INDIAN INSTITUTE OF TECHNOLOGY
ROORKEE

NATIONAL PROGRAMME ON TECHNOLOGY
ENHANCED LEARNING
(NPTEL)

Discrete Mathematics

Module-04
Discrete Probability
Lecture-02
Probability, conditional probability

With
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In this lecture I will start with an event.

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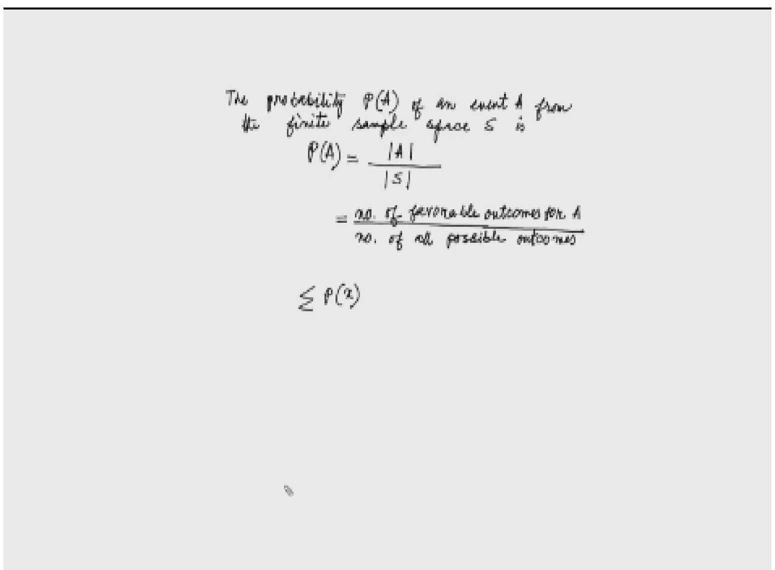

The probability P(A) of an event A from the finite sample space is given by the ratio between the cardinality of A and the cardinality of S. So it can be written in this way also number of favorable outcomes for the event A divided by number of all possible outcomes to find the probability of an event A, we sum all the probabilities assigned to the sample points in A. This sum is called the probability of A, and is denoted by P(A) which is like this.

Now from this we can easily say one thing that probability of the whole sample space is always 1. So we can write in this way that summation of P(X) X belonging to S=1.

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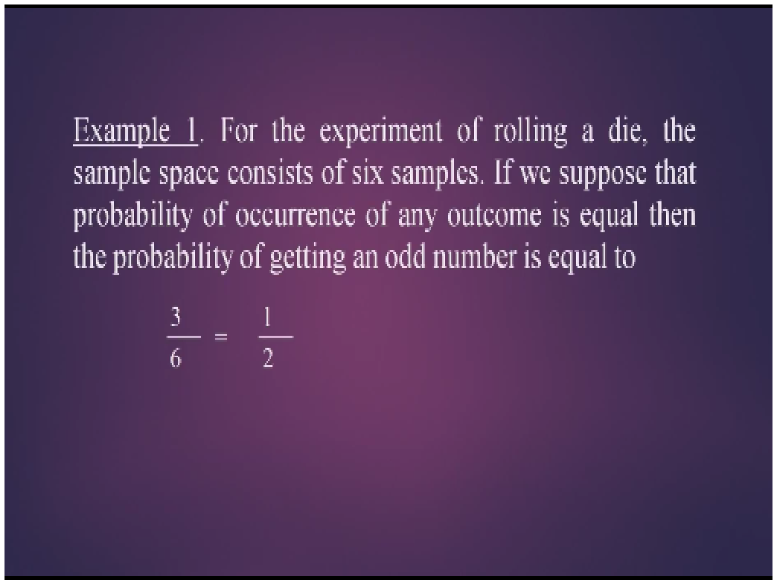

So let us consider one example in this context for the experiment of rolling a die, the sample space consists of 6 samples. If we suppose that probability of occurrence of any outcome is equal, then the probability of getting an odd number is equal to 3/6 that is 1/2, because there are only 6 possibilities in the sample space, so sample space contains 6 sample points. So that is why the denominator is 6, and the numerator is 3, because the number of favorable cases will be 1, 3, & 5 because these are the odd numbers available in a die.

So that is why it will be 3/6 which is 1/2. Now let us consider another example it is very, very important example let us.

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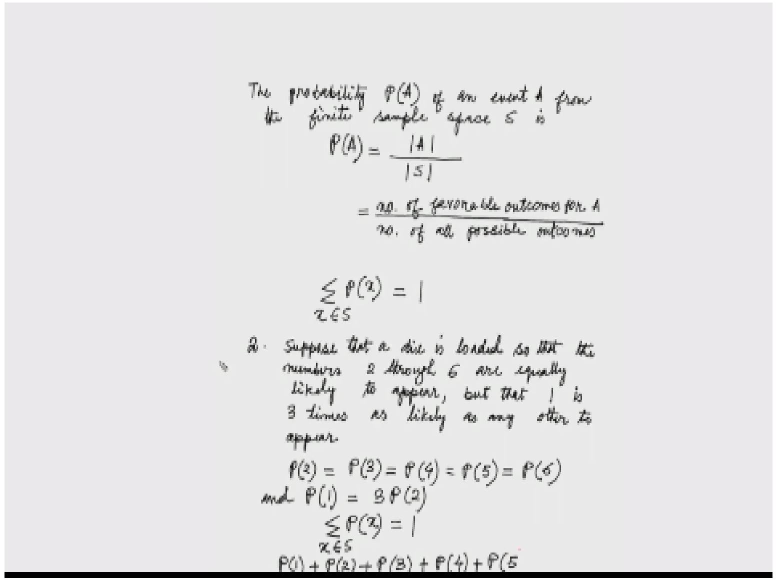

Suppose that a die is loaded, so that the numbers 2 through 6 odd equally likely to appear, but that one is three times as likely as any other 2 appear. Now to model this situation we should have P(2)=P(3)=P(4)=P(5)=P(6). And another condition is probability of 1=3 into probability of 2. Since we know that summation P(X) X belonging to S=1 that is why P(1)+P(2)+P(3)+P(4)+P(5)+P(6)=1.

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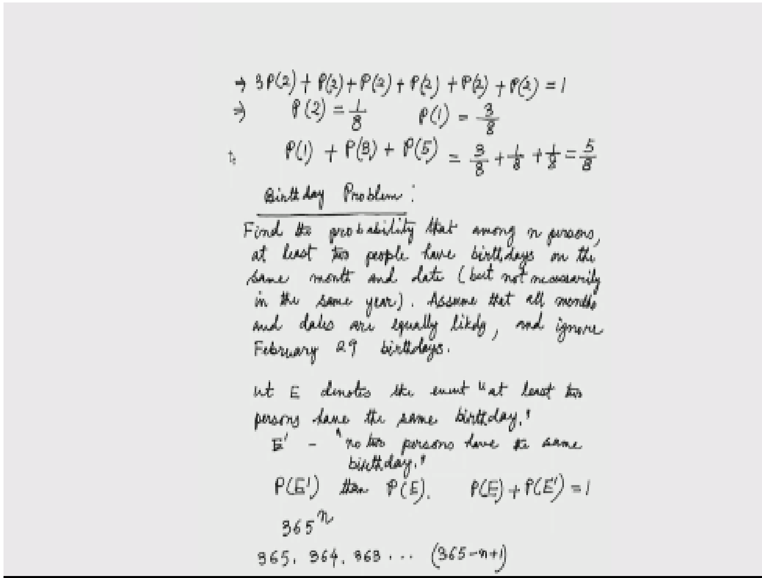

So we can write that the 3P(2)+P(2)+P(2)+P(2)+P(2)+P(2)=1. So from this we can find the value of P(2)=1/8. So we can find P(1)=3/8. So now if we have to find the probability of an odd number it will be P(1)+P(3)+P(5)=3/8+1/8+1/8=5/8. Another very known and important problem we can discuss that is known as birthday problem. So the problem is like this find the probability that among in persons at least two people have birthdays on the same month and date, but not necessarily in the same year.

Assume that all months and dates are equally likely and ignore February 29 birthdays. So let E denotes the event that at least two persons have the same birthday. So the event E' will be the event that no two persons have the same birthday. Now it is easier to compute P(E') than P(E). And we can find P(E) from the fact that P(E)+P(E')=1. Since all months and dates are equally likely, and we are ignoring February 29 birthday.

The size of the sample space will be 365^n, because there are n persons, so that is why the sample space will contain 365^n sample points. The first person's birthday can occur on any one of 365 days, if no two persons have the same birthday the second person's birthday can occur on any day except the day of the first person's birthday. Therefore, the second person's birthday can occur on any one of remaining 364 days.

So first person's birthday can occur in 365 days, then second person's birthday can occur on any one of remaining 364 days, then similarly if we proceed third person's birthday can occur in 363 ways or we can say it can occur on any one of the remaining 363 days. In this way if we proceed we can say that the size of the event that size of the event E’ that no two persons have the same birthday, no two persons have the same birthday the size of this event will be 365 x 364 x 363 in this way the last one will be 365 - n + 1 because they are in person’s.

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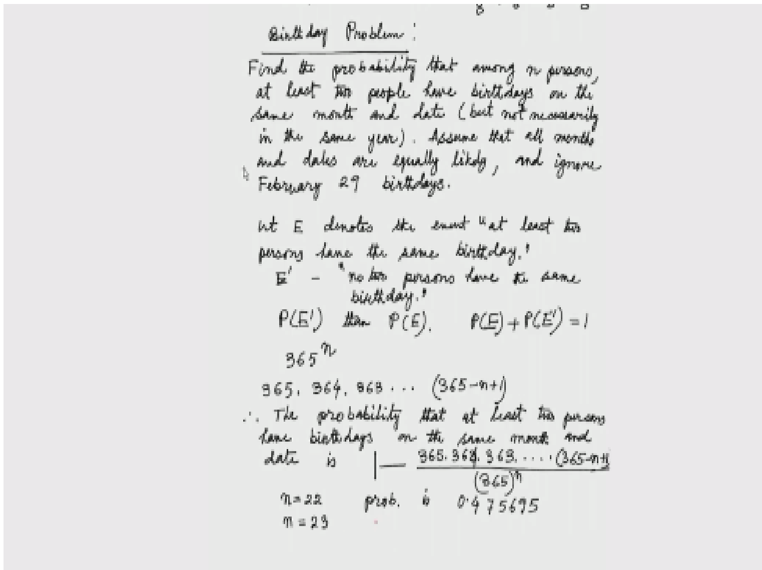

Therefore, the probability that at least two person’s have birthdays on the same month and date is 1 - 365 x 364 x 363 in this way 365 - n + 1/365^n. Now if we consider n = 22, so that means if there are 22 persons, then this probability will be, then the probability is 0.475695, if n = 23, then the probability is 0.507297.

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Thus, if n is greater than equal to 23 the probability is greater than 1/2 that at least two persons have birthdays on the same month and date. So from this problem this is our conclusion, so that is like this if there are more than or equal to 23 persons the probability is greater than 1/2 that at least two persons have birthdays on the same month and date.

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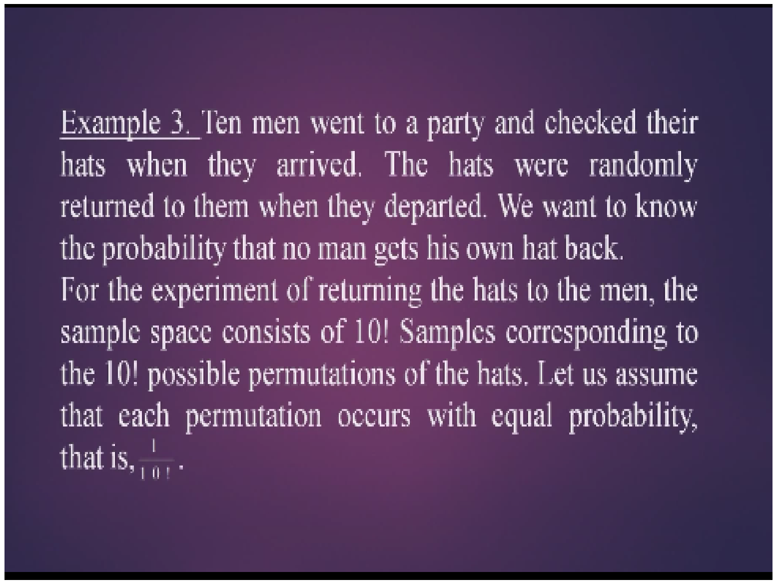

Here is another example this is also very interesting example 10 men went to a party and checked their hats when they arrived. The hats were randomly returned to them when they departed. We want to know the probability that no man gets his own hat back. For the experiment of returning the hats to the men, the sample space consists of 10! Samples corresponding to the 10! Possible permutations of the hats let us assume that each permutation occurs with equal probability, that is, 1/10!

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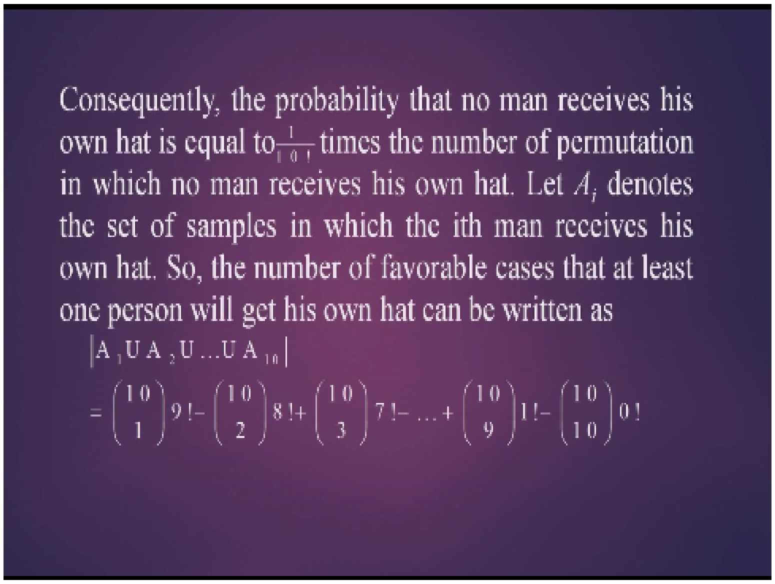

Consequently, the probability that no man receives his own hat is equal to 1/10! times the number of permutation in which no man receives his own hat. Let if Ai event, this is an event, let Ai denotes the set of samples in which the i^{th} man receives his own hat. So, the number of favorable cases that at least one person will get his own hat can be written as this cardinality of A1 U A2 U … U A10.

So now how can we find this cardinality, so that is the next challenge. So let us consider, so we have A1 U A2 U … U A10, so this can be written as ∑i=1 to 10 cardinality of Ai - ∑i<j of the cardinality Ai ∩ Aj + ∑ i<j<k cardinality Ai ∩ Aj ∩ Ak – so on, so it will continue in this way the last entry will be -1^{10-1} = 9. Cardinality of A1 ∩ A2 ∩ so on ∩ A10, so this is coming by inclusion exclusion principle which has already been covered in set theory.

So we have like this, now what is the summation of the cardinality Ai, so that means here we have to find the number of ways that one person will get the hat correctly, so that means number of ways one person will get his own hat. So let us come to now this slide, here ∑i=1 cardinality of Ai, the number of ways that one person will get his own hat is 10 choose 1. So for each of these cases the other arrangements can occur in 9! Ways.

So that is why the resulting number of ways will be 10 choose 1 into 9! In this way the next one is that that two persons will get their own hat and that number is 10 choose 2, and for each of these ways the other arrangements can occur in 8! Ways. So the resulting number of ways will be

10 choose 2 into 8! ways. So in this way the last one will be 10 choose 10 0 x 0! So there ten people will get their own hat in 10 choose 10 ways.

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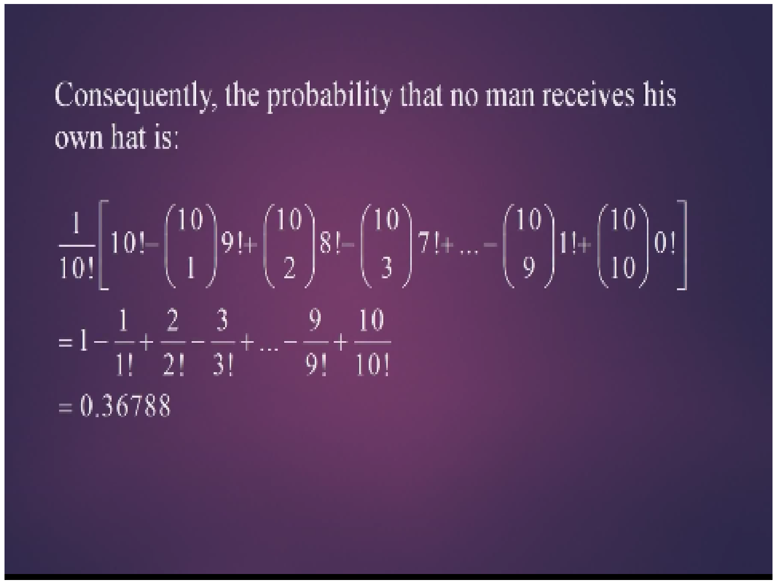

So in this way if we find the cardinality of A1, A2 and so on U A10, then we can find the probability that no man receives his own hat in this way. So it will be 1/10! X 10! - the cardinality of A1, A2 … U A10, so that cardinality which is nothing but 10 choose 1 x 9! X 10 choose 2 x 8! ... - 10 choose 10 x 0! So the term inside the bracket is coming in this way that number of all possible cases which we have already noted down that is 10!.

So 10! - that cardinality of A1, A2 U A10 will give this term inside this bracket which will give the number of ways that no man receives his own hat. And we are dividing it by 10! to get the probability of this event. So it will be 1 – 1/1!+2/2!-3/3! And so on up to +10/10!. So if we calculate this we will get the value 0.36788.

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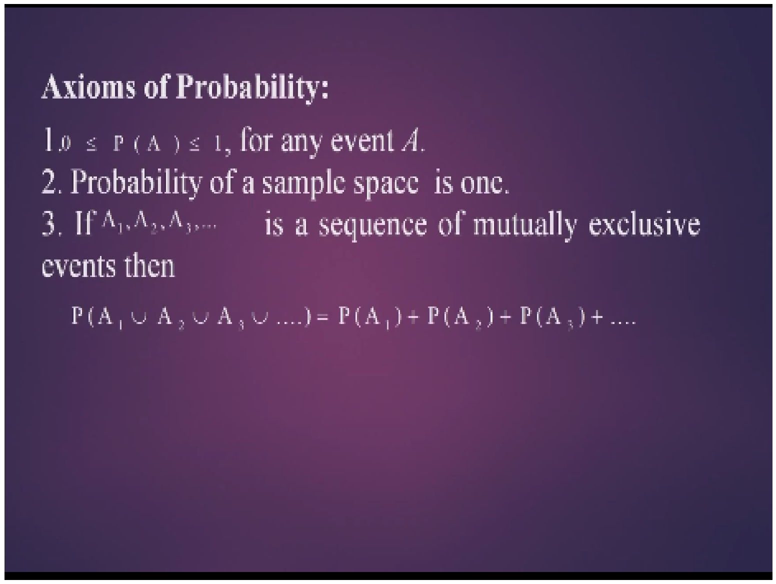

Now there are three axioms of probability, 0 less than equal to P(A) less than equal to 1 for any event a this is the first axiom, second axiom is probability of a sample space is 1 which we have already mentioned. Now if A1, A2, A3 and so on is a sequence of mutually exclusive events then P(A1 U A2 U A3 U … = P(A1) + P(A2) + P(A3) + …. So this sum of the probabilities of this events individual probabilities of this events.

So the first axiom is obvious, second axiom is also obvious, we have already mentioned now what about the third one. Before going to this axiom, let us prove one result here.

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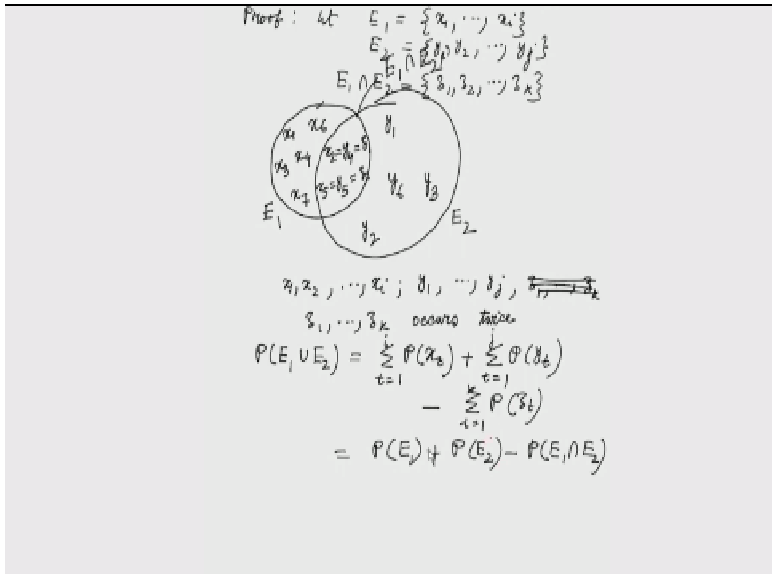

Let E1 and E2 be events then P(E1 U E2 = P(E1) + P(E2) – P(E1 ∩ E2). The proof is like this let E1 = x1, x2 …. xi, E2 = y1, y2 …. yj. Then E1 ∩ E2 is considered as Z1, Z2 … Zk. So let us describe this by Venn diagram. So it will be like this, so this is E1 event, this is E2, and this one is E1 ∩ E2. So E1 is having the members x1, x2, x3, x4, x5, x6, x7, and E2 is having six members y1, y2, y3, y4, y5, and y6 and E1 ∩ E2 will be like this it is having two members z1 and z2, and z1 is same as x2 which is same as y4.

So x2 and y4 are same, so those are denoted as z1, and x5 and y5 are same so this two are denoted by z2. So E1 ∩ E2 will contain two members z1 and z2. And assume that each set element is listed exactly one time per set, then in the list x1, x2 … xi, y1, y2 … yj, z1, z2 … zk, so in this list x1, x2 … xi, y1, y2 … yj, z1, z2 … zk occurs twice. So from this it follows that P(E1 U E2) = ∑P(Xt) t=1 to i + ∑ P(Yt) t = 1 to j - ∑ P(Zt) t = 1 to k, and this is equivalent to P(E1) + P(E2) – P(E1 ∩ E2).

Because we have already seen that z1, z2 … zk occurs twice if we list x1, x2 … xi, y1, y2 … yj, so that is why one time we are subtracting it from this list, so we will get P(E1 U E2).

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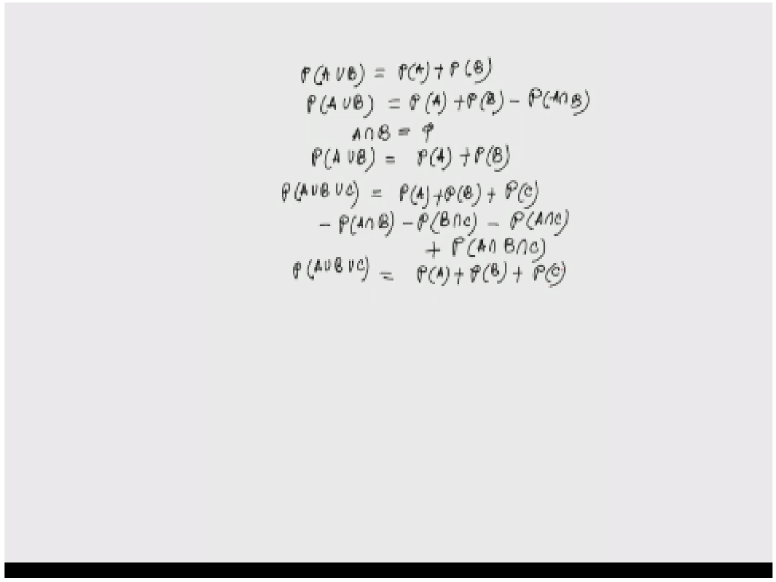

So once it is proved, then we can consider the third axiom of probability which says that P(A U B) = P(A) + P(B), when A and B are mutually exclusive events. So this will come from the previous one previous result that is, we know that P(A U B) = P(A) + P(B) – P(A ∩ B), so this result is in general. Now if A and B are mutually exclusive we know A ∩ B = ϕ, and that is why P(A U B) will be simply P(A) + P(B), because P(A ∩ B) will become 0.

In this way if there are three mutually exclusive events ABC then from the inclusion exclusion principle we know that P(A U B U C) = P(A) + P(B) + P(C) – P(A ∩ B) – P(B ∩ C) – P(A ∩ C) + P(A ∩ B ∩ C). So since ABC are mutually exclusive events all these probabilities will be equal to 0 and that is why we will get A U B U C = P(A) + P(B) + P(C).

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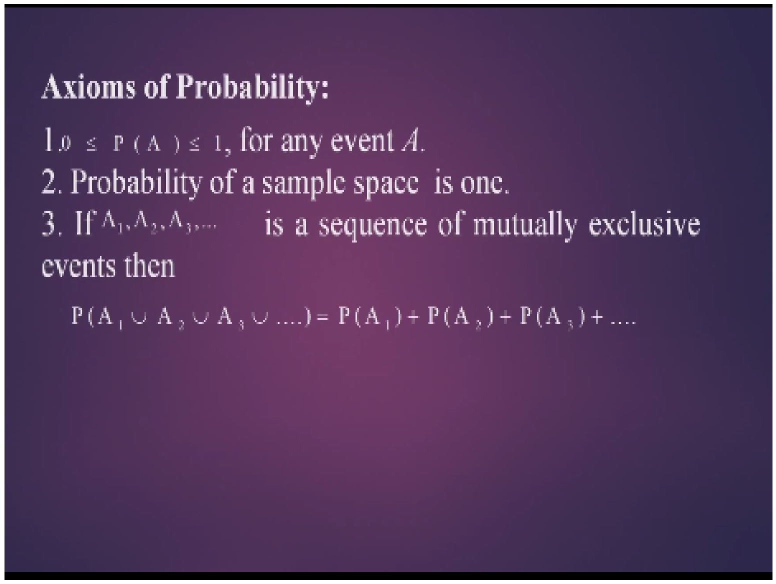

So let us consider the third axiom here it says this that there are mutually exclusive events A1, A2, A3. So in finite number of mutually exclusive events so P(A1 U A2 U A3 U …. So this will be the sum of probabilities individual probabilities of A1, A2, A3 and so on, because the same reason, the probability of the event A1 ∩ A2, A1 ∩ A3 will all vanish, and that is why we will get only the sum of this probabilities.

So let us discuss now the conditional probability, the probability that event A occurs given that event B has already occurred is defined as the conditional probability of event A given the occurrence of event B which is denoted by probability of A/B.

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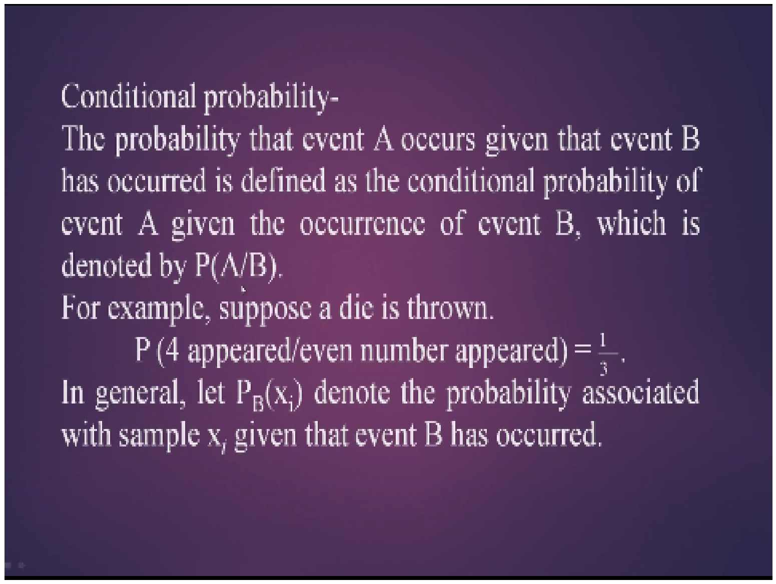

So this for example supposes a die is thrown, then if it is asked to find the probability that 4 will appear given that even number has occurred. So the probability that 4 appeared given that even numbered appeared will be equal to 1/3, because even numbers there are only 3 even numbers 2 4 6 so among this the probability that 4 will occur with probability. So probability that 4 will occur will be 1 / 3.

So in general let provide P subscripts P of X I denote the probability associated with exam with sample X_i given that event B has occurred.

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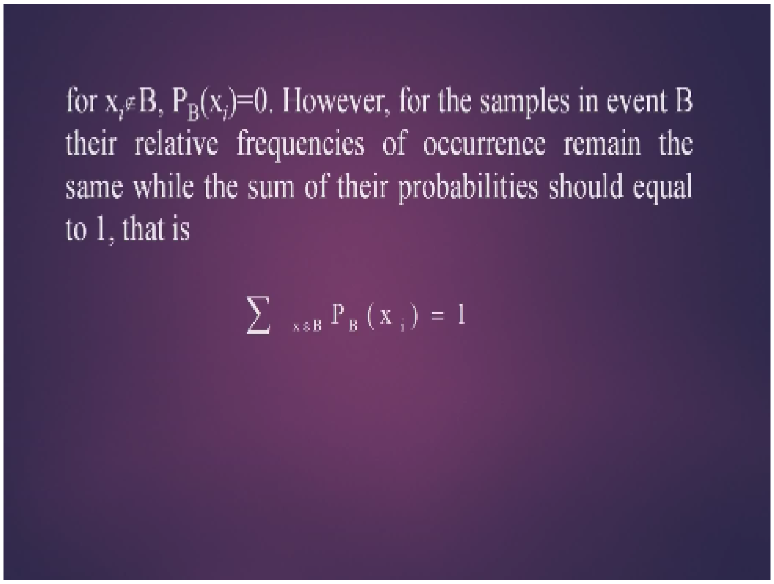

Now for X I does not belong to B probability subscripts P of X I P_B= 0, however for the samples in event B their relative frequencies of occurrence remain the same while the sum of their probabilities should equal to that is summation of probability subscripts P of X I such that this X I belongs to B = 1.

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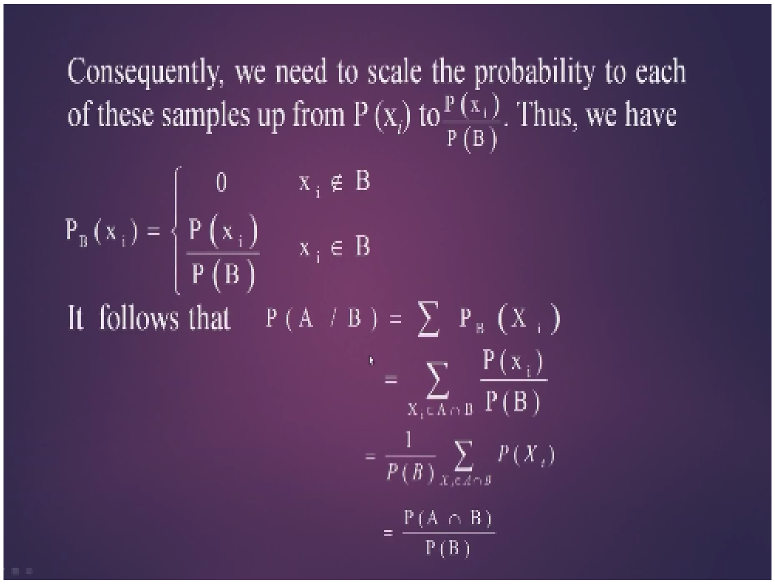

Consequently we need to scale the probability to each of these samples up from $P(X_I)$ to $P(X_I)$ / P(B) thus we have $P(X_I) = 0$ when X_I does not belong to B and $P(X_I)$ / p (B) when X_I belongs to B it follows that probability of a given B =∑ P B of X_I = ∑ X_I belonging to a ∩ B $P(X_I)$ / probability of B.

Now 1 / P(B) we can take out of this summation, so we will get 1 by P (B) into ∑ X_I belongs to ∩ B $P(X_I)$ = P(a ∩ B / probability of B, so we are getting the definition of conditional probability in this way, so probability of a given B is probability of a ∩ B / P(B) . So let us take one example in this context three dies where rolled given that no two faces were the same what is the probability that there was and s.

Let a denote the event that there was an s and B the event that no two faces were the same, now probability of B will be 6 that is nothing but 6 x 5 x 4 why is it this the first one, so what is B actually be the event that no two faces were the same. So the first eyes can give 6 outcomes, so there are six possibilities, now one outcome has already occurred from the first eyes and that can occur in six ways.

So once that occurs the second dice will give five possible outcomes because the number which has already occurred in the first dice that is discarded from the list, so that is why it will be 6 x 5, then once the first one and second one have occurred the third dice will give the outcomes in four possible ways because the first outcome and second outcome we have to discard from the list so there will be four outcomes remain.

So that is why we will have total number of outcomes 6 x 5 x 4 which is 6_{p3} and all possible cases what is the number of all possible cases it will be 6^3 okay. So 3 dice are there so that is why 6 x 6 x 6, 6^3 next we have to find the probability that A ∩ B that is both a and B will occur is the event that there was an ace and B the event that no two faces were the same, now if one is already fixed one is an ace so that is why there will be only 5 possibilities because a and B both will occur.

So that is why that outcome is gone from two dices, so that is why it will be 5_{p2} 3 x 5 p_2, so three dice are there so for each of this case we will have 5_{P2} possibilities so that will be so total number of cases will come 3 x 5_{P2} and the denominator is 6^3 because this is the number of all possible cases. So probability of a ∩ B will become 3 x 5_{P2} thus probability of a given B will become 3 x 5_{P2} / 6_{P3} if we calculate this we will get ½, so the conditional probability will be ½. So this is the end of this lecture thank you.

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