INDIAN INSTITUTE OF TECHNOLOGY
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NATIONAL PROGRAMME ON TECHNOLOGY
ENHANCED LEARNING
(NPTEL)

Discrete Mathematics

Module-05
Graph theory
Lecture-05
Shortest path problem

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In this lecture we will discuss the shortest path problem. Now before we go into an algorithm to solve the shortest path problem, we will introduce something called a weighted graph.

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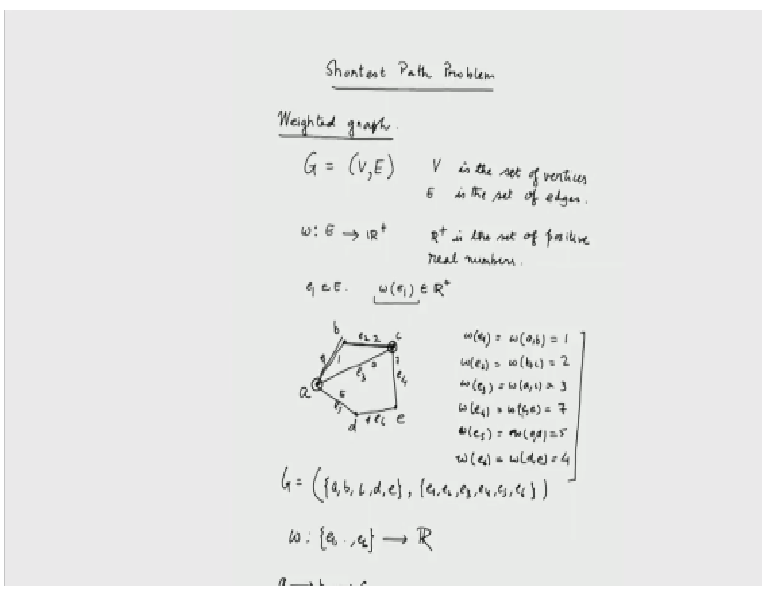

Now we have already seen that a graph is an ordered pair of sets. So I write G as a graph which is equal to V, E where V is the set of vertices and E is the set of edges. Now we know what are vertices and edges. Now when we have a weighted graph, then each edge has a weight. And in our discussion we will take the edge, the weights to be greater than 0. So strictly speaking V consider a function from E to positive real numbers.

So we consider functions from E the set of edges to R+, where R+ is the set of positive real numbers. And this function is called the weight function and individual values of this weight function will be called the weights of edges. For example, suppose we have an edge E1 belonging to E, we can operate the weight function on this edge E1 and we will get W(E1) which is a positive real number and which will be called the weight of the edge E1.

Let us look at an example, let us consider a graph like this. Suppose these are the vertices A, B, C, D and E and the edges. Suppose, E1, E2, E3, E4, E5 and E6, now each edge are assigned to some weights possibly the edge E1 is assigned to 1, 2, 3, 5, 4, and 7. So that means that W(E1) or in this case, since it is a simple graph we can also say that W(AB) = 1, W(E2) = W(BC) = 2, W(E3) = W(AC) = 3, W(E4) = W(CE) = 7, W(E5) = W(AD) = 5, and W(E6) = W(DE) = 4.

So these are the weights of the and this graph which was originally G = (a, b, c, d, e) and edges E1, E2, E3, E4, E5, E6 these graphs when we consider the function W from E1 up to E6 that is a set of edges to a set of real numbers define as over here becomes a weighted graph. Now our problem here is that suppose, I am given two vertices in a weighted graph, how do we find out the shortest path from the first vertex, the second vertex and by the shortest path I mean that a path such that if we add up all the weights of the edges belonging to the path will get the smallest value.

If we look at the graph that we just drew over here, if we consider the vertices A and vertices C and suppose we want to go from vertex A to C, we have got these three paths one is A to B and B to c we see the total – weight is 3 so we have got a to b and b to c.

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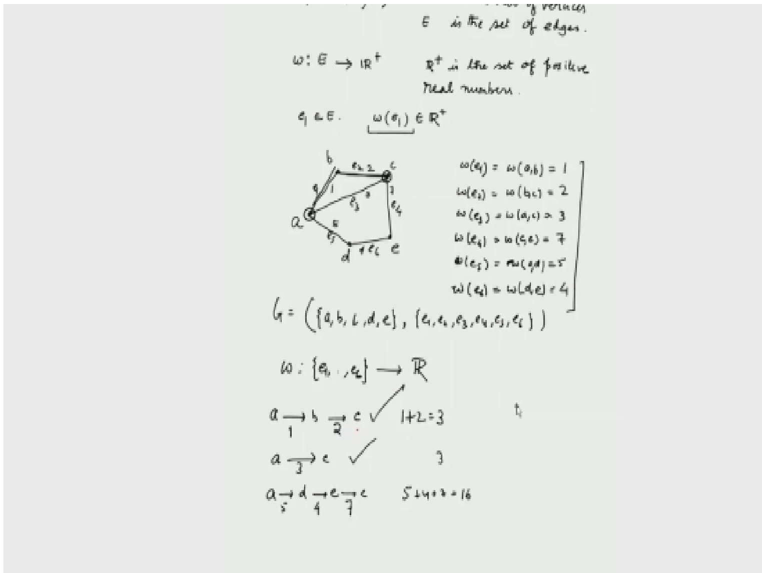

a to b is 1 b to c is 2 so the weight is 1 + 2 which is = 3 the total weight or otherwise we could have gone from a to c the total weight is 3 well and we other possibility is from a to d, d to e, e to c, a to d is 5, d to e is 4, and e to c is 7 so the total weight is 5 + 4 + 7 = 16 now of course the shortest paths are this or this now this is the small example and the graph is quite simple that is by observing the graph and checking the weights is possible to find out the shortest path but it may not be.

So when the graph is complicated we need an algorithm and Dijkstra's algorithm we sizably gives us that so to say Dijkstra's algorithm shortest path between 2 points now we move to Dijkstra’s algorithm.

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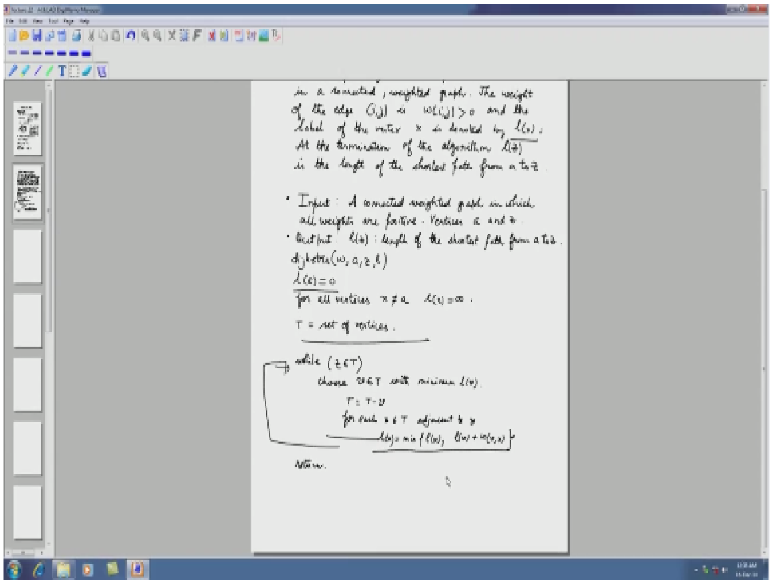

Suppose we have got a graph where we are labeling the vertices by a, b, c, d and the edges from a one vertex to the other by an odder pair so now this algorithm finds the length of a shortest path from the input vertices a and z, in a connected weighted graph the weight of the edge i, j is w(i, j) which is taken to be > 0 and the label of the vertex x is denoted by l(x).

Now here in this algorithm to each vertex assign a label which initially keeps on changing and after that comes to a fixed label and that label gives us the shortest length path by which I mean the path or which the sum of the ways is the smallest the shortest length path from the point a is a vertex a to x and this labels will keep on changing and ultimately when we take the first input as a that is we want a shortest path from the vertex a and when we start calculating the labeling at the end of the algorithm.

The label corresponding to z will give the shortest path from a to z so the labels are denoted by l(x) at the termination of the algorithm l(z) is the length of the shortest path from a to z now let us start discussing the algorithm so first step is being our input is A is a connected weighted graph in which all weights are positive and vertices a and z so we have got 2 inputs so to say 1 is is that graph on which we would like to apply the algorithm.

And two vertices in that graph between which we want to know the shortage path, now the output is Az which is the length of the shortage path from A to z now we come to the algorithm that we denote by dick straw w this w is for weight function of the graph g that we are using and

Az and the function L which we will see how to compute now in first step we will put l (A) = 0 and for all vertices x v = A.

Lx is infinity so what we are doing in the beginning is that we know from orient to that from which what else we have to calculate the shortage path we are putting the label of that vertex to be 0 that is the step lA = 0 and all the other vertices are label as input now T is a set vertices other than no we leave it as set of vertices and then we will process this T now in the first step we come like this that is y, z belongs to T we start this loop choose V belonging to T with minimum Lv and then remove V from T and build up a new T.

And then for each x belonging to T adjacent to v do Lx = minimum of Lx and Lv + w Vx and v turn so let us see what happens our starting set of vertices is a set of all vertices and then among them among that in Tv find the smallest weight vertex that is vertex with the smallest level and we choose it and then we delete it from T so then we will get set of vertices other than that vertex that we have deleted and then what we do is that we start scanning all the vertices which are adjacent to the deleted vertex and then we will calculate this quantity after calculating this quantity.

Now at least change I go again into this and we again to the same process so these loop will keep on happening and this loop also will keep on happening let us look at an example to clarify this matter.

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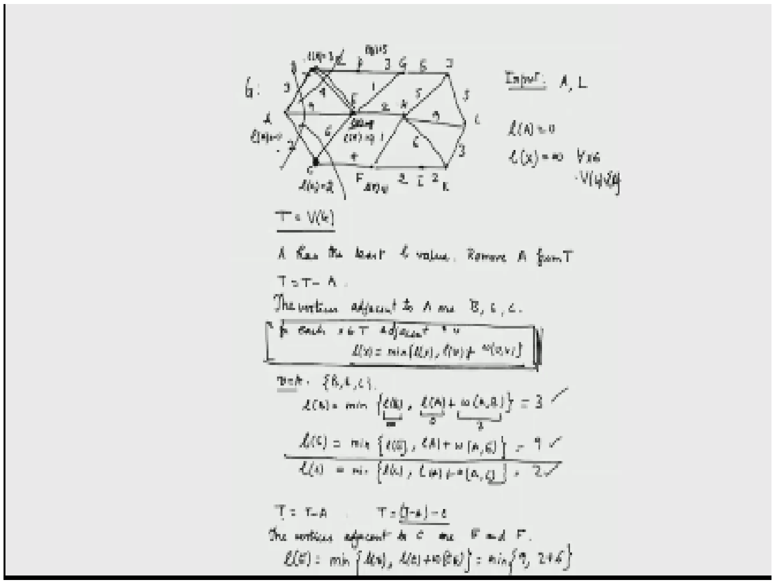

This is A and A is connected to B and the weight is 3 B is connected to D the weight is 2 d is connected to G the weight is 3 then B is connected to E the weight is 4 and here AE the weight is 9 now here A is connected to C and the weight of the A is 2 C is connected to R the weight of the edge is 6 after that R is connected to G the weight of the edge is. Now R is connected to H the weight being to and C is connected to F the weight being 9 F is connected to H.

The weight being 1 and here G is connected to J the weight being 5 and GJ the weight being 5 again and here J is connected to L the weight is 5 H is connected to L the weight is 9 F is connected to I the weight is 2 I is connected to k the weight is 2 again h is connected to k which is 6 and k is connected to l which is 3. Now suppose somebody tells me that my input vertices are A and L and I have to find out the shortest path between A and L how do I start? Now let us go according to the algorithm proposed by Dicks Shaw step wise.

In the first step my T is a whole set of vertices of the graph given graph so let us denote the graph by G. now and then what I do before that is that we our starting vertices L A so LA is put to 0 L of anything else x is infinity for all x belonging to Vg – l – a so other than a all vertices are put to infinity. Now we start with T now in the first step we will consider the vertex which has got the least level value and in this case it is A.

So we will remove from t the vertex A, so C A has the least L value so remove A from T, so I calculate T which is T – A. now in that we have to consider all the vertices which are adjacent to A, the vertices adjacent to A are B, E, and C. Now we use the formula that is for all X for each X belonging to T adjacent to V LX = minimum LX LV + WV, X. So this portion of the algorithm is to be use now, so what do I do?

Here my V is equal to A and the adjacent vertices are B, E and C, so I calculate L of B to be minimum of L of B, L of V that is L of A + W A – B, now we see that L of B is infinity and L of A is 0 W AB is 3 so we have got a choice between infinity and three so the minimum is going to be three, then L of E is going to be minimum of L of E and then L of A + W AE and using the same argument it is going to be 9 and similarly L of C is going to be 2 which is minimum of L of CL of A+ W AC.

Now we see that now we have a graph where I do not consider A anymore, so that is my new T and B has weight 3 or sorry B has level 3 E has level 9 C has level 2 and all the others have level infinity. Now we will look we go in to again the while loop and we chose from T the minimum vertex that is vertex with minimum level value and that level value is going to be C over here, so I can put the level of C to be equal to 2 incidentally level of A is 0 and then I will cut out C from T.

So now my T was T – A original T – A this is my T and now my next iteration my t will be essentially T – A – C so I will have I will also delete C, now when I have deleted C now I will calculate the new level values for the vertices adjacent to c. now what are the vertices of adjacent to c? The vertices adjacent to c are e and f. Now again we will this step of each of the vertices we have to calculate this quantity l(e). Now we know that e has the label value that is = 9, so I have to take the minimum of le and lc + w c_e.

Now look that it is minimum of 9 and lc is 2 + c to e is just 6, so I have to choose between 9 and 8 which is of course 8. So this see the label value of e is decreased. So we will put le = 8 and lf = minimum lf which is ∞ and lc + w cf, if we do lc w + cf than we have minimum between ∞ and lc + w cf is going to be 11, so it is 11. So now we come back to our original graph we see that, e is now label as level of e = 8 and level of f = 11.

And level of B remains at 3 and we have cut out c, now we go to the next intervention, we have to consider b, e, f and rest of the vertices are ∞ among them we see the b is the smallest. So we have to start with b which is 3 so we are essentially cutting out b and then we are re computing the labels. So b is adjacent to d, d is already ∞, so therefore this value ld is going to be 5 and when we come to le we will see that it is even shorter than before.

And because if we go from a to b that is 3 and b to e that is 7,s o I have compare between 7 and 8, so this value of l is going to change to 7. So I have got le = 7 and ld = 5 and these are the vertices which are adjacent to b. so after we curt off b then again we have to look at the graph.

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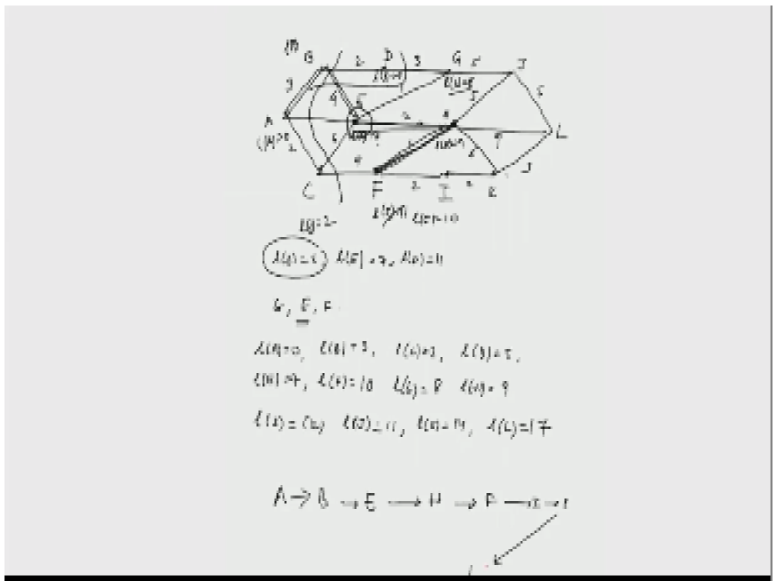

So I have got a this is A, B we have D over here then we have G over here, J over here, L over here and C, F we have E and then we have H, H is connected to J, H is connected K and F both and now the width is you see that 324235 and this is one and this is five this is five three yeah this is nine and this is one and now what you see this becomes two what you see over here we have taken care of three vertices we will find out how that will be equal to 3 let us see these are two not surprising but because of our algorithm we have come to understand that well e as a solid value 7 and d=5 lf=11 now let me look at this one.

Then we will see that now I do not have to consider ec and esc so now I will consider the values of e and f so ld is five le is 7 and lf is 11 now among this suppose I choose le I will get out le so this also be a ray and then I have to find out the vertices adjusted to solid adjutancy e and calculate the only vertex which is adjusted to be is g and its l value is going to be 5+3 that is lg will be eight and then so I will go for the next step I will have I have to consider g, e and f and then I consider g, e and f then I am going to get e on this stable.

So this is the vertex which is smaller in table okay so I took out e and then consider the vertices which are adjusted to e the vertex adjusted to e are h, g both a intersection of h when it is 8 and this is 7+9 so n=9 so in this way we have to proceed I keep on proceeding in this way eventually I will get the n values of the other vertices we have given n values of the similar that we list down I will get la=0, lb=3 lc=2, lb=5, le=7,le=10 when we calculate lf we will find f is 1 and h is 9+1 is 10 so lf will stage on to 10 and lg will be 8 lh will be 9 li will come to 2i lj will be 11 lk will be 13 and eventually ll will be 17.

And what is the root for 17 we have to go for e and then from e we will come back to e so A to B, B to E and then E to we will go to h like this we have to come back to h so s is 10 and then from f I will have to go to I to k and eventually k with l so we see that lk is 14 so l is 17 it keeps on going though corrections and we will get this so this is given as shortest path algorithm in action and this is the end of the today's class thank you.

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