Approximate Reasoning using Fuzzy Set Theory
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Lecture - 09
Boolean Algebra of Sets

Hello and welcome to the next of the lectures under this course titled Approximate Reasoning using Fuzzy Set Theory. A course offered under the NPTEL platform.

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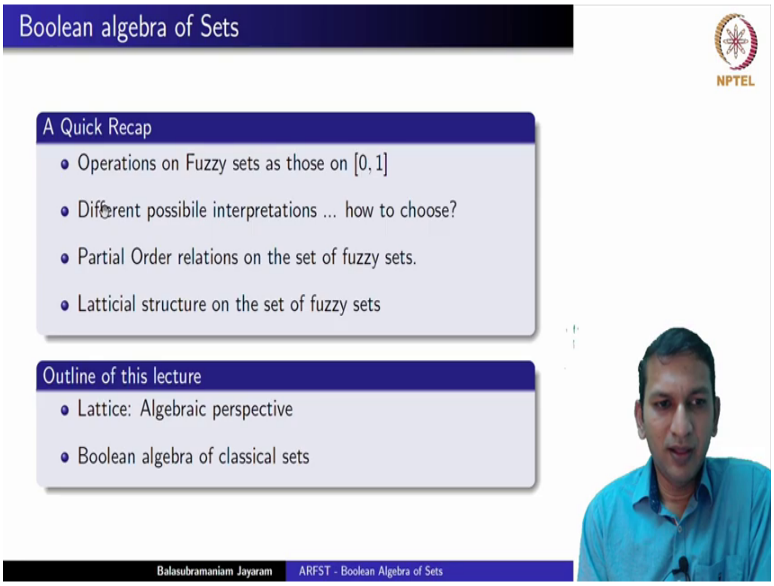

In the previous lectures during the week, we have seen how operations on fuzzy sets can be seen as those on the unit interval 0, 1. We saw that there were different possible interpretations for the operations and we were left wondering how to choose among them. In a quest to answer this question, we have seen how to define some partial order relations on the set of fuzzy sets, and in fact, we have gone on to see a latticial structure on the set of fuzzy sets.

However, if you want to understand more about the concept that we have generalized, it is perhaps wise to look at what is available with the concept or the object which we have generalized it from. In that sense what we will do is, so, far we have seen lattice from an order theoretic perspective. In this lecture, we will look at lattice from an algebraic perspective and move towards Boolean algebra which is a very specialized lattice and see that if you if you are given a set x the set of all subsets of x can be looked at as a Boolean algebra.

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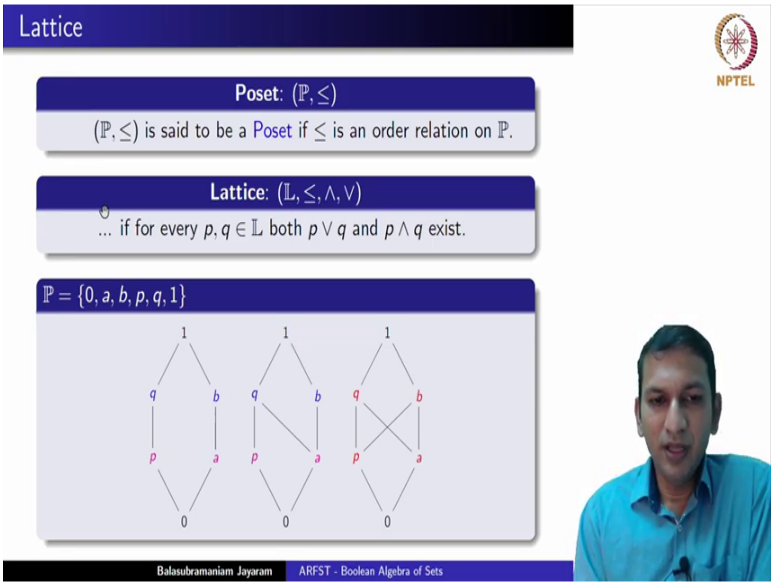

A quick recap of looking at lattice from an order-theoretic perspective. We know that a poset is a set with a binary relation defined on it, wherein the binary relation is an order relation; that means, it is reflexive, anti-symmetric and transitive. We also saw that a poset becomes a lattice, if every pair of elements has a supremum and then infimum. Supremum is the least upper bound with respect to the order relation we have defined on it, and infimum is the greatest lower bound that we have with respect to the order relation we have defined on the set, these are the familiar posets by now.

We know that on the same set P we can have different ordering relations, among these three posets we have also seen that the first two. The one of the left hand and the center they are actually lattices, while the last one on the extreme right is not a lattice, this was clear to us. If you take the pair of elements P and a, while they have a meet while they have an infimum which is 0, they fail to have a supremum. Similarly, if you take the pair of elements q and p they have a supremum, but they fail to have an infimum.

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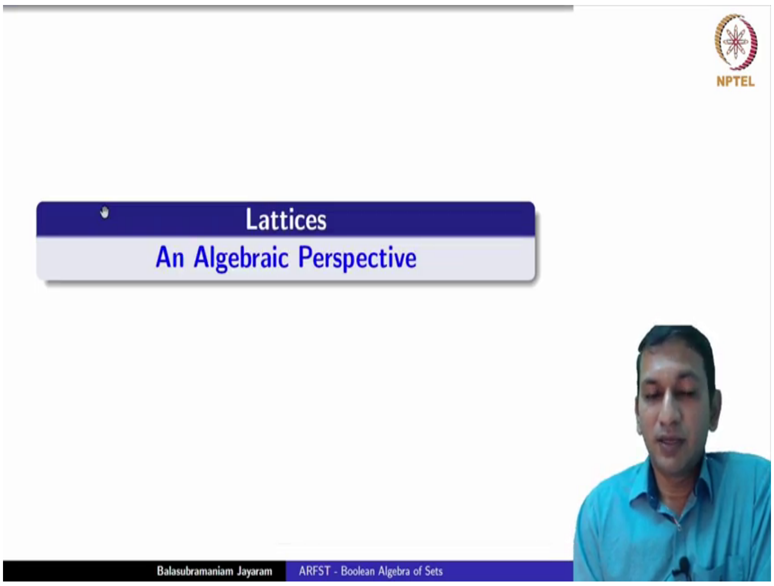

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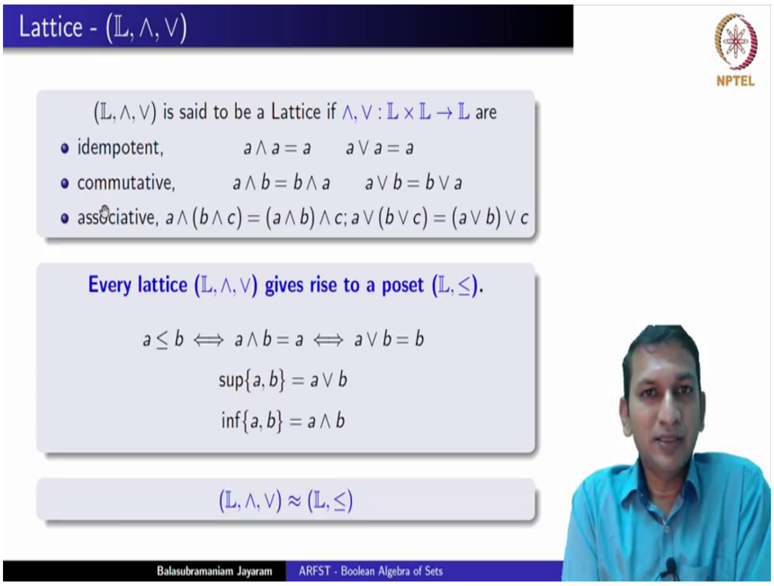

Let us look at lattices from an algebraic perspective. Now, what is an algebra? An algebra in general terms consists of a set along with a few operations, it could be binary, unary or ternary and we could have many operations on it. For a lattice we have a set with two operations and it is said to be a lattice if these two operations which are closed on L; that means, they are binary operations on L to L, they are idempotent, by idempotence this is what we mean.

When we take these two operations and operate on the same element, we should get the same element. It should be commutative we understand what this commutativity and it should be associated. So, if you have a set L with two operations two binary operations which are closed on the set; that means, they are these operations should act on L, binary operation should act on L and its codomain also should be L. Further this operation should be idempotent, commutative and associative, well we have seen lattice from an order theoretic perspective.

Now, we have defined it as an algebra is there a relationship between these two. In fact, it can be shown that every lattice gives rise to a poset; that means, on the set L we can define a relation a binary relation which turns out to be a partial order relation. How do we define this? This is how we define a partial order relation on the set L.

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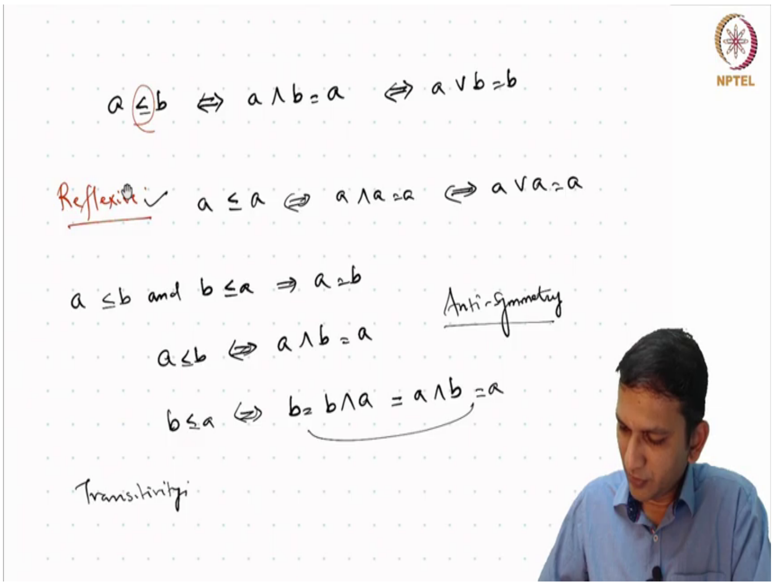

Now, let us look at whether this really is a partial order on the set L. Let us consider the relation, this is how the relation is defined a is said to be less than or equal to b. If let us call this operation meet and let us call this operation join. Now, we want to check if this relation is indeed a partial order relation; so, the first thing that you will ask is, is it reflexive?

Now if it were to be reflexive, we want that for every a, a should be less than or equal to a. Now, what does this mean from the definition we see that this means, a meet a should be a also a join a should be a, but we know this from the idempotence of these two operations; so, it is reflexive. Now, the next question is, is it anti-symmetric? So, for anti-symmetry what we want is if a is less than or equal to b and b is less than or equal to a they should actually imply that a is equal to b.

But now when you say a is less than or equal to b; this means, a and b is actually a. When you say b is less than or equal to a, this means b is actually b meet a; now, but b meet a we know that meet is commutative. So, if it is commutative, we can write this as a meet b, but we know that a meet b is actually a from these two we get that b is equal to a. Thus, we see that this relation is also anti-symmetric or satisfies the anti symmetric problem.

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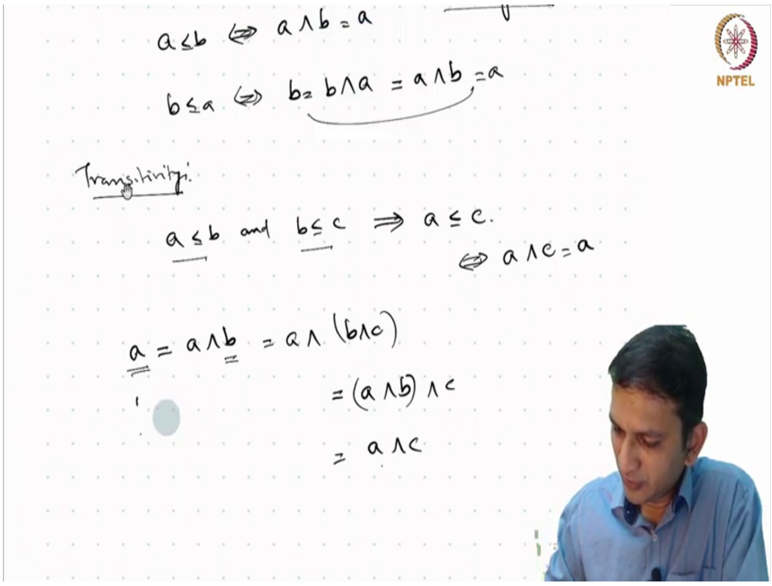

Now finally, for transitivity. Now, what do we have here for transitivity we are given that a is less than or equal to b and b is less than or equal to c, we need to prove that a is less than or equal to c. Proving this means, we need to prove that a meet c is actually a. Now, let us start with a; since we know that a is less than or equal to b, we can say a is actually equal to a meet b; since, b is less than or equal to c we say we know that b is b meet c.

But now if you substitute for b here, this is a meet b and c; however, we know that this meet operation is associated that is another property that we have assumed on meet and join. We can rewrite this as a meet b meet c, but what is a meet b is essentially a, and this is meet c and what is this is equal to a. So that means, you know we need to just remove this.

So, what we have is actually equal to a meet c this is what we wanted to prove which means it also has transitive line. Thus, we see that every lattice gives rise to a poset, but when we saw it from order theoretic terms, we saw that lattice was not just a poset it was a special poset. That means, every pair of elements should have a supremum and then infimum is it available, it has to be the greatest lower bound and the least upper bound with respect to the order that we have defined on the set error.

Now in fact, it can be verified without much effort that yes for any pair of elements both the supremum and the infimum exist; in fact, they turn out to be the join and the meet operation that we have defined on it. So, in that sense when you look at it, lattice can be looked at it from an algebraic point of view or an order theoretic point of view. Either we can take the L and define two such binary operations with them properties that we have on them idempotence commutativity and associativity.

Or we can look at lattice as a poset with special properties on any pair of elements these are that every pair of elements has a supremum and an infimum.

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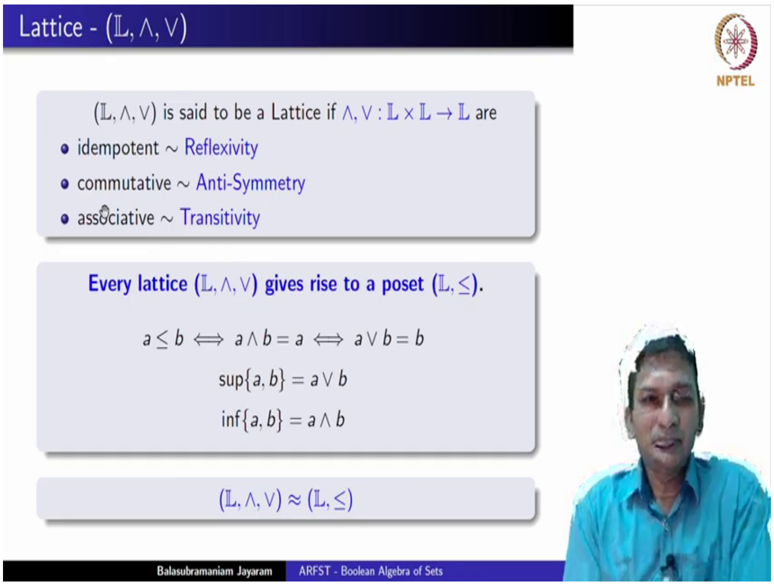

Note also this property that it was idempotence which ensured reflexivity, commutativity, ensured anti symmetry and associativity ensure transitivity. In that sense these properties are minimal, if you want to obtain an order as defined through this relation ok. So, now, we have seen lattice also from an algebraic perspective, let us look at some special types of lattices.

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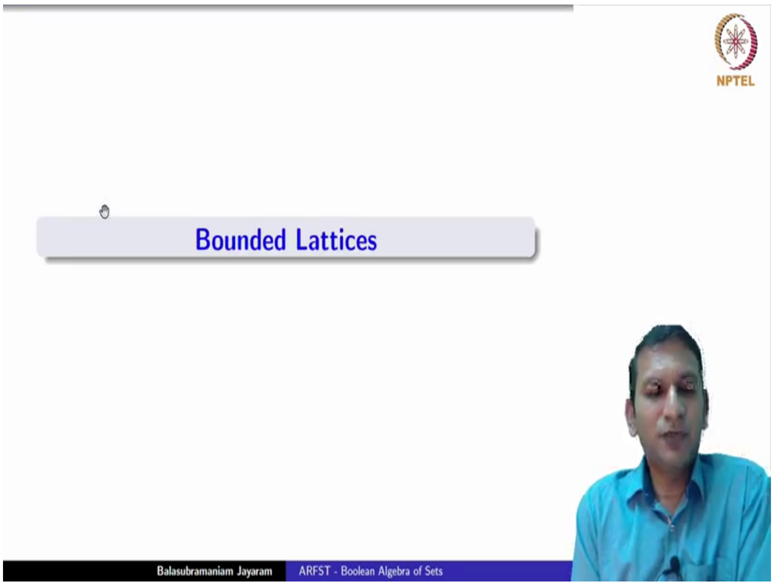

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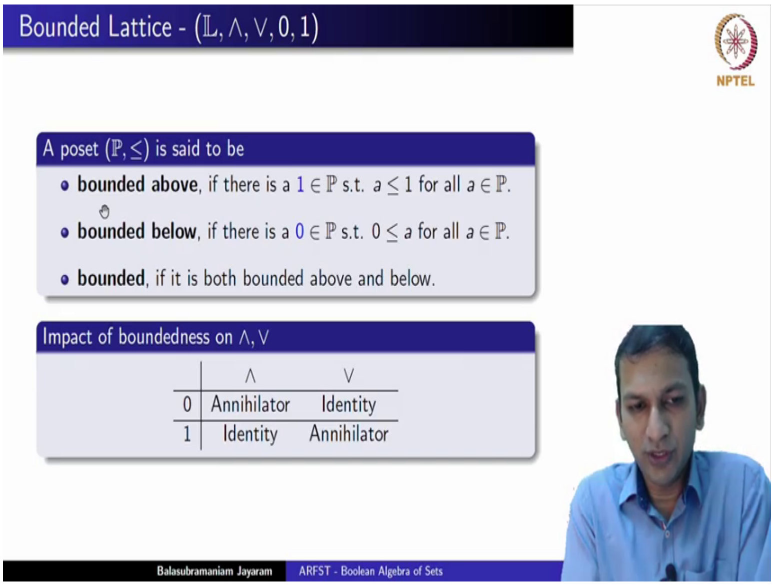

The first of them is a bounded lattice. Of course, a lattice is also a poset, if you see from order theoretic terms. So, the definition of boundedness remains the same, if we say it is bounded above if there exists a special element one which is the maximal element. That means, it is above everybody else it is bounded below there is a special element which we denote by 0 which is smaller than every other element essentially the minimum of the poset.

And we say it is bounded if it has both these minimum and maximum elements. While we have talked about boundedness in terms of the order with respect to the order. Let us look at the impact of boundedness on the operations of meet and join; it is quite interesting to look at this.

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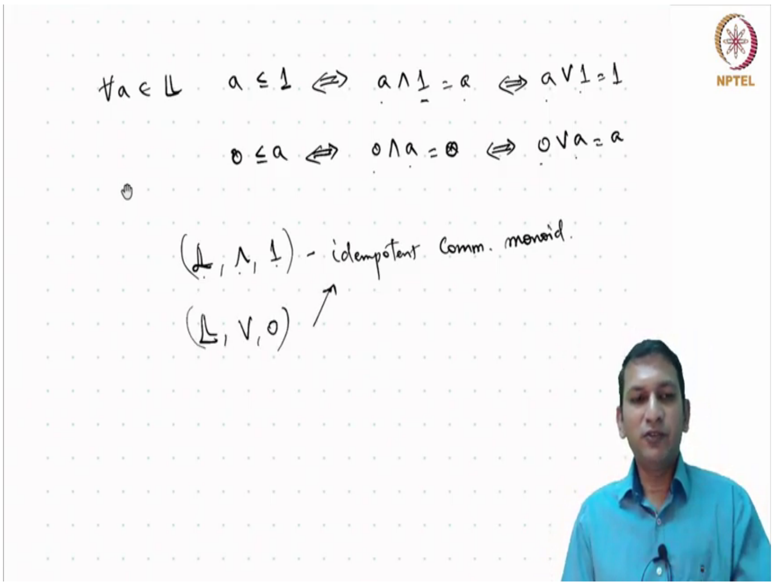

Now, if it is bounded below; that means, for every a in L, we know that a is less than or equal to 1; 1 is the maximum element, a less than or equal to 1 this happens if and only if, if a meet 1 is a and equivalently a join 1 is 1. Similarly, if 0 is the minimum element; so, for every a 0 is smaller than each of those elements. From our definition of order this means 0 meet a is 0 and equivalently 0 join a is a. Now, what it means is look at this; that means, 1 is the right identity of a and by commutativity; in fact, it is also the left identity of a and 1 becomes the annihilator of a both left and right.

Similarly, 0 the minimum becomes the annihilator of the meet operation while it becomes the identity of the join operation. In that sense algebraically speaking, if we consider this we know that L with this operation this operation is idempotent, commutative and associative this is the identity of this operation. So, what we get is an idempotent commutative not just a semi group, but a monoid.

Similarly, we could also consider this which is again an idempotent commutative monoid. So, this is what we have 0 access an annihilator for meet, but as identity for join; 1 access an identity for meet, an annihilator for join. So, this is the impact of boundedness a concept which is defined with respect to the order. Now, we see how it has been translated in terms of its properties on the operations of lattice essentially the meet and join operations of the lattice.

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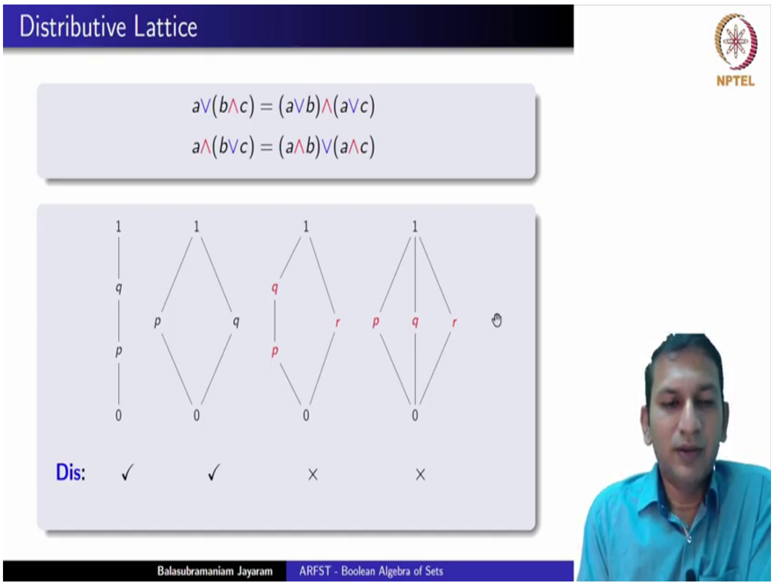

Let us look at another special lattice called the distributive lattice. What is a distributive lattice? Note that from an algebraic point of view lattice has these two operations, meet and join. We say a lattice is distributive if join distributes over meet and meet distributes over join, these are the usual equations of distributivity that we have.

Now, let us look at these four lattices and discuss whether these are distributive or not. Let us look at the first lattice, we know that this is a chain; that means, it is totally ordered that is any two elements are related by this partial order relation. Is it a distributive lattice? Yes, it can be seen that every chain is actually distributive lattice. Now, to see this let us look at the first lattice or in general a chain itself.

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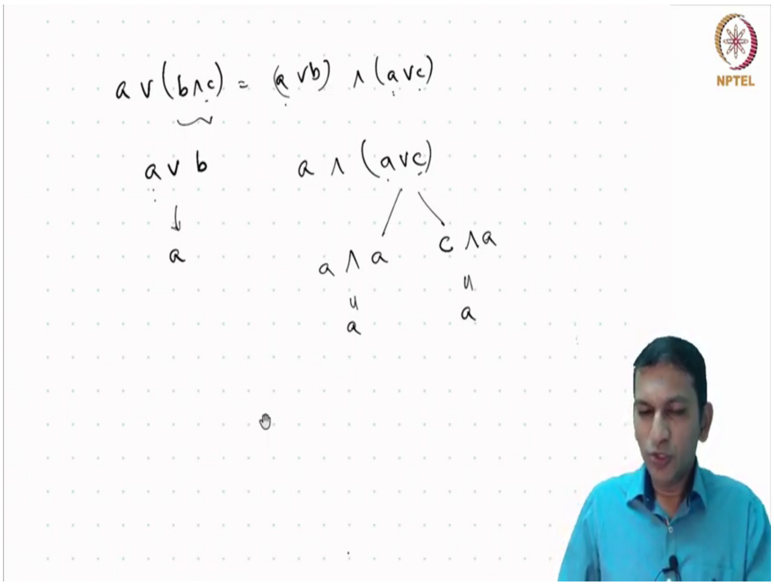

So, this is what we want to do a or (b and c) is (a or b) and (a or c). Now, in a chain since any two elements are related, b and c can either be a b or c it cannot be anything else; unlike, the other lattices that we have it. Look at this if you are looking at p meet q essentially, we know that the meet operation is also the infimum. So, that is infimum of p q with the greatest lower bound of p and q is actually 0, its neither p nor q; however, this cannot happen in a chain.

Thus, a or b and c if we have this b meet c it is either b or c; let us assume that, it is b and we have a join b. Now, once again there is some order relation between a and b, let us assume that a is bigger; so, now, from here what we get is a. Now, look at the right-hand side; so, it is again a join b we have just assumed that a is bigger than b; so, from here we would get a. Now, look at the order there is definitely an order between a and c; however, we are not sure of what that would be; so, let us take it as just a meet a or a join c.

Now, if a is bigger than c, then we would get a from here now a meet a this is actually by idempotence it is a. In case c is bigger than a, we know that a join c will be c, but once again we are going to take meet with a. Now, we know that c is bigger than a which means c meet a is a; so, every chain is actually a distributive lattice. Once again it can be easily verified that the second lattice is also distributive; now, what about the third lattice? Unfortunately, this is not a distributive lattice. So, to see this consider the three elements p q and r which are marked in red.

Let us consider p join q meet r, we want to see whether this is equivalent to p join q meet p join r. Now, what does q meet r? This is infimum of q and r we see that this is actually equal to 0, so; that means, this is p join 0. Now, p join 0, 0 is the minimal element; so, this translates into p. Now, we are looking at p join q; so, we know that q is bigger than p; so, p join q will be q itself from here.

And now we are looking at p join r, p join r; that means, we are looking at least upper bound and that is the supremum which is 1; so, we have 1. So, now, we are looking at q meet 1, q meet 1 is the infimum of these two elements this pair of elements which is actually q. So, now from the left-hand side we get p, right hand side we get q, but p is not equal to q.

Thus, we see that for this triple of elements the distributive law does not hold and hence this is not a distributive lattice; note that the above two equations should hold for every triple of elements. In fact, it can be proven by duality that it is sufficient to consider one of these equalities. However, for completenes's sake we have put the both the formula error. Finally, let us look at this particular poset, is it distributive once again unfortunately? No, it can again be seen by using this formula.

Once again let us consider p or q meet r, we are asking question is it equal to p join q meet p join r, we see that q meet r here it is the infimum of q and r which is actually 0. So, now we have p join 0 p join 0 will turn out to be p; here, p join q we know a cycle is the supremum of p and q which is 1 supremum of p and r is again 1. So, 1 meet 1 where idempotence is 1, but we see that p is not equal to 1, thus this lattice is also not distributed.

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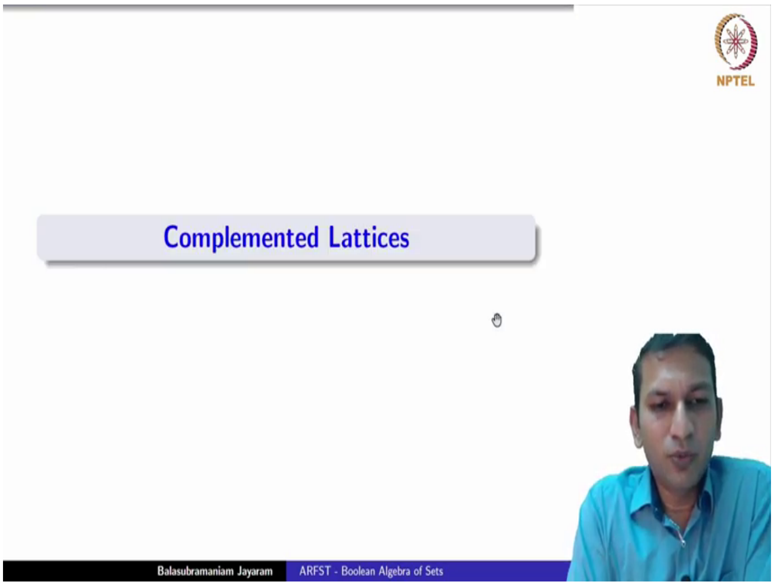

Let us look at one more special kind of lattice before we move to Boolean algebra which is the complemented lattice.

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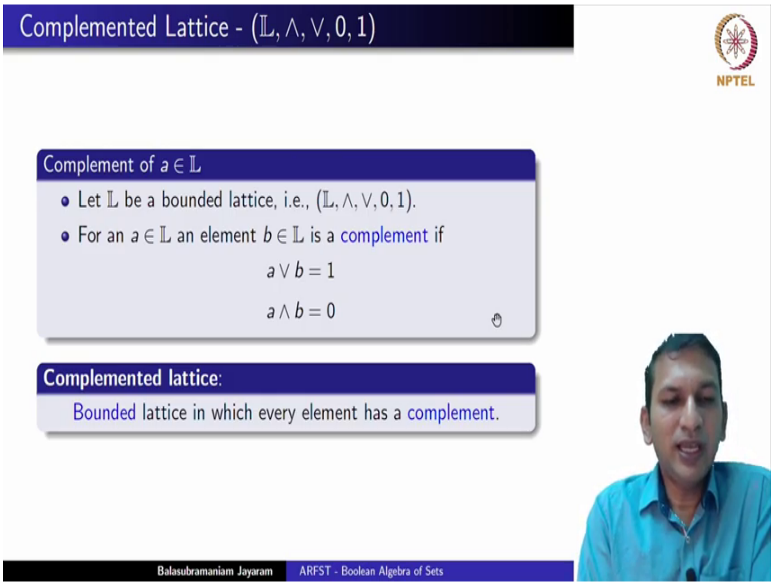

Now, what is a complemented lattice, to understand this first let us look at what is the complement of an element? Now, complemented lattice or complement of a lattice can be discussed only in the context of bounded lattices. So, to begin with L should be a bounded lattice; that means, order theoretically it should have both the minimum and maximum elements. From algebraic point of view there should be an identity for the join operation also an identity for the meet operation.

Now, if we take an element a, we see another element b is a complement of a, if these two properties are satisfied; that is, a join b should be 1 and a meet b should be 0. If you interpret this join and meet as actually union and intersection and 1 and 0 as the whole set x and empty set. And the complement as the set complement it is clear that we are looking at something like a union a complement is the whole set x, and a intersection a complement is the empty set.

Now, what is a complemented lattice? It is a bounded lattice in which every element has a complement. What we defined before was, what is the complement of an element a? We said if there exist element b such that a join b is 1, and a meet b is 0 then we say b is a complement of a, but dually we can also say a is a complement of b.

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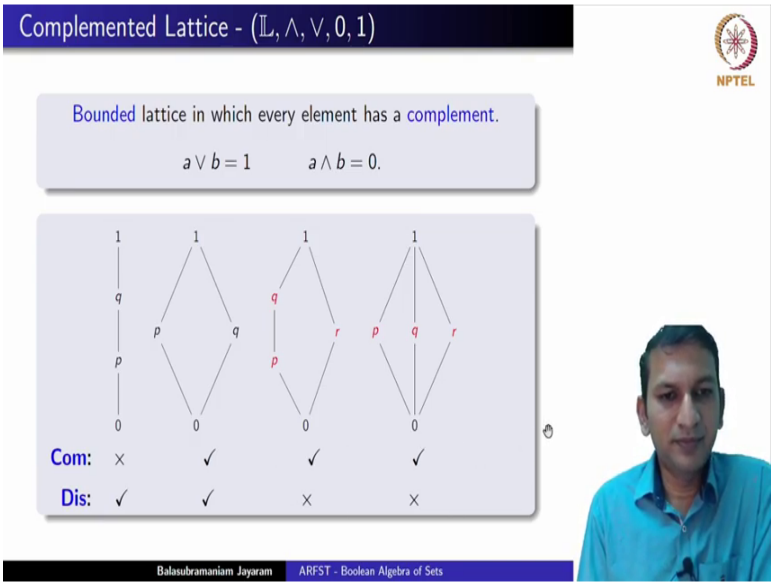

Let us look at the same four lattices that we consider; let us ask the question, is the first lattice which is a chain complemented? Unfortunately, it is not, why so? Let us look at this, let us consider. It is clear that the complement of 0 is 1 and for 1 it is 0; since, we will always consider bounded lattice to talk about complements. We have these two elements 0 and 1 which are always complemented.

It is for non-zero, non-one elements; elements other than the minimum and maximum elements that we need to discuss about, it is about the existence of its complement. So, let us take the element p.

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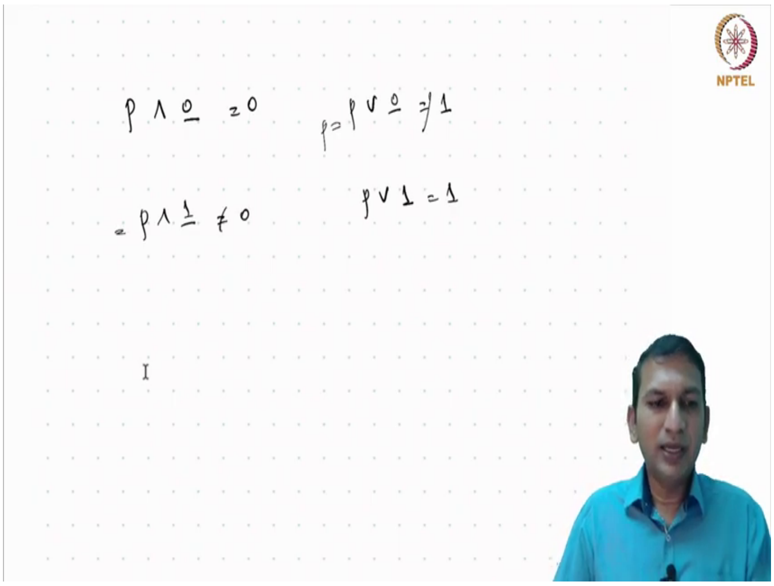

Now, you see here if p does have a complement, then with that complement p meet that should be 0 and p join that element should be equal to 1. Now, when you look at this the only element with p whose meet can give you 0 is actually 0 itself; so; that means, p meet 0 is 0. However, if you put this here p join 0, then it is not 1 because in a chain we know that p join 0 is actually p it is equal to p.

Now instead if you say ok maybe I can put 1, because p join 1 is 1, then you have to put the same 1 here. Now, this is not equal to 0, because we know that 1 is a maximum and minimum which means this is p. Now, this is not a complemented lattice, because at least there exist one element which does not have a compliment. If you consider a second lattice then clearly it is a complementary lattice, 0 and 1 are complements of each other.

If you look at p, q is the complement of p because for this pair of elements p and q the infimum is 0 and supremum is 1; so, p is a complement of q and q is a complement of p. What about the third lattice, we saw that it is not a distributive lattice, but is it a complemented lattice? Yes, it is. Let us look at the element p; now, we want other element with respect to which if you take the infimum, it should be 0 and if you take the supremum, it should be 1, we see that for p we have r to be such an element.

So, p meet r is 0, p join r is 1; so, p is a complement for r; that means, r also has complement which is p. What about q, does it have a complement; look at the same r, q meet r is 0 and q join r is 1; that means, q has a complement which is r. Now, interestingly while q and p have unique compliments, r can afford to choose among these two both q and p are complements of r. It can be easily shown that if the lattice is also distributive then the complements are actually unique; that means, every element has a unique complement.

So, since r does not have a unique complement, it is clear that this lattice is not actually distributed and that is what we have found out before. Similarly, if you look at this lattice once again you can say that it is a complemented lattice; for p you have q, acting as a complement or for p you can also see that r acts as a complement. Similarly, for q whether p or r can act as a compliment and same way r, for r, p and q both can act as compliments.

So, among these four posets that we consider only the chain is not complemented other three are complemented. Now, we have seen that in the case of distributivity, the first one is distributive; so, is the second, but the last two are not distributive. Now, this shows that when we consider bounded lattices complementation and distributivity are in fact, mutually independent they do not one does not include the other; we have all sorts of examples here.

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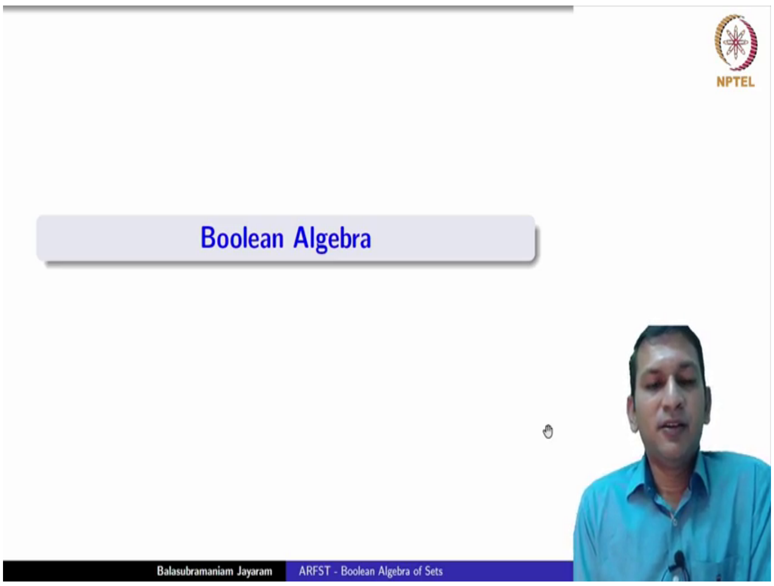

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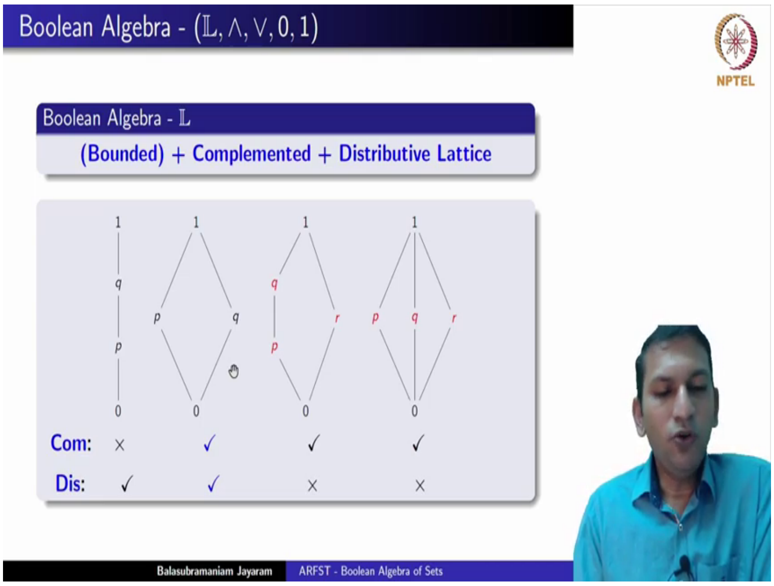

Finally, we are in a position to define what is Boolean algebra? A Boolean algebra is a bounded, complemented, distributive lattice. We have put the word bounded in brackets, because a complemented lattice can be discussed only in the context of bounded lattices. However, for emphasis we have also added the word; otherwise, it is sufficient to say a Boolean algebra is a complemented distributed lattice.

So, all these three special types of lattices we have seen and when you put them all together, we see that it actually gives us a Boolean algebra. Now, if you look at these four lattices that we have considered, you see that this all of them are bounded, but; however, only this lattice is both complemented and distributed hence among these four only this becomes a Boolean algebra. Now, how important is a Boolean algebra.

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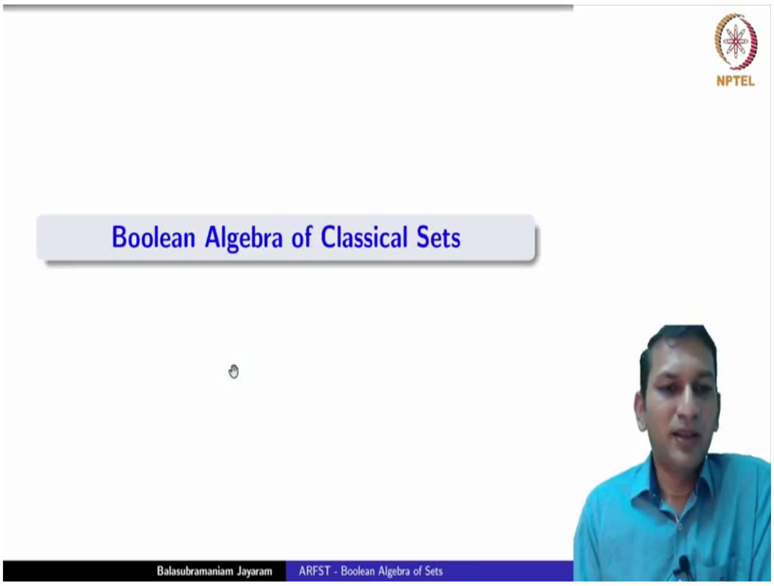

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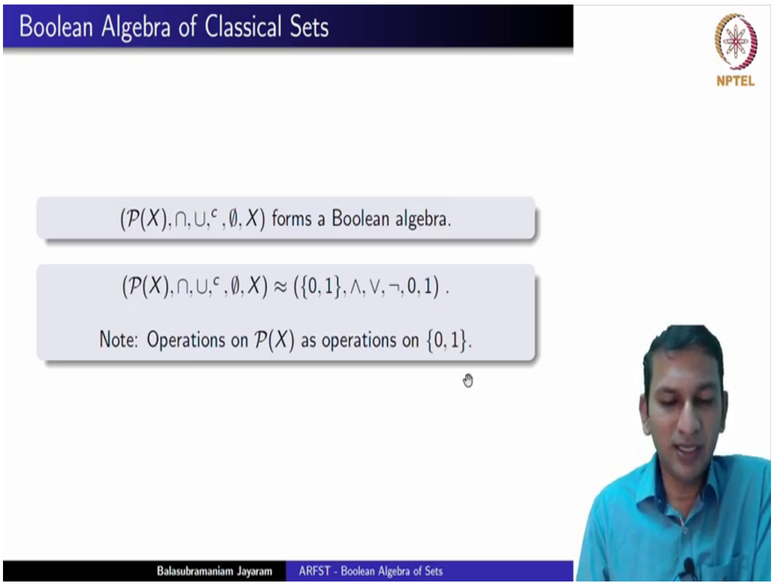

Consider the set X and the set of all subsets of X and for the operation consider intersection and union and the usual complement and the empty set phi and the set X itself. It can be shown perhaps already we know it from our knowledge of mathematics from school that the power set of X the intersection operation is idempotent, commutative, and associative; so, it is the union operation.

And you we do have a complementation operation with respect to which a, intersection a complement is phi and a union a complement is X. That means, this is a complemented lattice note that phi and X are the bounds with respect to the usual subset hood order which is what we would get if you use the intersection meet. Thus, the set of all subsets of next it forms a Boolean algebra.

Now, what is interesting is we know that in terms of the characteristic function representation, every subset of X can be represented in terms of its corresponding characteristic function. And we know the operations on classical sets can be mimicked on the corresponding co-domain of the characteristic function which means there exists an isomorphism between this algebraic structure and this algebraic structure; so, this is a Boolean algebra and so is this.

Here we have only a two element set, this is the usual meet which you can now safely look at it as a minimum, this join as the maximum, the negation is actually the complementation, for 0 it is 1, for 1 it is 0, there are only two elements and these are the bounds. Note that this was possible only because we can look at operations on P of X as operations on 0.

So, immediately it should ring a bell we know that operations on f of X set of all fuzzy sets can also be looked at as operations on the unit interval 0 1. So, perhaps it gives us a queue as to studying the operations on just the 0 1 interval and that is what we will do in the next lecture.

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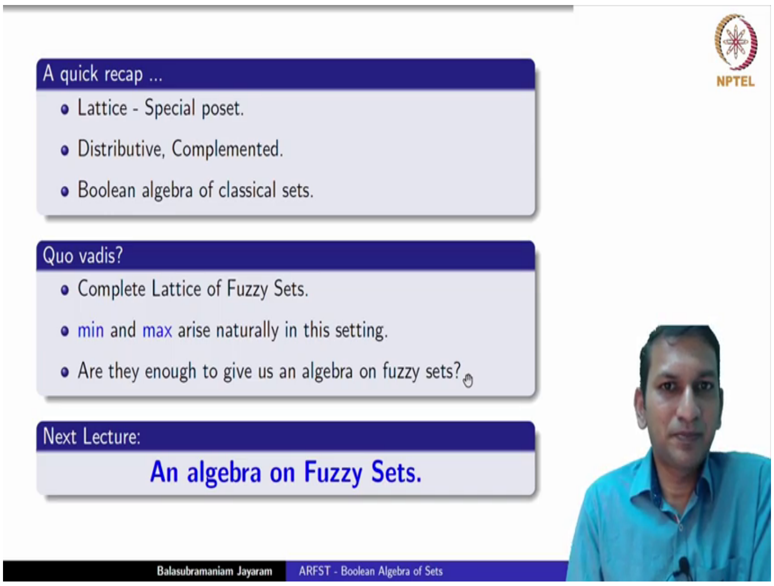

A quick recap of this lecture, we have seen lattice itself as a special poset. But in this lecture, we looked at it from an algebraic point of view and we saw some special types of lattices distributive, and complemented. And we saw that they lead to a Boolean algebra which is a complemented, distributive matrix. And if you consider the set of subsets of the set X, they actually form a Boolean algebra under the operations of intersection, union and usual complementation with phi index acting as the bonus.

Now, what next in the previous lecture we had seen a complete lattice structure on the set of fuzzy sets and we have seen that min and max quite naturally arise in the setting. The question now is, are these operations enough to give us an algebra on fuzzy sets or do we need to take help of other interpretations.

In the next lecture we will study this in depth and we hope that our quest for choosing among the different interpretations will have more light crown on them when we study this algebra on fuzzy sets. Thank you for joining me once again I hope to see you soon in the next lecture.

Thank you.