adjacent C matrices. So what we are looking at as matrices of relations are in a sense equivalent to adjacent C mattresses of digraphs and let us look at this example.

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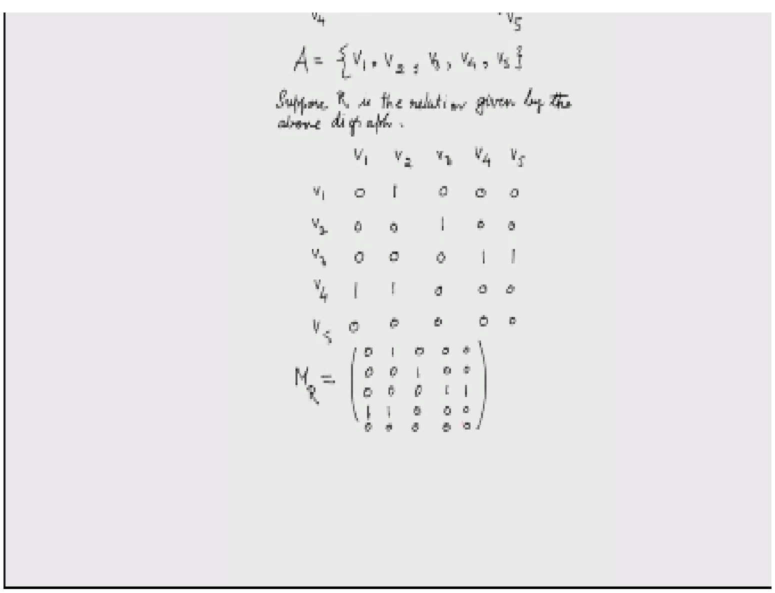

Now we have five vertices V_1 then V_2 then we have V_3 and then we have V_4 and we have V_5 and we have an adjoining V_1 to V_2 and V_2 to V_3 and then we have an adjoining V_T to V_5 and an adjoining v_4 to v_1 and v_4 to v_2 right and there is a adjoining v_3 to v_4 so we write we write a draw a line from V_T to v_4 and put an arrow head towards B v_4. So this is the graph that we have got, so the underlying set of vertices or the set on which the relation is defined is v_1 v_2 v_3 v_4 and v_5 and we would like to know the relation.

Let us suppose that the relation given by the digraph big denoted by R suppose R is the relation given by the above digraph alright and we would like to find out mr for that we again label the rows and columns of a matrix by the elements of A according to the order given here we could have basically chosen any order but we have to specify the order at some place, possibly at the beginning.

So let us now right V_1 up to V_5 and V_1 up to V_5 as the labels of the rows and then let us start checking here see V_1 is related to V_2, therefore I put a 1 here and V_1 is not related to anything else therefore I am free to put 0 in all the other places. Now let us look at V_2 V_2 is related to V_3 well I put a 1 there and otherwise V_2 is not related to anything else therefore I will put 0 as the other entries of the second row.

And then we come to V_3 that is a third row and in the third row V_3 is related to two elements v_4 and v_5 so V_3 is not related to V_1 so I put a 0 over here V_3 is not related to V_2 I put a 0 over here V_3 V_3 is well V_3 is not related to itself, so I put a 0 here but V_3 is related to V_4 therefore I will put a 1 over here and V_3 is letter to V_5, so I put a 1 over here again and then we go to V_4 where we find that V_4 is related to V_1 therefore I put a 1 over here and also it is related to $V_{2\,I}$ put over here and rest of the places I put 0s.

And V_5 is not related to any other element therefore I put all 0 in the fifth row thus we have the basic matrix corresponding the relation R and we can write if we choose to forget the ordering that is to say suppose we fixed the ordering once and this say that okay fine be John we do not write the ordering beside the rows and columns of the matrix all the time, therefore we can write M_R as 0 1 0 0 0 0 0 1 0 0 0 0 0 1 1 and 1 1 0 0 0 and at the end we have another row that is all 0 row that is corresponding to V_5.

So this is the matrix corresponding to R, now we move on to see what we can do with these matrices, sometimes these matrices are useful in computing the results of operations on relations for example we have seen before that given several relation on a set, we can take Union and intersection and composition of those relations and sometimes what happens is that, if we are able to construct the corresponding matrices then we can determine the Union intersection composition of several relations very quickly and possibly by using a computer.

Therefore let us see how to deal with unions, for that we have to just see the see some general definitions, so suppose R and s are two relations on a set A.

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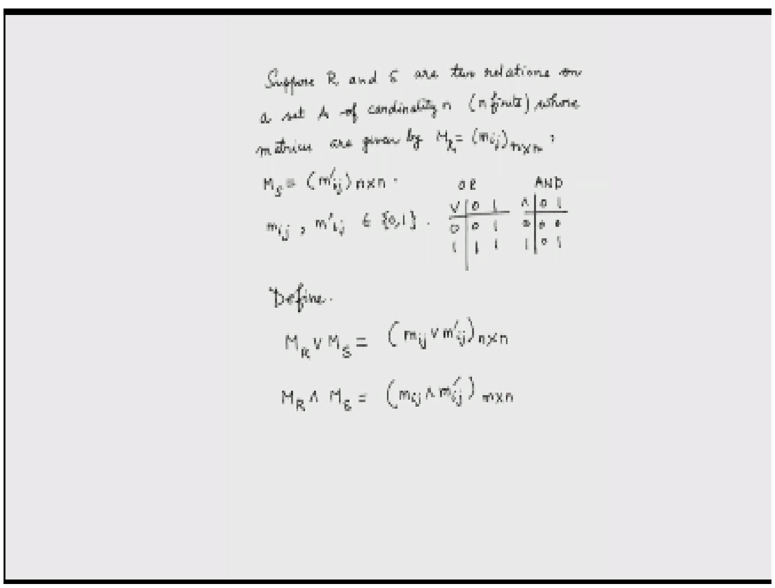

Suppose R and S are two relations on a set A of cardinality n and finite all right whose matrices are given by M $_R$ well which is I will write it as m_{ij} this is, so it is in this is n x n and we have a Ms which we denote by let us say m ' $_{IJ}$ n cross m, then we know that all these m $_{IJ}$ s and m' $_{IJ\,s}$ are elements of the set 0 1 and on the set 0 1 we have the usual operations or an and given by this operation is known as our and this operation is known as AND.

We use this elementary operation to define operations on the matrices of relations we will define Mr or Ms as m_{ij} or m ' $_{hi\,j}$ n x n that is we take these two matrices and then we take element wise or to get another matrix, which we call or of these two matrices similarly we will take these two matrices Mr and Ms and take element wise and to determine another matrix which we will call the AND of these two matrices.

Now the question is that what is the use of doing this we will shortly see that and then all of these two of the AND all of these two matrices correspond to Union and intersection of the relations, let us look at it by an example. So we check one example.

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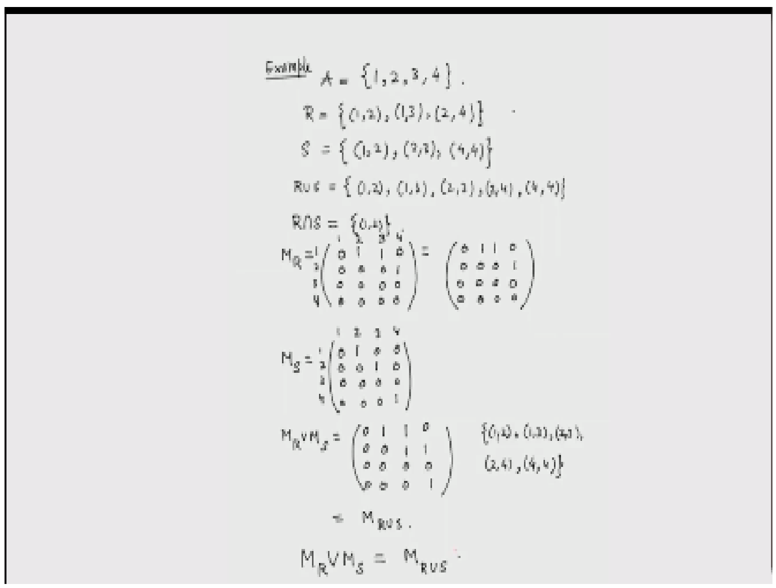

We consider a set A which is 1, 2, 3, 4 and then we will consider a relation on A given by 1 2, 1 3 & 2 4 we consider another relation S on A given by 1 2, 2 3 & 4 4 alright. Now let us look quickly at what is R union S, R union S is given by 1 2, 1 3, 2 3, 2 4 & 4 4 and our intersection S is the single turn given by 12 now let us see what happens if we consider the mattresses instead of these sets Mr = now let us label the rows 1 2 3 & 4 1 2 3 & 4.

So 1 is related to 2 therefore I put a 1 over here and 1 is related to 3, so I put a 1 over here so it is 0 and this is 0 & 2 is related to 4 and that is all and there is no pair inside are therefore I complete the matrix in this way alright, so I write it again so the matrix corresponding to M_R is 0 1 1 0 0 0 0 1 0 0 0 0 and 0 0 0 0. Now let us find out the matrix corresponding to M s again I have to label first 1234 and 1, 2, 3, 4 and if I write like this then in M s 1 is related to 2.

So I will put a 1 over here and 2 is related to 3 so I will put a 1 over here and then I have 4 is related to force I put a 1 over here rest of the entries are 0s, so I will get something like this and 3 is not related to anything therefore I will put all 0s over here all right. So I have got this one now I would like to take Mr or Ms for that process these two matrices now I check the first entry in Mr and Ms both are 0.

So 0 second entry both one, so 1/3rd and 31 0 10 is one therefore it is 1 and 0 0 is 0, so therefore it is 0 then in the second row we have 0 0 0 0 0 1. So therefore we have got 1 over here and then 1 0 so we have got one over here and similarly if we check that third row we will find that this is all 0s and then at the end in the 4th row please see that if we compare the elements of Mr and Ms then first three entries are 0s in both the matrices therefore the result is 0.

Only in the last one in ms the four for entry is 1 so therefore we will put a 1 over here and rest of the cases is 0. Now let us try to construct the relation corresponding to this matrix with see that in this relation will have this is 1 is related to 2 and then the entry 1 3 is 1 therefore 1 is related to 3, then in the second row 2 3 is related to 3 and in the again in the second row 2 4 is related to a 2 is rated to 4 and then at the end 4 is related to 4.

So we see there the relation corresponding to the matrix Mr or Ms is 1 2 1 3 2 3 2 4 & 4 4 which is exactly same as the relation R union S. So the matrix that we have obtained is nothing but the matrix corresponding to R Union S and this is in general true therefore we have one result which says that Mr or Ms gives me the matrix corresponding to m R union S. Now what about the intersection and let us look at the same pair of matrices.

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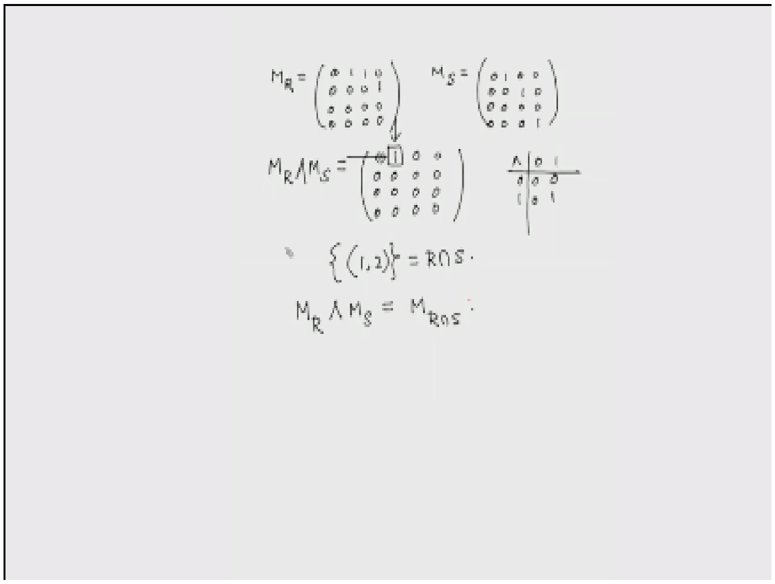

So we have again M R which = 0 1 1 0 0 0 0 1 then 0 0 0 0 and 0 0 0 0 and we write Ms which is 0 1 0 0 0 0 1 0 and then the third row is totally 0 and last row is all 0 except, at the last entry where we have 1 and now we check and of these two relations Mr and ms which gives me well I can compare in the same way I can see that the first entry of Mr and first entry of Ms is 0 so I put a 0 over here first in the first row second column entry is 1 in both the matrices, so I put one over here and the rest two are 0s.

Please note here that the in the previous example we were using element wise or so we 0 or 0 is 0 0 or 1 =1 or 0 =1 or 1 which =1 but in this case the element wise operation is and therefore it is basically making most of the things 0. So I have got 0 0 is 0 0 1 is 0 1 0 is 0 and 1 1 is 1 and that is a rule that I am using so we see that if we look at the second row and element wise I see that there is no common 1 has entries.

Therefore all the entries will be 0s and of course the third row is 0 in the first and also in the second Matrix therefore all 0 and the fourth row is 0 in the first matrix therefore it is 0 therefore we have a single turn and that is this singleton is the entry first row second column.So it is 1 2, therefore the matrix the relation corresponding to the matrix M R well here is a mistake this is and so let me write it down well here this is and so Mr and Ms =1/2 and which is our intersection s therefore we can write M R and M s =M R intersection S.

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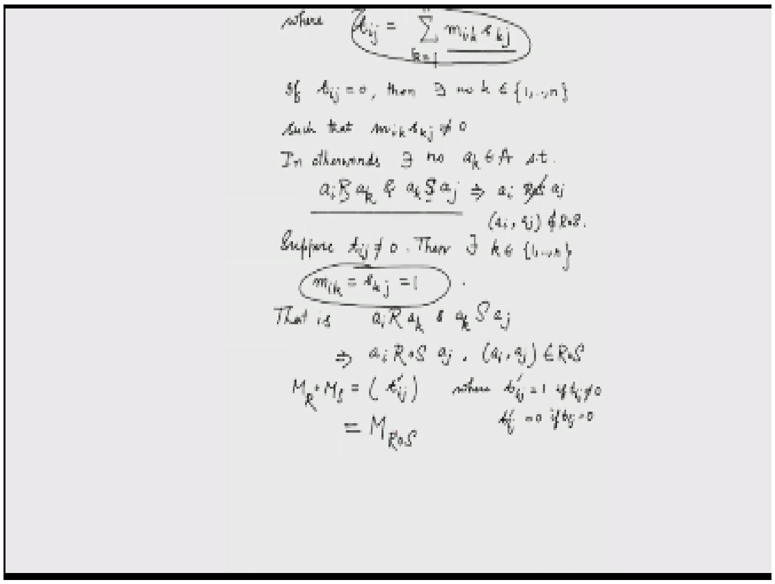

Next we will look at the scenario of what happens if we take product of two matrices corresponding to two relations the question is that, whether it connects to some basic operations on relations. So let us look let us look at what happens if we just take matrix product, so again we are considering a set A which consists of n elements and we have a matrix R matrix of are defined as M_R which is essentially m_{IJ} it is an n x n matrix we have another matrix Ms which is let us say is given by s_{IJ} which is n x n.

And now if I just take a product of these two matrices usual matrix product, then we will get M_R M s which =some t_{IJ} n x n where t_{IJ} =σ K going from 1 to n m_{ik} s_{KJ} of course this is usual matrix multiplication and well we have to remember that in this case the summation that we see in the right hand side of the equation containing t_{IJ} is a summation over integers. Now we ask a question that what happens if t_{IJ} is 0 for a pair $_{IJ}$.

So that means if t_{IJ} =0 then there exists no k belonging to 1 to up to n such that m_{ik} s_{kj} is not 0, so that means that in other words in other words, there exists no a_k' belonging to a right such that

a_i related to a $_K$ and a $_K$ related to a $_K$ related by s well a k related to a J we have to remember here that this relation is S and this relation is R, so there is no a $_K$ such that this holds but that means that this implies that a $_I$ is not related by R composition S to a_j or if we want to be a bit more specific then we will write a_j e ki a_j does not belong to R composition S.

Now suppose we have a situation where the some $t_{ij} \neq 0$ suppose $t_{ij} \neq 0$ then please note that this summation is sum of non negative integers because $m_{i\,k}$ & s_{kj} are non-negative integers, therefore if t_{ij} is not 0 there has to exist at least one K such that such that well $m_{i\,k} = s_{kj} = 1$, so let me write that then there exists k belonging to 1 up to n such that $m_{i\,k}$ & s_{kj} both are ones that means that is a_i related by the relation R to a $_K$.
And a $_K$ is related by the relation S to a $_J$ which implies a_i R composition R a_i or if we like to write a_i comma a_j the ordered pair a_i aj belongs to our composition S, thus this proves that Mr at M_R product M s has something to do with the matrix corresponding to a composition of R and S what we do is that we slightly modify the definition of product and we write M $_R$ then a small dot and then M s, we write this that this =let us say t just put a small dash over it i $_J$ where t_{IJ} ' is one if t $_{IJ}$ as we have defined above $\neq 0$ t $_{IJ}$ - is 0 if t $_{IJ}$ =0.

So we define a new matrix M $_R$ composition Ms in this way and it is clear from the discussion that we had that this is same as the matrix corresponding to M_R composition S. Let us look at an example of this composition rule. So let us again go back to the matrices that we were talking about here as you can see that this M $_R$ is this and M s is this matrix we will write a fresh these matrices.

And take the product M $_R$ product M s which =0 1 1 0 0 0 0 1 then 0 0 0 0 then 0 0 0 0 and here we have this new operation on matrices and then 0 1 0 0 0 0 1 0 0 0 0 0 0 00 1. Now we start checking the elements if we see that the first row first a first for first column is 0 second row first row second column also 0 but then if we go to the first row third column then we get a one and then 0.

And then if we check the other entries we will see that they are 0001 and rest are 0, so either one can compute this directly or one can take a product the usual product of these two binary matrices such that the elements other are calculated over the sum is calculated over integers and then, the then modify the product matrix by the rule that if the entry is 0 it is 0 that even if the entry is non 0 put 1 then you will get this matrix.

And this is the matrix corresponding to M_R composition S that can be directly checked we have done this example in one of our previous lectures. So therefore we have our composition s is given by 1 3 & 2 4. In this lecture we have defined relation as the matrix corresponding to a relation and we have basically seen that this matrix is nothing but adjacent C matrix of a digraph that is the digraph corresponding to the relation.

So we have a situation like this we have digraphs then we have matrices let us say binary matrices and we have relations and what we have done is that we have developed connections among all these different concepts. We have seen that essentially they are same and then we have seen that the basic operations and compositions of relations can be expressed very nicely as operations over binary matrices, this is what we do in this lecture so thank you.

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