INDIAN INSTITUTE OF TECHNOLOGY
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NATIONAL PROGRAMME ON TECHNOLOGY
ENHANCED LEARNING
(NPTEL)

Discrete Mathematics

Module-05
Graph theory

Lecture-03
Walks, paths and circuits. Operations on graphs

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In this lecture we will discuss walks, paths, circuits or cycles in the context of graphs and then we will move on to discuss operations on graphs. So first walks, paths, and cycles.

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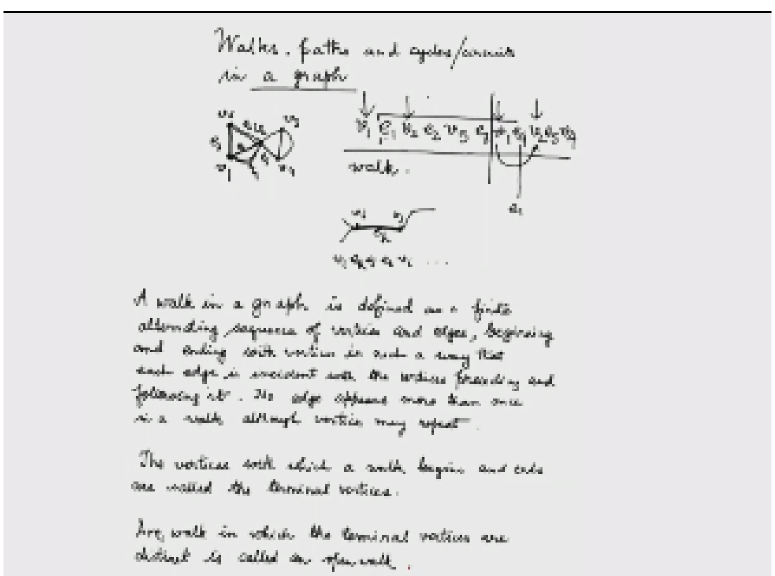

Walks, paths and cycles or circuits in a graph, now suppose we consider a graph like this then we may like to traverse on this graph for example starting from let us say this vortex let us say V_1, I may like to go to V_2 call it V_2 let us call it V_3 V_4 and V_5 suppose there are some multiple edges as well like this suppose I want to go from V_1 to V through V_2 through this edge say even then I go from here to V_5 let us say this is E_2 then we go from V_2 V_1 let us say this is E_3 then I may like to take another edge let us say E_4 and reach V_2 again and then possibly go to V_4 through another edge let us call it P 5 then what do we have we have what is known as a walk.

Now the question is that how do we specify a walk here we see that we have started from one vertex which is called V_1 and move to V_2 through one edge e_1 and then from each V from V_2 we have taken another edge E_2 and moved to V_5 from V_5 we have taken one more edge E_3 and move to V_1 again and then from V_1 we have taken an edge e 4 and move to V_2 again and then from V_2 we have taken an edge e_5 and move to V_4.

So this whole sequence of actions that we have done can be specified by a sequence of alternating vertices and edges starting from a vertex and ending at another vertex not necessarily distinct from the initial one. Now this is what we will call a walk, the question is that whether in a walk a vertex can be repeated the answer is yes a vertex can be repeated like we see that V_1 is repeated twice over here and V_2 is also repeated now another question is that can an edge be repeated the answer is in our definition we do not allow edge to be repeated in a walk in the literature in some books you will find that people allow reputation of edges as well in a walk and define something as a trail which does not repeat edges.

But in our definition we are fixing that we are not going to repeat edges because if we repeat edges suppose here when we are coming again to V_1 and we are going to V_2 by E_4 suppose that instead of E_4 we had taken E_1, then suppose instead of E for this is E_1 then we could have started the whole process from here itself. So what we see is that if a edge is repeated then whatever happened in between the repetition of two edges can be removed and we will get essentially the same thing this will let us remove certain cases like this like suppose I have got a graph over here and suppose I have got some V_I and V_J and there is some edge E_k and suppose I am spec suppose I allow repetition of edges then I will I can have a sequence like V_I E_K V_I E_K V_j E_K V_I and so on that is I go from here to here to here to here to here like this.

I do not want such a thing therefore in our box we would not get edges repeated we can also do away with repetition of vertices but for the time being we are not going to do that and we will we will introduce a different terminology for the walks where vertices are also not repeated, but now let us write the definition of walk in a formal way, a walk in a graph is defined as a finite alternating sequence of vertices and edges beginning and ending with vertices in such a way that each edge is incident with the vertices preceding and following it with we have to also specify that no age appears more than once in a walk no edge appears more than once in a walk although vertices may repeat so that is a walk.

Now the vertices at which a walk begins and a and ends, so there are two special vertices in a walk a vertex at which it begins and a vertex at which it ends these two special vertices are called the terminal vertices of the graph. The vertices with which a walk begins and ends are called the terminal vertices now we have to be careful that the terminal vertices may not be distinct, so there is a classification of walks in terms of the fact whether the terminal vertices are distinct or not if we have a walk in which the terminal vertices are distinct then they are then it is called an open walk a walk in which the terminal vertices are distinct is called an open walk a walk.

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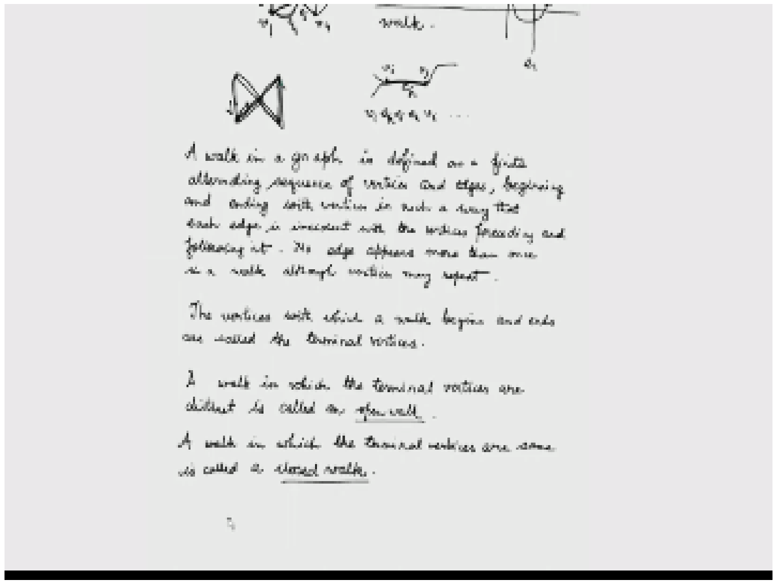

A walk in which terminal vertices are same is called a closed walk, so we have got the concept of an open walk and a closed work. So for example if we look at the graph that we were dealing with again, so I have got a graph like this if I start off from one vertex like this go to vertex like this then go to like this for example I go like this then this then come back like this like this then go like this and come back here starting from here I arrive here it is a it is a closed walk now if I start from here and let us say move like this and come here then it is an open walk.

Now we come to the concept of a path and this answers our question of what happens when a walk is such that no vortex is repeated a part a wok or more specifically an open walk.

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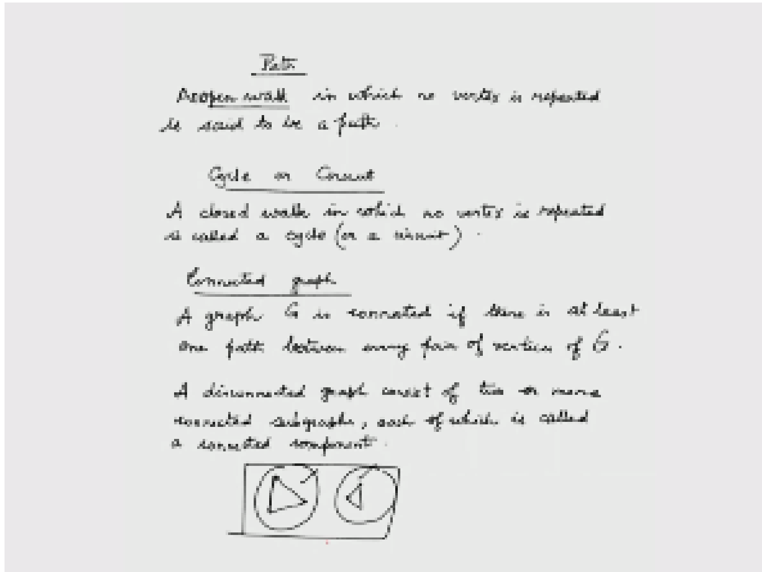

In which no vertex is repeated is said to be a path now then what is a cycle or a circuit cycle or circuit we will in use these two words synonymously a cycle is a closed walk a closed walk in which no vertex is repeated he is called a cycle or a circuit. Now once we have known the concepts such as walk path and circuit or cycles we are ready to investigate the idea of a connected graph or a connected component of a graph the basic interest here is that given two vertices in a graph I would like to know whether I can move from one vertex to the other through some walk or a path.

So if in a graph I can do that for any two pair of vertices then I call that a connected graph, and if I cannot do that then it I call that disconnected graph. But whatever be the case even any graph I can find out so called connected components that is I can start from a vertex and see how much I can cover starting from that vertex call that a connected component and then like that find out all the connected components. So let me write the definitions connected graph a graph G is connected if there is at least one path between every pair of vertices of G.

A disconnected graph consists of two or more connected sub graphs each of which he is called a connected component. Now this is easy to see suppose I have a graph like this, now this part is definitely a connected sub-graph and this is also a connected sub-graph my graph consists of the complete set of vertices and edges. So these are connected components of the graph under consideration.

Now there are some results related to the connected components connected and disconnected graphs that we will see right now.

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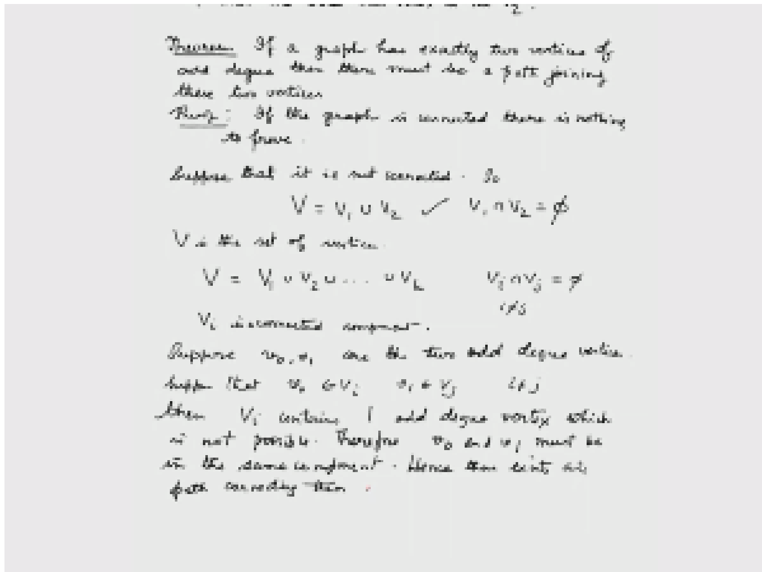

Now we move on to some theorems theorem a graph G is disconnected if and only if it is vertex set V can be partitioned into two non-empty disjoint subsets V_1 and V_2 such that there exists no edge G whose one end vertex is in V_1 and the other end vertex is in V_2. So this is somewhat very straightforward theorem which says that if you have a disconnected graph then your set of vertices are going to be partitioned into two subsets and you do not have an edge from starting off from one of those subsets and ending at and the other one.

Now we move on to the next theorem which states that if a graph has exactly two vertices of odd degree then there must be a part joining these two vertices proof. Now let us look at the statement, now we are considering graphs with only one restriction that in this graph there are only two vertices of odd degree and rest of the vertices are of even degree now suppose this graph is connected then there is no problem because then of course any two vertices have a path joining them and therefore these two odd vertex odd degree vertices have paths joining them.

So I can write if the graph is connected there is nothing to prove, now suppose that it is not connected then by the previous theorem I can split the set of vertices into two disjoint sets such that there is no edge connecting an element of the first one with the second one. So we the set of vertices is equal to $V_1 \cup V_2$ where R even $\cap V_2$ is empty and V is the set of vertices. Now I can keep on doing this process and ultimately end up with connected components. So ultimately what can happen is that the set of vertices V is split up into let us say some $V_1 \cup V_2 \cup$and so on up to some V_K where $V_I \cap V_J$ is 5 for I not equal to J and V_I is connected is a connected component.

Now we have repeated this process over and over again and therefore we know that there is no edge between V_I to V_J, now the question is that where the odd degree vertices will rely, so suppose small v-0 and small Z with 1 are the two odd degree vertices now what we claim is that these odd degree vertices cannot lie on two different components because if that happens then that component as a sub graph will have only one odd degree vertex which is not possible by using the first theorem that we have proved which true which says that any graph in any graph the number of odd degree vertices have to be even.

Suppose that V_0 belongs to V_I and V_1 belongs to V_J for some I not equal to J then V_I contains one odd degree vertex which is not possible therefore v-0 and V_1 must be in the same component hence there exists a part connecting them which is what we wanted to prove. Now we move on to another theorem related to connectedness which gives me an upper bound of the number of edges that a simple graph with K connected components can have, now let us move on to the theorem.

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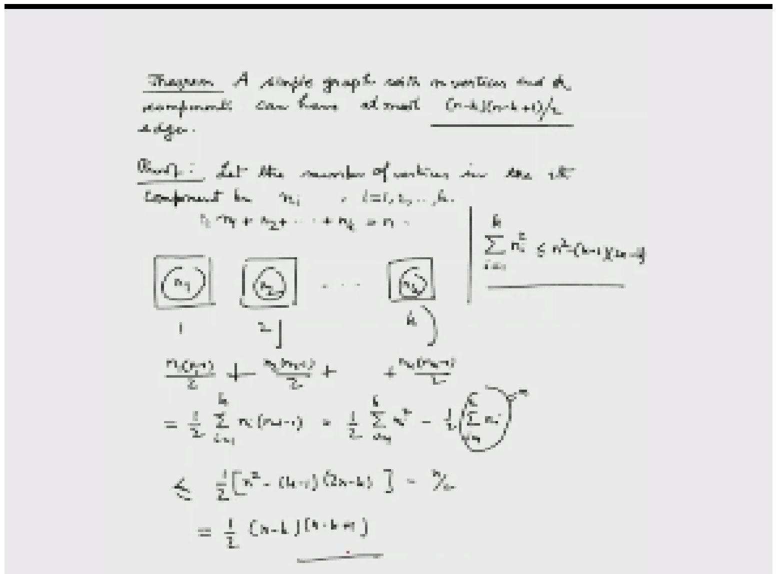

A simple graph with n vertices and k components can have at most n - K x n - k + 1 / 2 edges before going into the proof let us recall what we mean by a simple graph a simple graph do not add a simple graph does not admit self loops and multiple edges or parallel edges. Now we realize that this theorem is not going to work for a graph in general because even if I have got a graph with only two vertices I can keep on increasing parallel edges or self loops and blow up the number of edges.

So here I am allowed to have only one edge between two vertices if at all and no self loops are allowed and in this context we see we say that if we have if we have K components the maximum number of edges is given by n - K x n - k + 1 /2. Now we start off by assuming that we have a graph with K components and the number of vertices in the I^{th} component is N_I where I varies from 1 to K.

So let the number of vertices in the I^{th} component be N_i and I varies from.. K therefore we have N1 + N_2 + up to so on up to N_K = N we will use an inequality from algebra which is this that Σ_I = 1 to k $N_i^{\ 2} \leq n^{\ 2}$ - K - 1 2 N - K you will use this a little later. Now let us check this picture, so I have split up my graph into K components 1 $_2$ and K and inside these there is a connected sub-graph inside this there is another connected sub-graph inside this another connected sub-graph ensue or so on and the number of vertices is n_1 number of vertices $n_{\ 2}$ and here number of vertices NK.

I question that what is a maximum number of edges possible when you have got in 1 many vertices the answer is n_1 x n_1 - 1 /2 the question is why it is exactly the number of ways I can choose 2 vertices out of N_1 many vertices so that is n 1 choose 2. So I have got max n_1 x n_1 - 1 /2 many vertices over here sorry too many edges over here it is n_2 n_2 - 1 /2 many edges max so here it is N_K x N_K - 1 / 2 many edges.

So I have to sum up all these things then I will get some like this which is half of Σ I = I to k Ni I – 1 which is well equal to ½ of Σ I = 1 to K N_i^2 - ½ of Σ and I = 1 to K and I realize that I can use this in equality and if I plug in this inequality I am going to get ½ n^2 - K - 1 x 2 $_n$ - K - n / 2 because this sum is equal to N and finally if we simplify we will see that we'll get n - K n - k + 1 and which is the answer thus we have got an upper bound on the number of edges of a simple graph with n vertices and K components. These are more or less the results on connected graphs connected components that we study in this course and now we move on to another topic called operations on graphs.

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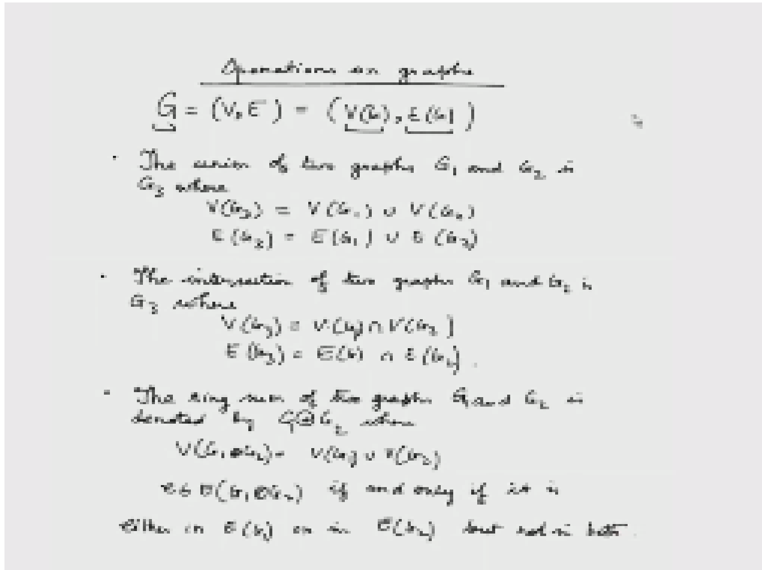

Now we can think of several operations on graphs when we consider graphs as objects these operations are ∪∩ then ring sums and then deletion fusion and so on. So I will define these operations one by one and try to provide some examples here when we have a graph G we will consider it as a ordered pair of the set of vertices and set of edges we can be even more specific and write V_G and E_G V_G is a set of vertices of the graph G and E_G is a set of edges of the graph G graph G is over here.

Now the ∪of two graphs G_1 and G_2 is G_3 where V of G_3 that is set of vertices of G_3 is equal to $V_{G_1} \cup V_{G_2}$ and G of G_3 is $E_G 1 \cup E_G 2$ this is straight forward the ∩ is also straight forward the ∩ of two graphs G_1 and G_2 is G_3 where V of D_3 is $V_{G_1} \cap V_{G_2}$ and E of G_3 is $E_{G1} \cap E_{G2}$, now we move on to another operation which is slightly more complicated than these ones that is the ring some the ring some of two graphs G_1 and G_2 is denoted by G_1 o + G_2 where V of G_1 o + G_2 is V of G_1 ∪V of G_2, so there is no change over here from ∪but now the change comes he belongs to e of G_1 o + G_2 if and only if it is either in E_{G1} or in E_{G2} but not in both. Now we come to the composition.

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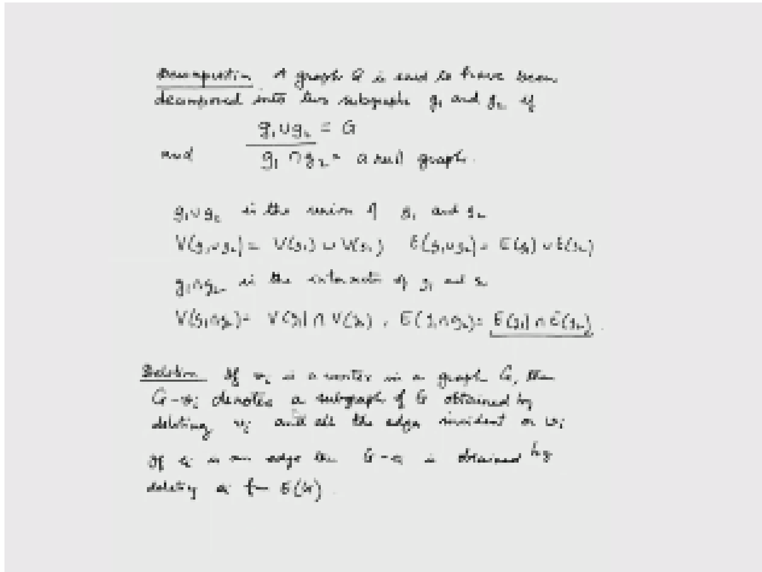

A graph G is said to be a graph G is said to have been decomposed into two sub-graphs G_1 and G_2 if $G_1 \cup G_2 = G$ & $G_1 \cap G_2$ equal to a null graph. Now the question is what do we mean by $G_1 \cup G_2$ $G_1 \cup G_2$ is the $\cup$of G_1 and G_2 that is what we denoted by G_3 in the definition, so V of $G_1 \cup G_2$ is V of $G_1 \cup$V of G_2 and E of $G_1 \cup G_2$ = E of $G_1 \cup$G of G_2 similarly $G_1 \cap G_2$ is the $\cap$ of G_1 and G_2 so V of $G_1 \cap G_2$ is V of $G_1 \cap$ V of G_2 and e of $G_1 \cap G_2$ is e of $G_1 \cap$ G of G_2.

So when I say that $G_1 \cup G_2$ is G that means that the $\cup$of the set of vertices of G_1 and G_2 is going to give me the set of vertices of G and that $\cup$of edges is going to give me the set of edges in G and when I say that $\cap$ is a null graph that means that there is no common edge between G_1 and G_2. Now we come to deletion if V_I is a vortex in a graph G then $G - V_I$ denotes a sub-graph of g obtained by deleting V_I and all the edges incident on V_I if V_I is an edge then $G - E_I$ is obtained by deleting E_I from E_G.

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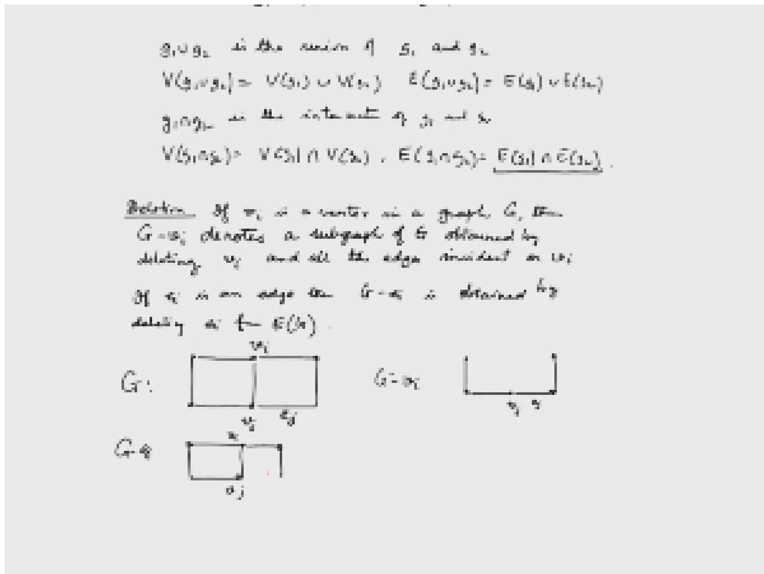

Now let us look at an example that suppose we consider a graph like this and suppose this is V_I this is V_J and this is let us say A_J now if I delete V_I then the graph $G - V_I$ will be like this like this whereas if I delete E_J the graph will be like this and lastly I have another idea or another notion that is fusion a fusion means that you can fuse two vertices and make it a one vertex and then all the edges which are incident on both these two vertices will be combined as incident edges on the new vertex.

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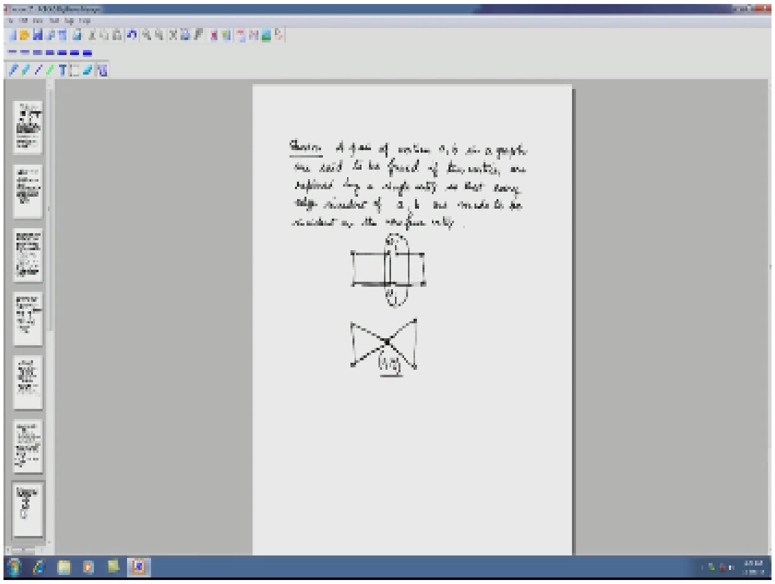

So fusion a pair of vertices A, B in a graph are said to be fused if two vertices are replaced by a single vertex, so that every edge incident on A B are made to be incident on the new fused vertex. Now let us see how fusion works we consider the previous graph like these this is V_I and V_J and then goes like this and suppose we want to fuse V_I and $V_{J,}$ so we shall make it a single vertex it will move like this and see that these two edges are now incident on this vertex and these two edges are now incident on these vertex so this is the fused vertex which can be read which might be denoted by V_I $V_{J,}$ this brings us to the end of today's lecture thank you.

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