INDIAN INSTITUTE OF TECHNOLOGY
ROORKEE

NATIONAL PROGRAMME ON TECHNOLOGY
ENHANCED LEARNING
(NPTEL)

Discrete Mathematics

Module-03
Mathematical Induction
Lecture-01
Mathematical Induction (1)

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In today lecture we will study the principles of mathematical induction.

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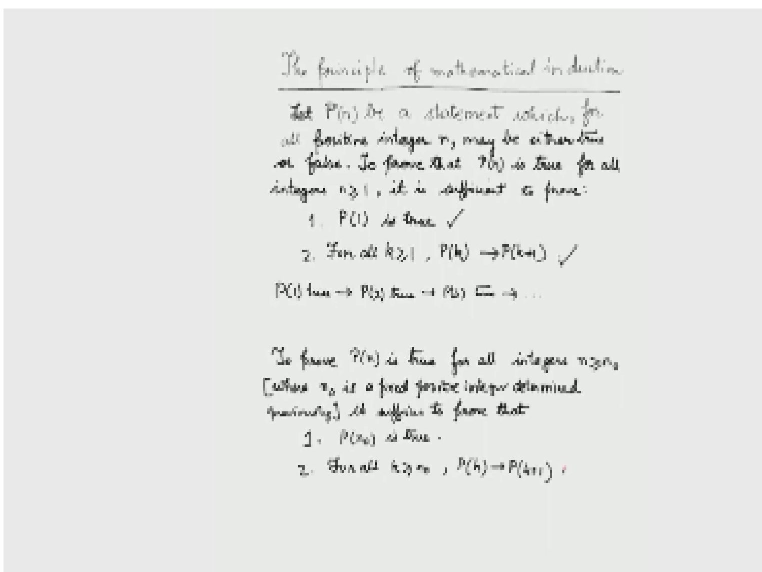

Now the principle of mathematical induction provides us a very powerful technique to prove several mathematical results. The basic idea is that suppose we have a statement or more precisely a predicate which depends on positive integer value n and we expect it to be true or false for all n greater than or equal to certain fixed positive integer.

Then what we can do is to prove this statement for n equal to that fixed integer and assume that the statement is true for some positive integer k which is greater than or equal to that fixed integer and assuming that it is true for k proves that the statement is true for k+1. Now let me write formally. Let P(n) D a statement which for all positive integer n may be either true or false.

To prove that P(n) is true for all integers n greater than or equal to 1, it is sufficient to prove 1 P(1) is true to for all k greater than or equal to 1 P(k)→P(k+1). Now thus we see that if we start from 1 and suppose P(1) is true and suppose for all k greater than or equal to 1 P(k)→P(k+1), this means that if P(k) is true, then P(k+1) is true. Now if we can prove these two facts, then since P(1) is true and P(k)→P(k+1) therefore P(2) is also true. Since, P(1) is true P(3) is also true.

So we will start a chain like this that P(1) is true implying P(2) is true which in turn implies P(3) is true and so on. And this is all for the two facts that we have already proved that P(1) is true and for k greater than or equal to 1 P(k)→P(k+1). Now what we observe over here is that this number one is very specific and we can relax the situation a little more. So it may so happen that some statements may not be true for 1, 2, 3 after you have fixed positive integer.

But after that the statement may be true for all the other integers greater than that specific integer. In order to bring this slight generalization into our framework, we state this whole principle in a slightly different way. We state that to prove P (n) is true for all integers $n \geq n0$ where n0 is a fixed positive integer determined previously it is a to prove that 1) P (n0) is true to for all $k \geq n0$ P(k) → P(k+1) so this is our slide generalization of the principle of mathematical induction that we stated in the beginning of the lecture now we want the requires steps in a proof which uses the principle of mathematical induction.

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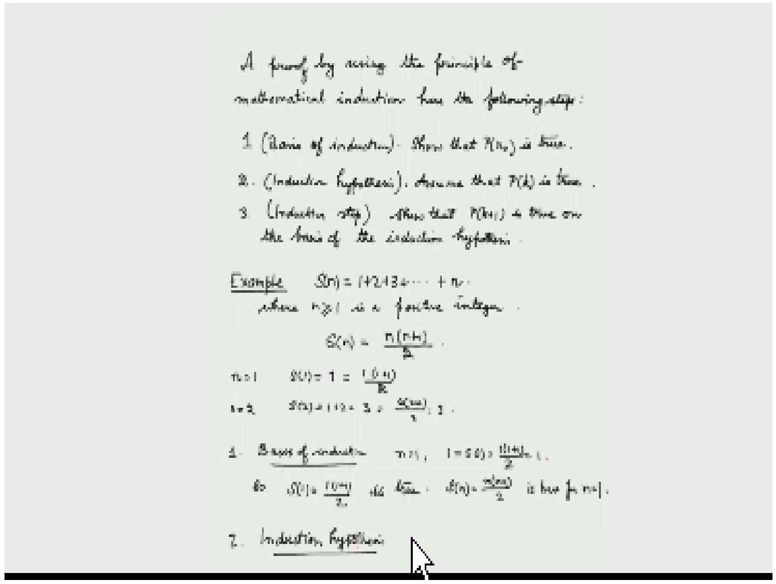

A proof by using the principle of mathematical induction has the following steps 1) the 1st step is called the basics of induction in the basics of induction we have to show that P (n0) is true if we are unable to show this then it cannot start induction because it will be meaningless 2) induction hypothesis that induction hypotheses is a hypotheses that P (k) that statement that we get by putting the value of n = ka is true.

So we assume that P(k) is true 3) inductive step show that P(k+1) is true on the basics of the induction hypotheses if we can successfully complete these 3 steps then we will have a proof by using mathematical induction now let us look at one example now we are considering the sum of first n positive integers let us write S(n) = 1 +2+ 3+ and so on up to n now suppose we have a conjunction that S(n) is = n (n+1) / 2.

And suppose you want to prove this first of all let us see that for small values of n this formula works so for example if I put n = 1 then S1 = 1 which is equal to 1 into 1 + 1 /2 n = 2 S2 = 1+ 2 which is = 3 and if you look at the formula it should be 2 2+1 /2 this is also = 3 therefore we see that at least for n = 1 and 2, Sn = n+n/2 sorry n+n+1/2 works so if we can form the basics of induction 1 basics of induction for n =1.

1= S1 = 1 into 1 + 1 / 2 which is equal to 1 so S1 = 1 1+/2 is true or in other words Sn = 1 n into n + 1 / 2 is true for n = 1 now we move on to the induction hypothesis right the induction hypothesis we have to choose or we will be say that suppose for k > = 1 sk = sk into k + 1 /2 is true this is my inductive hypothesis, now we come to the third point which is the inductive step what we do here we will start of case a +1 and write Sk + 1 as it is defined this is 1 + 2 +3 + n so on + k + 1 now here we observed that we can always some this first k entries by using that formula that we have already assumed therefore I can write this is equal 1 + 2 + 3 and so on up to k + k + 1.

This is equal to K into K + 1 /2 + K + 1 and that we will sum this expression to get 2 k into k + 1 +2 times k +1 = 2 in the denominator and the numerator K + 1 into k + 2 the numerator can be written as 2 k + 1 and then k + 1 + 1 thus we see that the formula that we wrote over here holds for K + 1 if you assume that it holds for K, so this means very strictly speaking Sk = k into k + 1 / 2 implies Sk + 1 = k + 1 into k +1 /2 thus this formula is said equal to n into n + 1 /2 true is going to be true for all positive integers > = 1 which is because you know that it is 2 for 1 and we know that it is 2 for k.

Which is going to be k for k + 1 it is going to be true for true since it is T for true then it will be going through for 3 and so on as possible integers, now we just look at another example where problem is solve is by using mathematical index.

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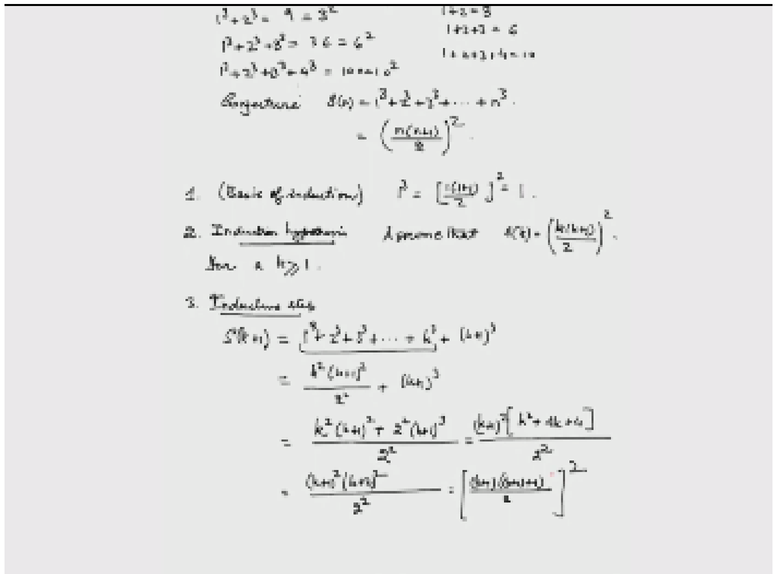

The problem is as follows find and true a formula for the sum of the first n cubes that is 1^3 $+2^3$ $+3^3$ and so on up to n3 well this is of course say difficult problem in the sense that nobody has given a formula this is the formula is given I can quickly check what I can do by using mathematical induction for that formula we need some imagination and something which cannot be really quantify, let is check by experiment what happens and so $1^3 = 1^2$ $1^3 + 2^3 = 9$ which is equal to 3^2.

$1^3 + 2^3 + 3^3 = 46$ which is equal to 6^2 $1^3 + 2^3 + 3^3 + 4^3 =100$ which is equal to 10^2 now if you go on in this way we will find that whenever we are taking sum of cubes of n which is becoming a part if square we can check few more task then somehow we can argue of course without any possible complex cube that probably whatever the sum may be it is part of square but square of what that we do not know again if we just sum up to n terms we will see that 1 = 1 1 + 2 = 3 1 +2 +3 = 6 1 + 2 + 3+ 4 = 10 surprisingly.

We see that this sum of cubes is looks like as it square of the sum of that usual sum, now of course this is not a prove but this may lead us to a connector like this well this is just a conjecture now what we can see that it is a very neat conjecture and it is what is checking by using mathematical induction whether this is indeed true for that we will start again from basis of induction, here we see that 13 is indeed 1 1 + 1 / 2 ^{2} = 1 thus we have prove the basis of induction and now we come to induction hypothesis. Now induction hypothesis we assume that sk = k x k + 1 / 2 ^{2} yeah that is it for the k ≥ 1 ^{3}.

Now we have inductive step they can like before we consider k + 1 so if I have sk + 1 and write explicitly the sum I will get $1^3 + 2^3 + 3^3$+ and so on up tok^3 + k + 1^3 again I absorb that the first k terms are essentially sk and therefore since have assume sk = k + 1 k x kl + 1/ 2 ^{2} I can write this as k^2 k + n1 ^{2} / 2^2 + k + 1 ^{3} and of course I can simplify this expression as putting denominate 22 in the denominator and on the numerator we have k^2 k + 1^2 + 2^2 k +1 ^{3} this gives us 22 and here we will have k + 1 ^{2} k^2+ 4 k + 4.

Now this gives me 2^2 k + 1^2 and this is k + 2 ^{2}+ by doing again a small manipulation we get a k + 1 and k + 1 within bracket +1 2, so the expression k + 1^2 x k + 2^2 / 2^2 = k + 1 x k + 1 within bracket + 1 / 2 and the whole expression is squared and we see that is conference exactly with the formula that I predicted that is sn = n + 1 n x n + 1 / 2 ^{2} therefore we can conclude that the conjecture is true.

Now we move on to one more application of mathematical induction and of course there are several applications of mathematical induction and here will use this mathematical induction to prove something related to logic specifically de Morgan's laws that we have studied in previous lectures. Let us go to the example.

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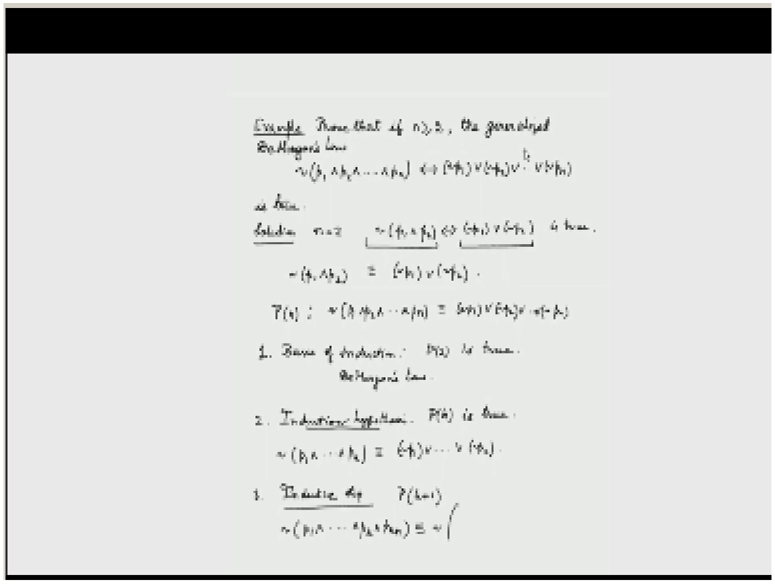

Prove that if $n \geq 2$ then the generalized de Morgan's law that is 0(p1) and p2 and so on and pn by conditional 0(p1) or 0(p2) or 0(pn) is true, have we gone to the solution. Now here we see that it is somewhat meaningless to start from $n = 1$ it is from $n = 2$ because the de Morgan's law the one that we have already studied involves two prepositions.

So for $n=2$ we have nor r p1 and p2 by conditional not of p1 or not of p2 is 2 in fact we can use our previous knowledge to write that these two statements not intersection p2 and not intersection with n here and not of p1 and p2 because the bi conditional is tautology can be prepositions therefore we can write that p1n p2 not of that is equivalent to not of p1 or p2 so this essentially calls induction hypothesis I am sorry this essentially falls the bases of induction.

So I write the statement as this I have got a statement pn which is not of p1 p2 and pn equivalent to not of p1 or not of p2 or not of pn first step bases of induction 2 is true which essentially de Morgan's law now the second step is induction hypothesis states that pa is true which means that not of e1 and so on to pk is equivalent to not of p1 and so on up to not of pk now equal to induction step now we start with pk+1 which is not of p1 and n up to pk and then n pk+1.

And now we see that this is equivalent to not of p1 and pk we can put a bracket and enclosing pk up to pk that is because after all we know that then we put pk+1 and once we have this if we see that we have one preposition p1 and up to pk and then another voice and not of that and we can use de Morgan's law for the original de Morgan's law the reason is that we have proved e2 is true therefore we will write not of p1 and so on up to pk or not of pk+1

And now the induction hypothesis to write that not of p1 and up to pk is not of p1 or not of p2 and not of pk we can off course put and enclose this by bracket and then it is not of pk+1 and now we know that we can remove brackets so we can write not of p1 or not of p2 or so on pk or pk+1

Because we see that I have to write equivalent over here so we see that pk+1 is true so pk implies pk+1 for k<2 thus we see that pn is true for all n is equal to 2 here we see that we have used is slightly different technique in that we are not only depending on the truth of pk but we are depending on truth of pk/2 in the basics of induction statement by this we end today lecture thank you.

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Acknowledgement
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