INDIAN INSTITUTE OF TECHNOLOGY
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NATIONAL PROGRAMME ON TECHNOLOGY
ENHANCED LEARNING
(NPTEL)

Discrete Mathematics

Module-06
Relations
Lecture-03
Graph of relations

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We will be discussing graph of a relation. Now first let us see what is a graph?

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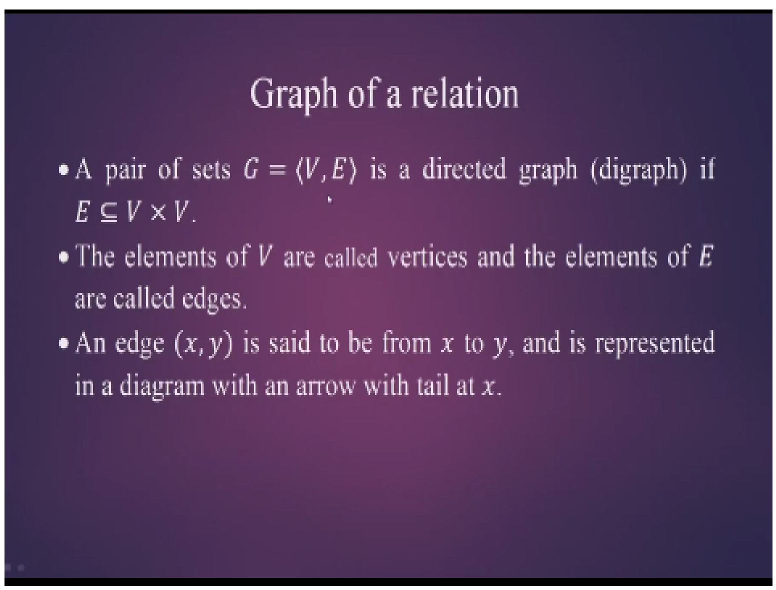

If we have a pair of sets G=(V, E) where VE are two sets we call it as directed graph or in short a digraph if this set E is a subset of V x V. The elements of V are called vertices and the elements of E are called edges. An edge (x, y) is said to be from x to y, and is represented in a diagram with an arrow with a tail with tail at x, and of course the head at y. Now let us look at examples of such graphs.

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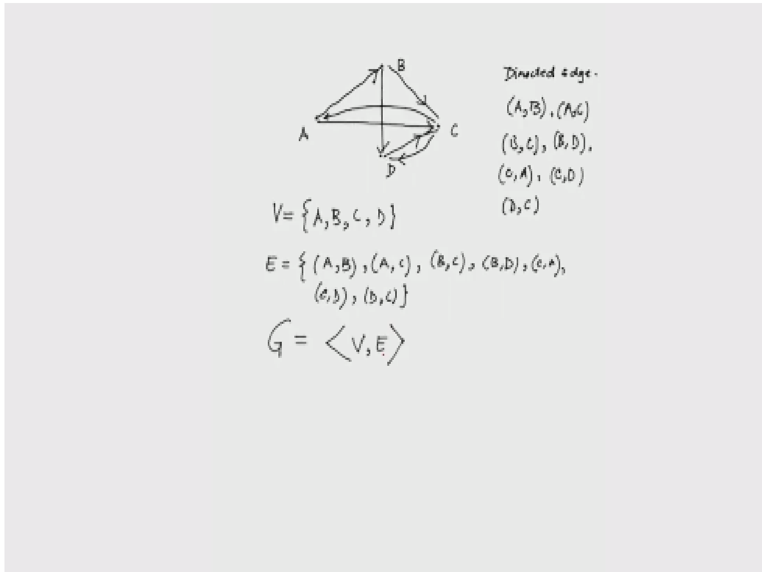

So first we will be writing the vertices as points on a plane. So let us say I draw a point call it A, another point let us say I write B, another point I write C, and let us say D is another point. Now we can in fact draw any directed edges from any vertex to another. So starting from A let us suppose we write AB, so it is a directed line from A to B this is what we will call a directed edge. And this is exactly what we will denote by the ordered pair AB.

Now let us draw one more AC and we can also write an edge from C to A, then possibly B to D, and D to C, and also C to D, and B to C. So what we have done here is that we have drawn some arbitrary directed edges from the vertices. We can write them as ordered pairs as the first one as I have already written A to B, and then A to C, then let us see nothing more on A, then we go to B, then we have an edge from B to C, then we have an edge from B to D, BD, then from C we can go to A, and also to D, and finally we have an edge from D to C.

Now the set of vertices are the set of vertices is A, B, C and D and the set of edges is (A, B), (A, C), (B, C), (B, D), (C, A) then (C, B) and (D, C). We combine these two sets to write the directed graph which is VE. What we note here is that this is naturally a relation, because let us recall suppose we have the set V which is as we see A, B, C, D, and suppose we talk about relations on V.

So a relation on V is nothing but a subset of V x V and the set E is exactly a subset of V x V, therefore it is a relation on V. Thus a directed graph gives me a relation on the set of vertices. On the other hand it is quite natural that if I have a relation, then I can I can take the set on which the relation is defined, and then take the, that relation itself will give me the set of edges of the directed graph.

So with respect to a relation we always get a directed graph. Now we here there is something we have to note that sometimes instead of a relation on a set A, we would like to look at more general situation where we are talking about relations from A to B.

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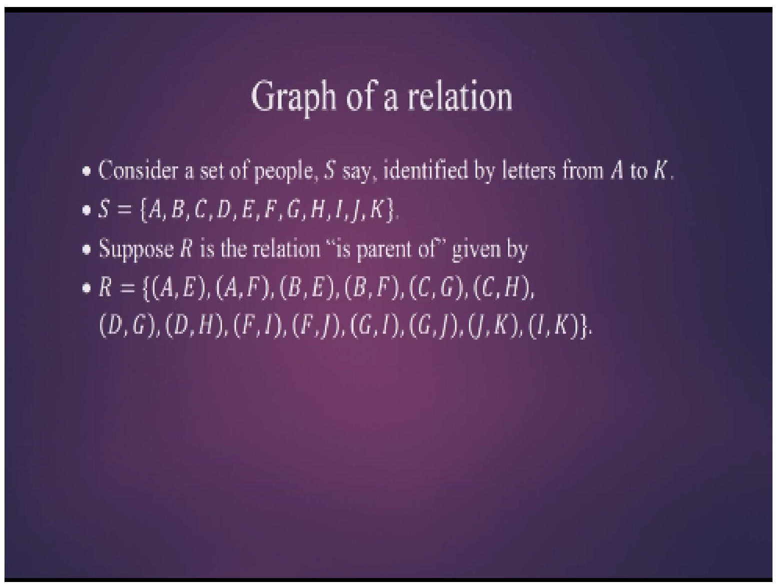

And that also does not make any make much difference over here, because what we can do is the set of vertices can be chosen to be the union of the two sets A and B, that is suppose I have a relation from A to B, I may choose the set of vertices to be union of A and B. And then I can consider R not as a subset of A x B, but a subset of A U B x A U B. And it is not difficult to see that a subset of A x B can be embedded in a subset of A U B x A U B.

And thus from there we can get the digraph corresponding to the relation from A to B. Now let us try to build graphs of some relations. Now let us take one graph sorry, let us take one relation on set of people let us call that relation, "is parent of". Suppose we denote a set of people by (A, B, C, D, E, F and so on up to K), and the relation R is, "is parent of" relation and suppose we have a listing like this.

So we have R which is a subset of ordered pairs like (A, E), (A, F) and so on up to (I, K). Here (A, E) means A is parent of E, (A, F) means A is parent of F, (B, E) means B is parent of E, (B, F) means B is parent of F, and so on. Now we would like to draw a graph of this relation, first we write the set AB set S.

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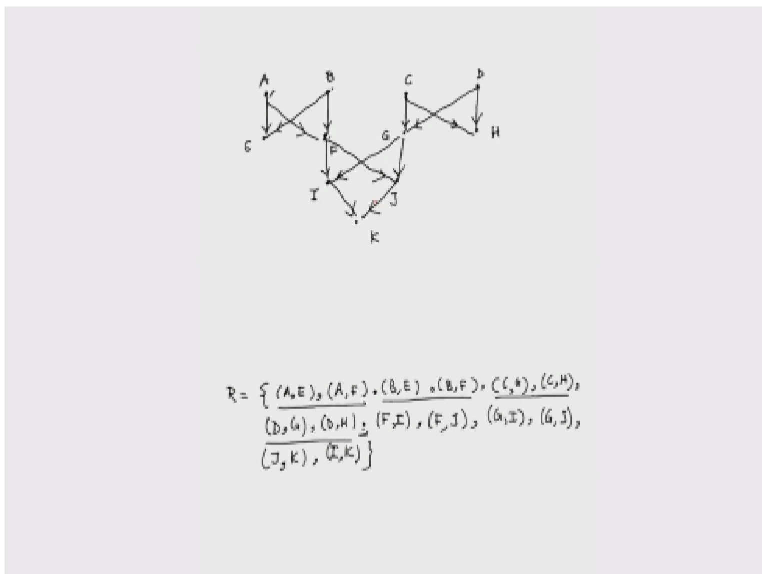

Now see this is A, the vertex A, and the vertex B, and then the vertex C, and the vertex B. And now let us write the listing over here, here we see that R=(A, E), (A, F), (B, E), (B, F), (C, G), (C, H), then we have (D, G), (D, H), (F, I), (F, J), (G, I), (G, J), (J, K), and (I, K). So let us look at these two ordered pairs. So (A, E) and (A, F) are in the relation that means that we draw two more vertices name them as E and F, and join A to E, that means A is parent of E.

And then join A to F which means that A is a parent of F. Then we see that we have (B, E) and (B, F), therefore we join B and D draw a line over here, B and F draw a line and an arrow. So this means this B is a parent of F, and B is a parent of E. Now we come to the next two pairs this is (C, G) and (C, H), and after that we have (D, G) and (D, H). So we write G over here and H over here.

So we join (C, G) and (C, H) which means that C is a parent of G, and C is a parent of H, and then we join (D, G) and (D, H), and that exhausts the relations up to this. Now then next we have (F, I) and (F, J), and (G, I) and (G, J). So we draw I over here (F, I), and we draw a point J over here we have (F, J), then we join (G, I) and (G, J), and lastly we have (J, K) and (I, K). So we draw another vertex over here which is K and join and I to K this is (I, K) and this is (J, K). Thus, we have a graph corresponding to the relation is parent of, and where the listing of the people is given in the set S.

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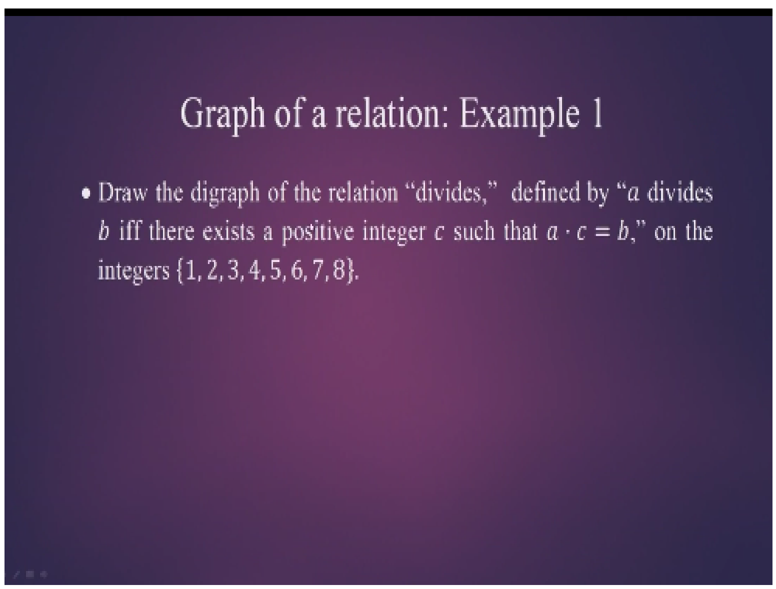

Now let us look at another example here we are asked to draw the digraph of the relation "divides," defined by "A divides B if and only if there exists a positive integer C such that A x C=B," and the set of integers on which this relation is defined is 1, 2, 3, 4, 5, 6, 7, 8, so that is integer positive integers from 1 to 8. Now this divides is a usual division on the set of integers. So first thing that we have to do in order to solve a problem like this is to write all the ordered pairs corresponding to this relation, so we shall do that shortly.

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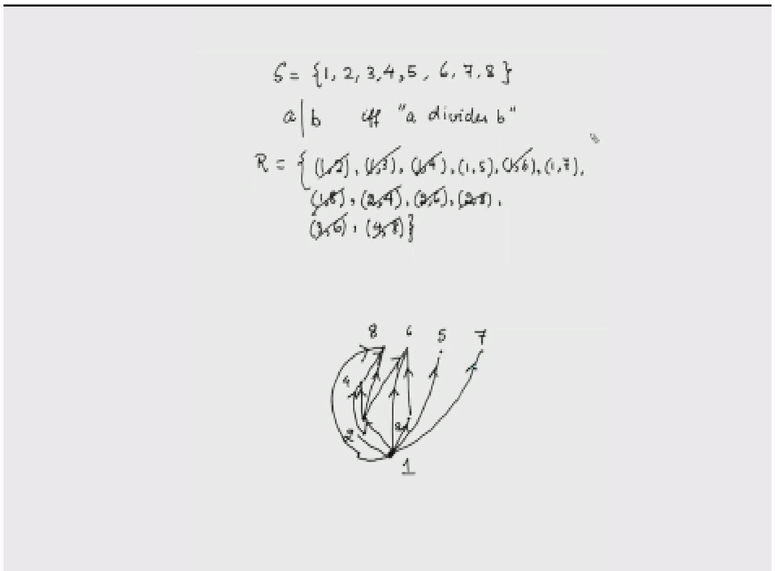

Now let us consider the set, let us call it for the time being S = (1, 2, 3, 4, 5, 6, 7, and 8) and we write the relation in this way A/B if and only if A divides B, we are denoting the relation by a vertical line. Now let us start to list all the ordered pairs corresponding to this relation. So we have got R, now let us start with 1, 1 divides all the integers in the set S, so (1, 2), (1, 3), (1, 4), then (1, 5), (1, 6), (1, 7) and lastly (1, 8).

Then if we start with 2 we have 2, 2 does not divide 1, but 2 divides 4, it does not divide 3 either, 2 does not divide 5, but 2 divides 6, so we have got (2, 4) and (2, 6) 2 does not divide 7, but 2 divides 8, so we have got (2, 8), so we finish up with 2, then we start with 3. So 3, 3 divides 6, and nothing else then we come to 4, 4 divides 8, and 5 divide, 5 does not divide any other number in S, 6 does not divide any other number in S, neither does 7 or does 8.

So we have the complete listing over here. Now we have to be a bit careful about drawing the graph because it may become very messy if we are casual about putting the vertices. So first let us observe one is related to many elements and no element is related to 1. Therefore, what we do is that we draw first the point 1 over here at the very bottom and write 1, and then we see that all the other elements 2, 3, 4, 5, 6, 7 and 8 are related to 1.

So first we write 2 like this 2 and draw an arrow, then we try to see from here where that 2 is related to something else S, here 2 is related to 4, so we write 4 over here and join by an arrow, so 1 to 2, 2 to 4, and then 6 is also there, and 8 is also there. But here we see that 4 is also related to 8, so what we do is that we draw a point the point 8 over here join from 4 to 8, and also join 2 to 8.

So we have completed 1 going to 2, 2 going to 4, 4 going to 8, and 2 going to 8. And now we start off with 3 and 6, see 3 is related to 6, and 2 is also related to 6, so we have to keep space for that. So we may like to draw 3 over here all right and possibly it will be a good idea to draw 6 over here, and join 2 to 6, and 3 to 6, at the same time I have to join 1 to 6, because that is also there.

And we must remember to join 1 and 8, because it is also in the relation, therefore see we have done (1, 3), then (2, 6), then (3, 6), then (1, 6), and we have also taken care of (1, 8). So we have not taken care of (1, 4), but we shall take care of that shortly this is (1, 4) and after that we are left with (1, 5) and (1, 7) that we can do by drawing two other points 5 and 7 and joining 1 to 5, and 1 to 7.

This is the graph corresponding to the relation a divides B on the set (1, 2, 3, 4, 5, 6, 7, 8). Now let us look at more examples on graphs of relations namely this.

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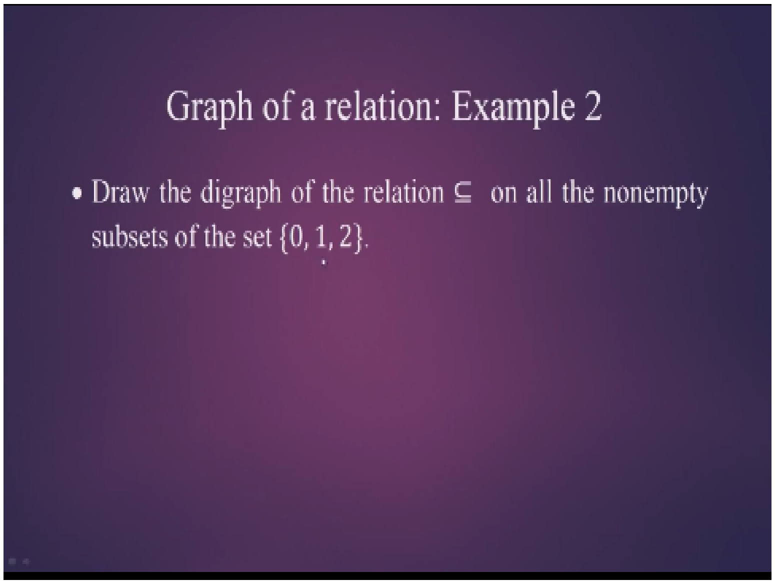

Now our set is (0, 1, 2) the three elements set (0, 1, 2) and a digraph we have 2 a relation is defined on it which is the subset of relation and not on the set (0, 1, 2), but on the set of all non-empty subsets of (0, 1, 2). So our problem is to draw the digraph of the relation subset of or equal to on the non-empty subsets of the set (0, 1, 2), let us try to do that now.

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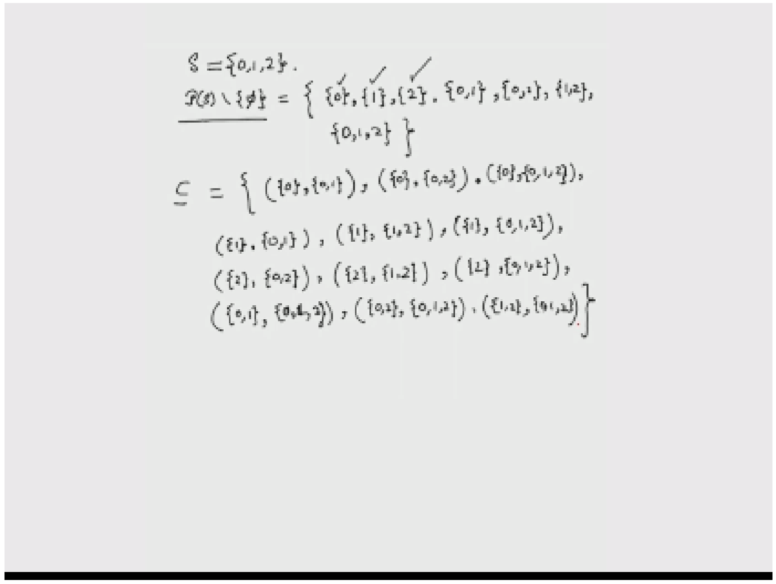

Here let us write the set (0, 1, 2) the first step over here is to list all the non-empty subsets of (0, 1, 2). If we call this set S, then the set of all non-empty subsets will be PS minus the set containing ϕ the empty set. Now let us start listing first we will write all the singleton subsets, then we will write the subsets of size 2 or subset of cardinality 2, (0, 1), (0, 2), and (1, 2), and finally we write the subset (0, 1, 2).

Now let us write the elements which are contained in the relation subset equal we are denoting the relation by the symbol this is after all a subset of the power set minus the set containing ϕ, so it is equal to ordered pairs, let us see, let us start with the element set 0, this element is a subset of (0, 1), then (0, 2), and (0, 1, 2). So therefore, in the ordered pairs we will have (0, 1), (0, 1), set 0, set (0, 1), then we will have set 0, set (0, 2), then we will have set 0, set (0, 1, 2).

Next we will start with the singleton 1, and then the singleton 1 is a subset of again (0, 1) next we consider the singleton 1 it is a subset of (1, 2), then we consider the singleton 1 again and of course this is a subset of (0, 1, 2) all right. Then we have got singleton 2, 2 is a subset of (0, 2) then 2, 2 is a subset of (1, 2) right, and then 2, 2 is a subset of (0, 1, 2). Finally we start taking, we have exhausted by taking starting with the singleton sets, and then we take sets containing two elements.

So we have now (0, 1) and we see that (0, 1) is contained in (0, 1, 2) so the pair (0, 1), (0, 1) so the pair (0, 1), (0, 1, 2) will appear over here, and then we will have (0, 2) that is contained in (0, 1, 2), and finally we have (1, 2) which is contained in (0, 1, 2), and we close the bracket. So these are the relations, these are the elements of the relation subset equal on the set containing all the non-empty subsets of the set (0, 1, 2).

And now we would like to draw the digraph corresponding to this relation. Here again in practice we start from an element which is related to all the elements, but no other element is related to itself, and that element in fact there is no single element like that over here there are three elements like that which are essentially we so see.

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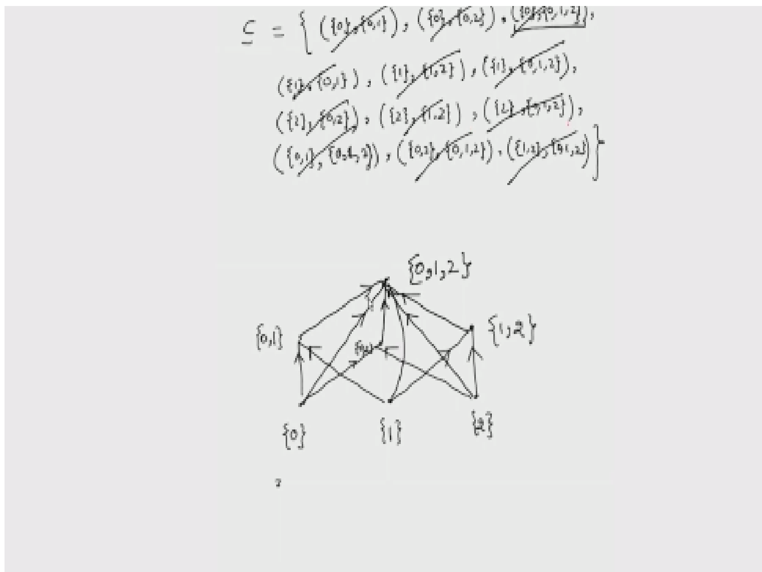

We have written 0 over here alright, and 1 over here, and 2 over here. Then we have written (0, 1) over here, and we have joined these lines okay. The (0, 2) we will write over here you will join (0, 2), (0, 2) and here (2, 2), (0, 2), and then we will write (1, 2) over here this is the point and join (1, 2), (1, 2) and (2, 2), (1, 2). Thus we have a nice shape in fact when we are drawing a digraph of a relation we have to take this into mind.

As well that we have to keep in mind that whatever diagram we are drawing it should look nice, and now we have only one element left that is again, let me remove this, we have got only one element left that is (0, 1, 2) and then we see that (0, 1) is a subset of (0, 1, 2), (0, 2) is a subset of (0, 1, 2) and (0, 1, 2) is also a subset of (0, 1, 2). Thus we get the digraph corresponding to the relation.

Wait a moment it is not complete yet there are some more things that we have to do we have missed out few things. For example, we have missed out this one. So we have to connect 0 to (0, 1, 2) because they are related S, we have to connect 1 to (0, 1, 2) and we have to connect 2 to (0, 1, 2). Now we have got all, we can check that because we have got B starting from 0 to (0, 1) we have done this (0, 2), (0, 2) we have done this then (0, 2), (0, 1, 2) we have done this, then 1, (1, 2), (0, 1) we have done this (1, 2), (1, 2) we have done this, and (1, 2), (0, 1, 2) we have done this also.

Then (2, 2), (0, 2) we have done this, (2, 2), (1, 2) we have done this, and then (2, 2), (0, 1, 2) we have also done this. Now (0, 1) is related to (0, 1, 2), (0, 2) is related to (0, 1, 2) and (1, 2) is related to (0, 1, 2), thus we have completed or we have completed the scanning the whole list and drawing the corresponding directed edges.

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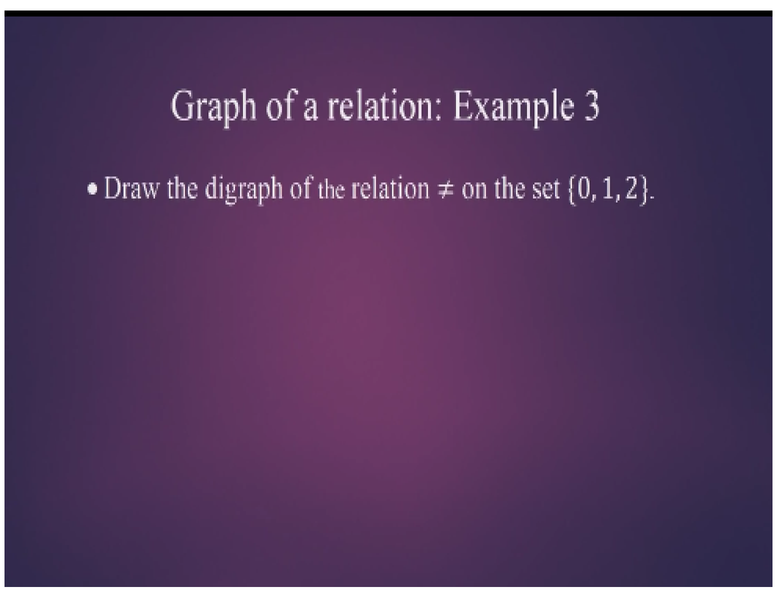

Now let us look at another example we are asked to draw the digraph corresponding to the relation not equal to on the set (0, 1, 2).

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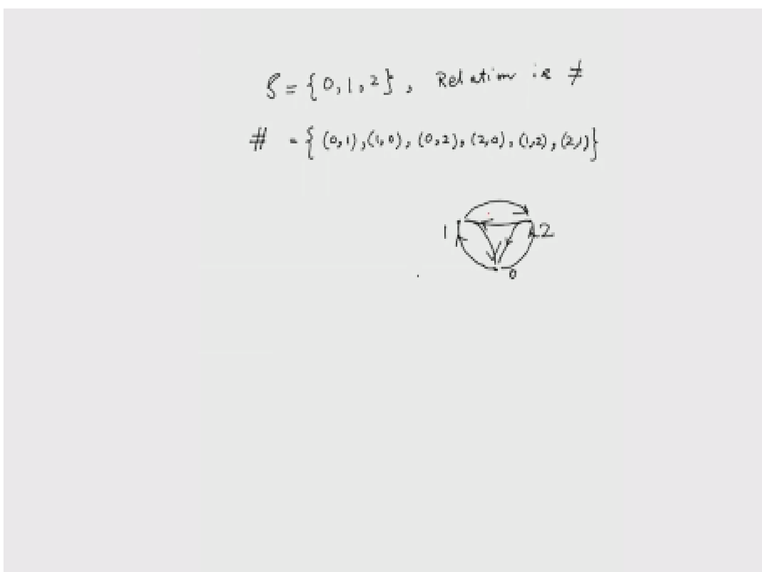

How will we do that, here we are starting from the set let us call it S which is (0, 1, 2), and our relation is not equal to. So if we try to write the set corresponding to the relation let us write like this, so 0 is related to 1, and 1 is related to 0, then we have 0 is related to 2, and 2 is related to 0, then 1 is related to 2, and 2 is related to 1. Then we have got 3 points on the set, so we write the 3 points.

Suppose like this is 0, this is 1, this is 2, so here we see that 0 is related to 1, so we have to write 1 is like this 0 to 1, I also write an edge like this 1 to 0, then 0 is related to 2, and 2 is related to 0, so I write 2 more edges. And then we see that 1 is related to 2, because 1 is not equal to 2, and 2 is related to 1 since 2 is not equal to 1. Thus we have got a nice set over here of the digraph. Now we look at some examples of digraphs.

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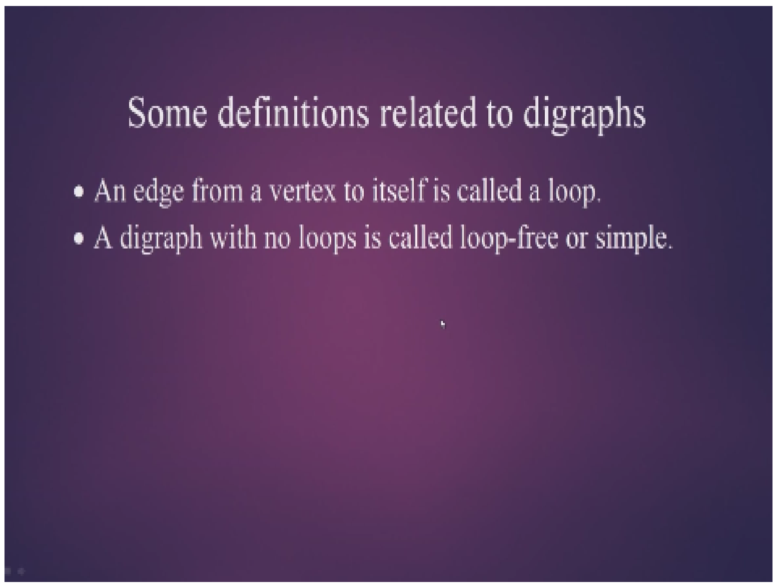

Now we will check some definitions related to digraphs, an edge from a vertex to itself is called a loop. So we will see examples of loops, a digraph with no loops is called a loop-free digraph or a simple digraph.

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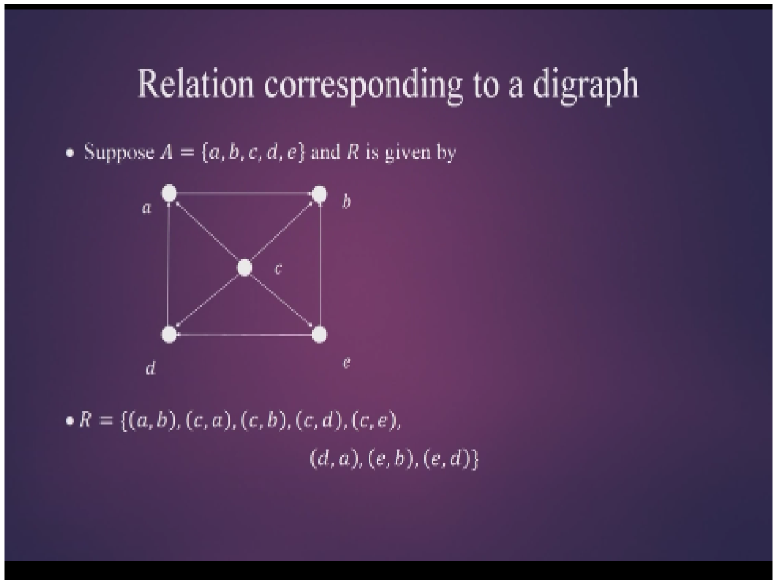

Next we will try to write down some relations from a graph but let us let us say check now, we have already seen this digraph do you have. So in the last example we have already seen this diagraph (0, 1, 2) like this now this is loop-free because there is no edges which starts from a vertex to another vertex but if we have an edge like this, for example in this way, so then it is it is a digraph with a loop this is a loop similarly we have a loop like this.

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Here we have two we observe that suppose we have a relation which is reflexive then each of the vertices of that relation will have a loop around it, for example if we make another loop over here then the corresponding relation will become reflexive because that will mean that each element is related to itself. Now we may also have some problems where we are given a digraph on a set of vertices and asked to write down the corresponding relation.

For example here we are starting with the set of vertices a, b ,c, d, e and then we are given a digraph now it is quite straightforward problem because we see that we can start with a and of course ab the pair ab is in the relation and no other pair starting from a so we write ab over here then we start from b, there is no edge coming out from b therefore we have we do not list anything then we come to c, we see that there are four edges coming out of c so (c, a) c, b and c, d so we write over here and then we have got d, a and then e, b and e, d. So we have the relation corresponding to this digraph. There are some more definitions related to digraph.

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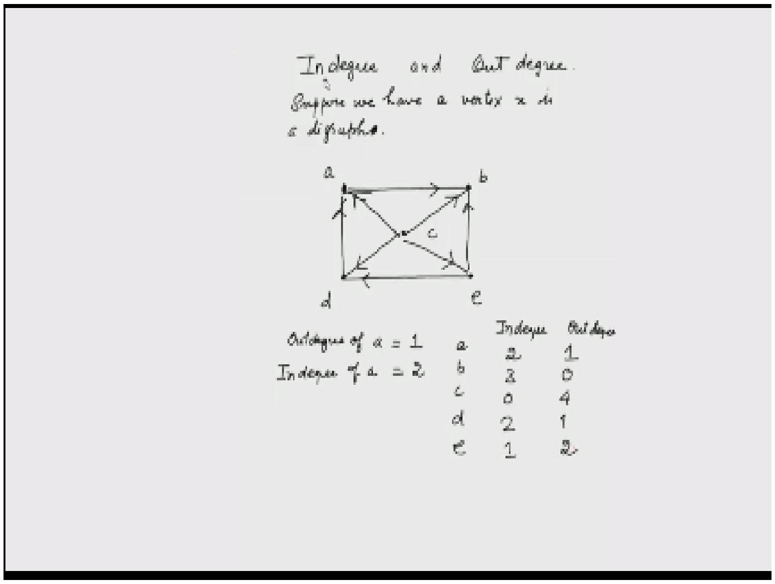

That is in-degree and out-degree, suppose we have a vertex x in a digraph the number of edges coming out of x is called it is out degree and number of edges incident on x is called it is in degree. So if we look at a graph that we have already studied let us see let us say this one all right, so we have a graph now let us consider the vertex a. So number of edges emanating from a is 1, so out degree of a is 1 and in degree is the number of edges incident only.

Here we see that this is (1, 2), so this is 2 similarly we can make a list of how in degrees and out degrees, in degree out degree and vertices a, b, c, d, and e we have seen that for e, a it is 2 and 1 for b let us consider b there is only in degree, so that it is 3 and out degree is 0 then in c all are out degrees, so we have got 4 and no in degree be in degree is 2 out degree is 1 and e in degree is 1 and out degree is 2. Finally we come to the idea of isomorphism of digraphs.

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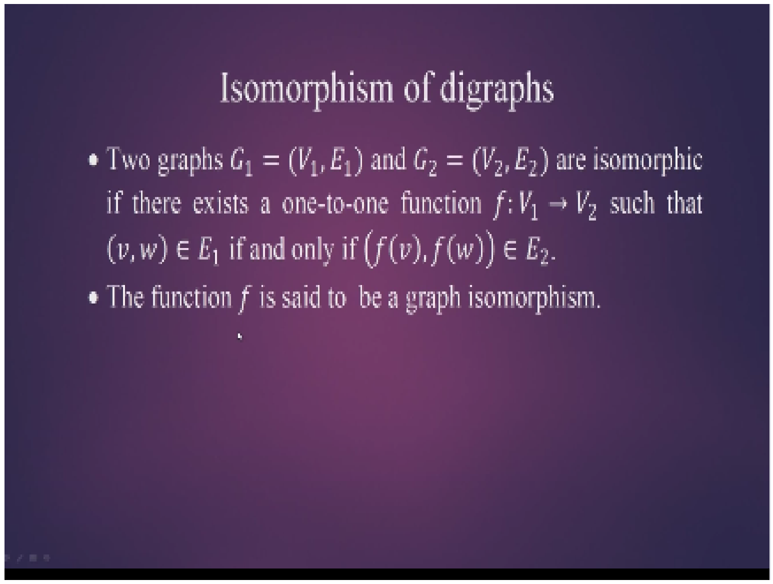

Suppose we have two digraphs G1 given by the payer V1 E1 and G2 given by the pair V to E2 now we will call it call the two graphs isomorphic if there exists a one-to-one function if from the set of vertices of G 1 namely V1 to V2 the set of vertices of G2 such that the ordered pair VW belongs to E1, if and only if the ordered pair FV and FW belongs to E2 and this function F is said to be a graph isomorphism in general it is a difficult problem to decide whether two graphs are isomorphic or not.

But there are some properties of graphs which are called invariants through which we can make these decisions at least towards one direction. So here we introduce an idea of invariance.

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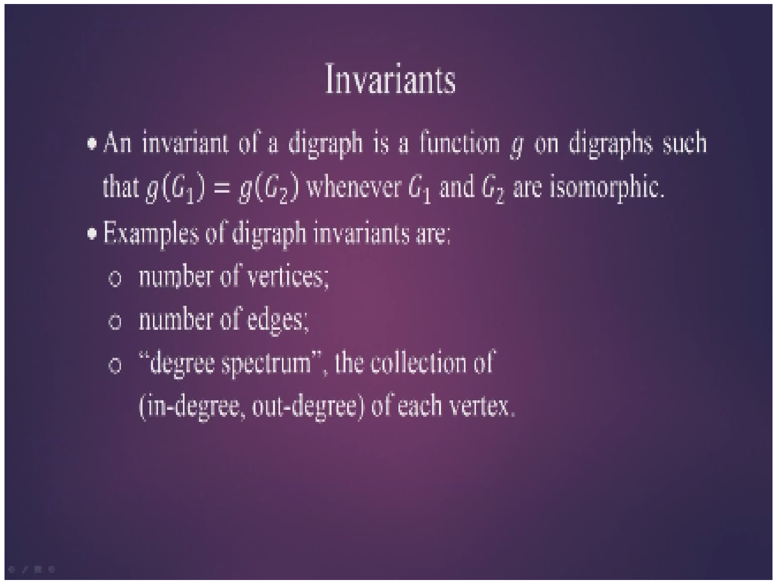

And invariant of a graph is a function G on digraphs such that G1 = GG - whenever G1 and G2 are isomorphic. Now the invariants can be number of vertices, the number of edges or the degree spectrum which is the collection of all in degree out degree pairs of all the vertices. So this means that given a graph I can always compute these things for example I can always compute the number of vertices or number of edges or the degree spectrum.

Suppose there are two graphs for which these numbers do not match then we are sure that these graphs are not isomorphic but if the match we cannot be sure we cannot take any decision however it is known that if two graphs are isomorphic then all the numbers will match therefore if the numbers do not match we can say they are not isomorphic but if the numbers match we can we cannot say anything because there are examples of non isomorphic graphs, for which all the all these numbers match.

So in this area it is an important problem to find out different invariants, so that we have got good resolution among the set of graphs. I will stop the lecture now we have studied how to build digraphs from relations and we have also seen given a relation we have also seen we given a graph how to convert that to a relation, we have also seen some examples of graphs of relations and relations from digraphs and finally we have discussed certain properties of these diagraphs this is all for today thank you.

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