Approximate Reasoning using Fuzzy Set Theory
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Lecture - 07
Posets on Fuzzy Sets

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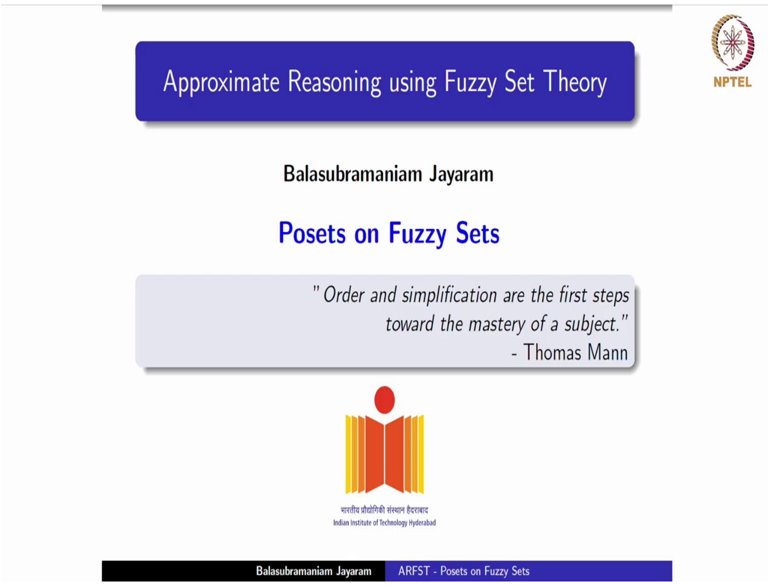

Hello and welcome to the next of the lectures under this course titled Approximate Reasoning using Fuzzy Set Theory.

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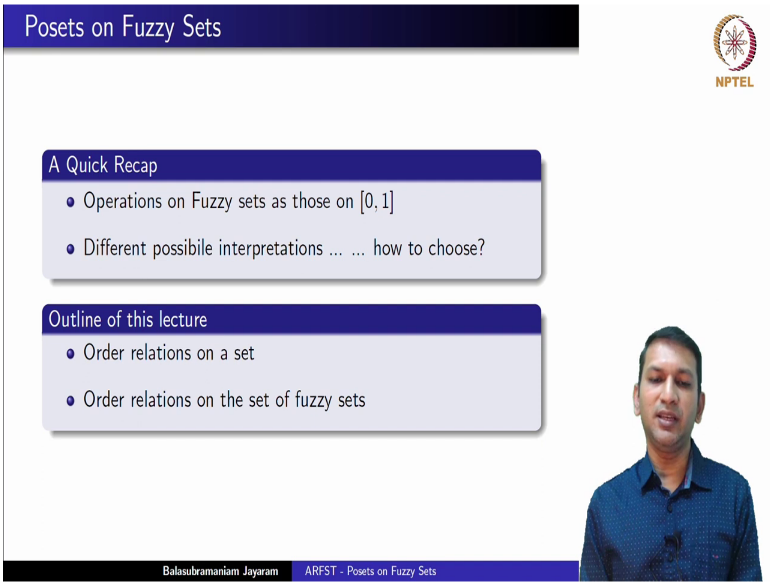

In the last lecture, we have seen that operations on fuzzy sets can be seen as the operations on the corresponding unit interval [0, 1] which is its older one. We have also seen that there are different possible interpretations available to us, for these operations and we wondered how to choose among them.

In this lecture, towards enabling us to answer this question we will look at a very preliminary and primary structure that of an order structure on the set of fuzzy sets. To begin with, we will look at order relations on a set and move on to looking at some order relations in the set of fuzzy sets.

You might remember that we have seen that there are two types of orderings that are possible among fuzzy sets. However, we would like to also confirm whether these relations that we call as quadrants on fuzzy sets whether they conform to classical, order theoretic concepts.

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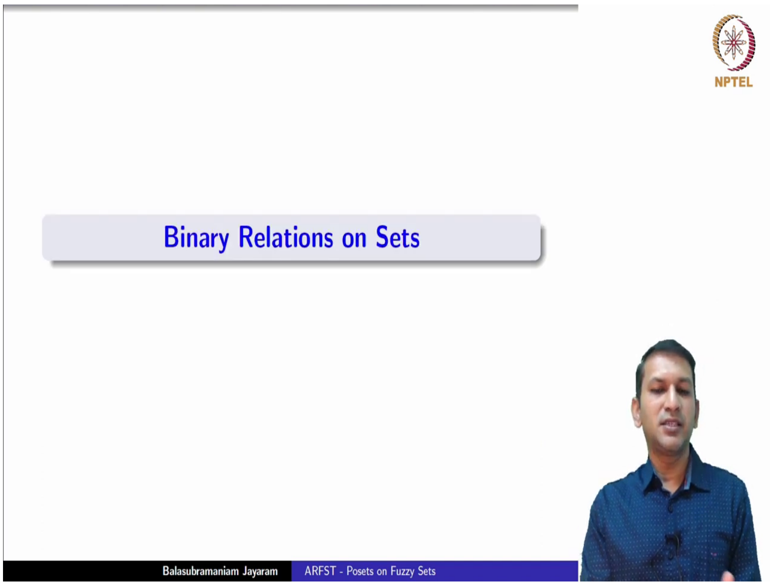

Let us begin with Binary Relations on Sets.

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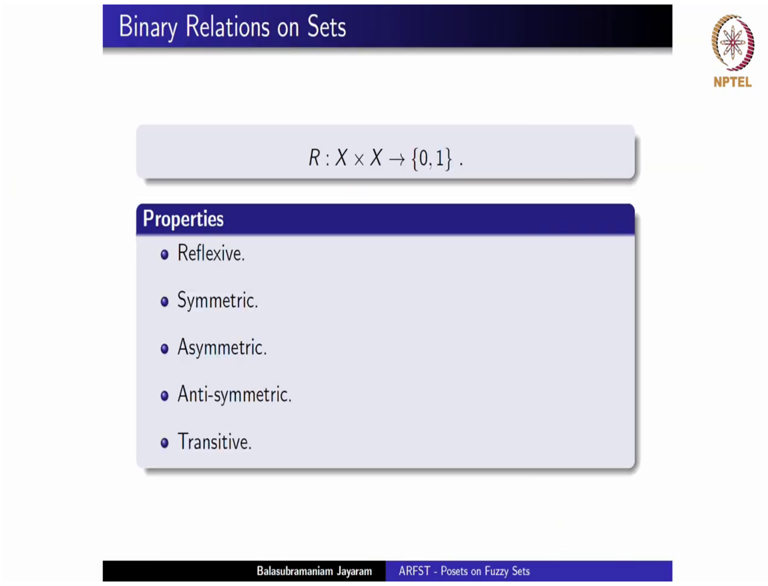

You all know a binary relation on a set X can be thought of as a function from X cross X to just the N set {0, 1}.

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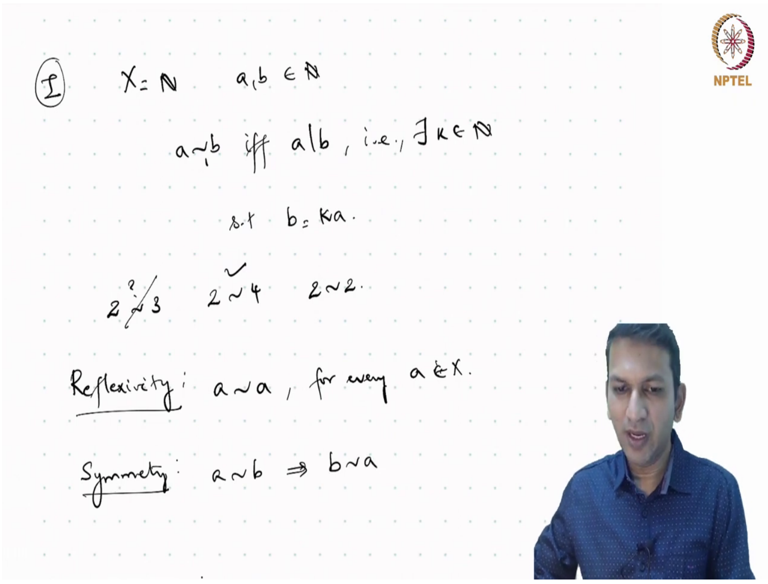

For instance, if we consider X to be the natural numbers we can define for any given a, b element of N, we can define a relation as follows a is related to b if and only if a divides b. That is there should exist a K element of N such that b can be written as K times a.

Now, you will see that if you define such a relation binary relation by the symbol tilde. You can ask the question is 2 related to 3, but clearly 2 does not divide 3. So, this is not true. What about 2 and 4, are they related? It essentially means asking the question does 2 divide 4. We know that yes, 4 is 2 times 2. So, if you see here these are binary relations on the set of natural numbers.

Now, we can have many types of binary relations, some of the properties that are often used and employed are this. What do we understand by reflexivity? We see a binary relation is reflexive if a is related to itself i.e., for every a in the set. You will see immediately that the relation that we have defined above that of divisibility is actually reflexive because 2 is related to itself.

The next of the properties is that of symmetry. Since, it is a binary relation we can talk about symmetry. What does symmetry mean? It means if a is related to b then we want that b is also related to a. Now, you can ask the question; is this true of the relation that we have defined above. Well, we know that 2 is related to 4. However, is 4 related to 2; clearly no.

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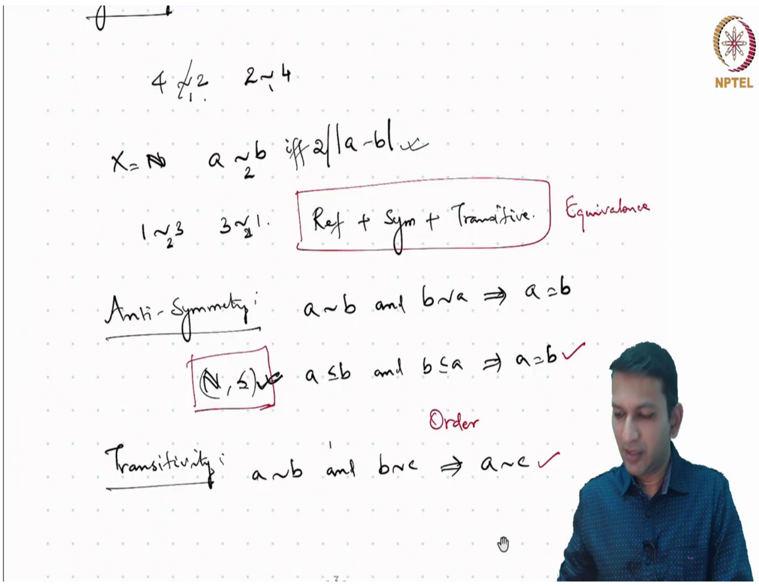

So, the above relation is not symmetric. But of course, there are many symmetric relations. For instance let us consider once again X to be N and let us define that a is related to b if and only if mod a minus b is divisible by 2. This 2 divides mod a minus b. Clearly, you will get a partition of N equivalence classes of those of even and odd numbers. It is clear that if I number this as relation 2, 1 is related to 3 and so is 3 related to 1. So, this is a symmetric relation.

There are also asymmetric relations and our first relation of divisibility falls under that. Since, 4 is not related to 2, but 2 is related to 4. So, asymmetry essentially means a related to b does not imply b is related to a. What is anti-symmetry? Anti-symmetry says that if a is related to b and b is related to a, this implies a is actually related to b. We all know that a usual order of N has this property; that means, if you have a less than or equal to b and b is less than or equal to a, then you know that a actually is equal to b.

Finally, the property of transitivity: Transitivity says that if a is related to b and b is related to c, then this should imply that a is related to c. Clearly, this relation is transitive, so is this relation.

What does transitivity say? It says that if a is related to b and b is related to c, this should imply a is related to c. Clearly, the ordering on natural ordering on natural numbers does have this property. So, thus the second relation where 2 divides mod a minus b, a binary relation that we have defined on the natural numbers here.

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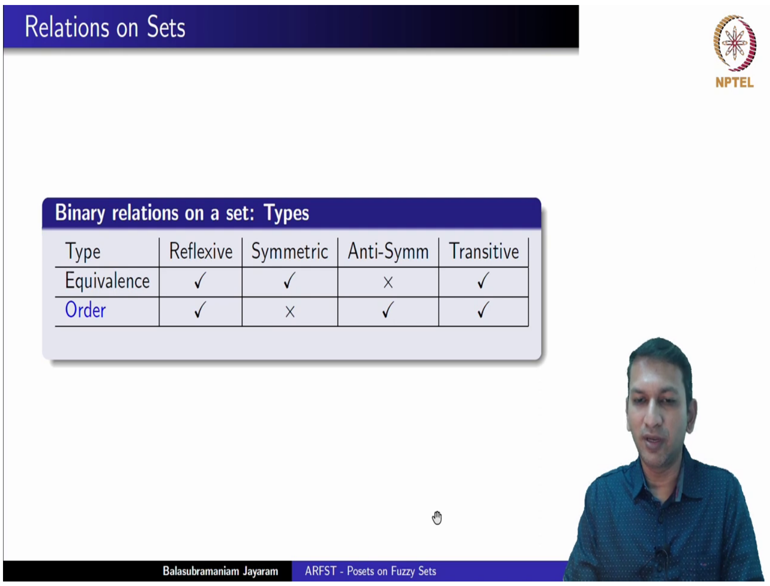

There are different types of binary relations that we can think of. For instance, if a binary relation is reflexive, symmetric, and transitive, we call it an equivalence relation. For instance consider the relation that we have defined here that on the set of natural numbers, a is related to b under this relation if and only if 2 divides mod of a minus b. It can be shown clearly that it is reflexive plus symmetric plus transitive. So, this is actually an equivalence relation.

We can also have relations which are reflexive, but not symmetric, but anti-symmetric and transitive. Such relations we call order relations. For instance consider the relation of the usual ordering on the set of natural numbers. We know that it is reflexive anti-symmetry and also transitive, thus it forms what we call an order relation.

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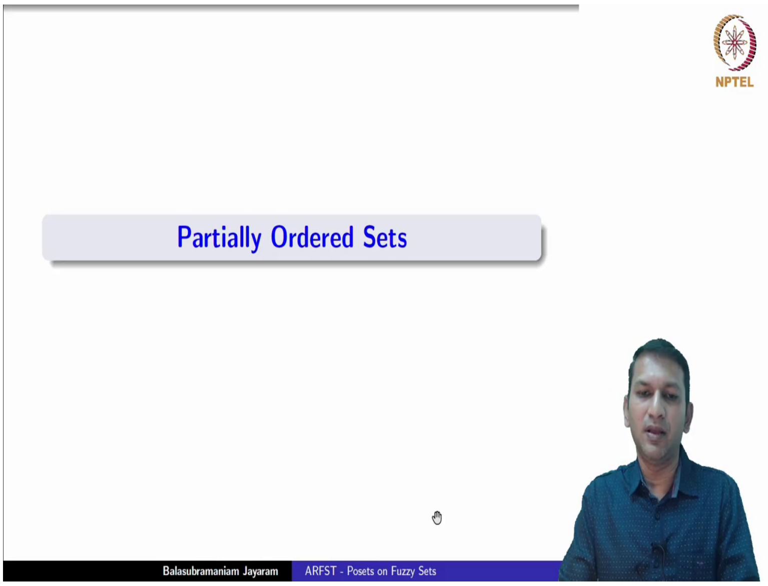

Let us look at what are partially ordered sets.

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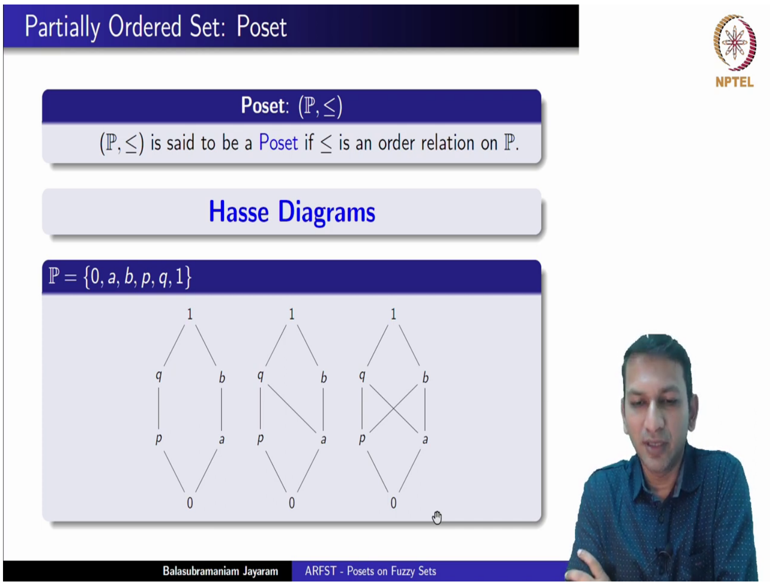

Let us consider a set P which is non-empty with a binary relation defined on it. We call this couple the set with the relation. We call this a poset if this relation is an order relation. It is typical to indicate order relations by this less than equal to symbol or some variant of it, and that is what we will also form. So, a set with this relation binary relation is called the poset, if this relation is an order relation on p.

Now, it is also typical that if the set P that we are considering is actually finite, then to show the order it is you know convenient to use what are called Hasse diagrams. What are they? Allow me to explain this to you. Consider the set P which has 6 elements, one possible ordering of it gives us this Hasse diagram. How do we decipher this Hasse diagram? If you take 2 elements, if there is a line between them, then these two are related as either predecessor and successor relationship. So, look at 1 and q. So, 1 is related to q immediately and q is related to p, p is related to 0.

But now we are talking about orders; that means, there is an ordering between them and it is common to use the language that 0 is less than p or p is less than q. So, as you go from the bottom to top, you can think of it as an increasing relationship. It is clear from this diagram that p and b are not related, nor as p and a. Similarly, q is not related to either a or b, under the ordering that we have given on the set.

But we could also have another ordering which gives us this kind of a Hasse diagram. Note that by the transitivity once you indicate these relationships, you could also extrapolate and find out determine the relations between any other two elements. For instance in both of these posets we know that 0 is related to q, 0 is also related to b.

And in fact, 1 is related to every element of the set and 0 is also related to every element of the set. Clearly, in both these posets, q is not related to b. We could also have a poset on the same b, where the relationships are such that the Hasse diagram that you get is as you see on the third diagram on the screen.

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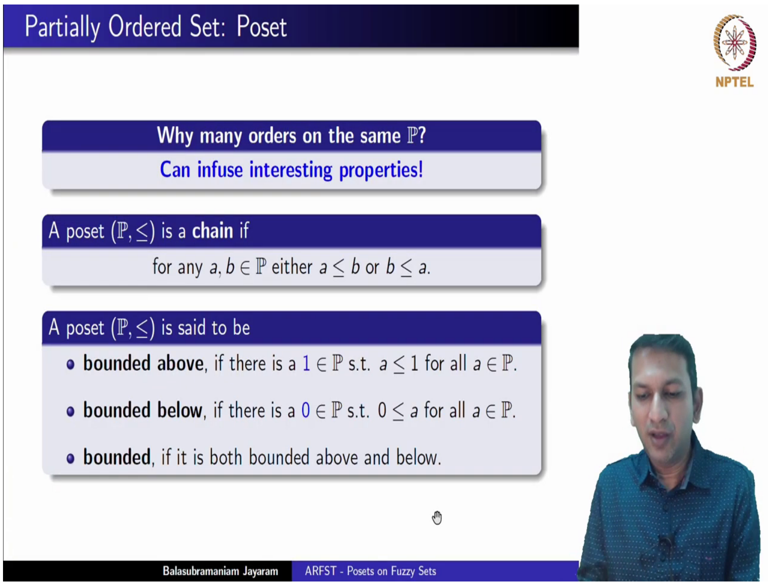

An immediate question would be why so many orders on the same set P. Well, the answer to that would be it can infuse very interesting properties. What do we mean by that? Let us take a couple of examples. A poset is called a chain or totally ordered if you pick any two elements a and b, then they are related; that means, either a is less than or equal to b or b is less than or equal to a.

What are the common examples of chains that we know?

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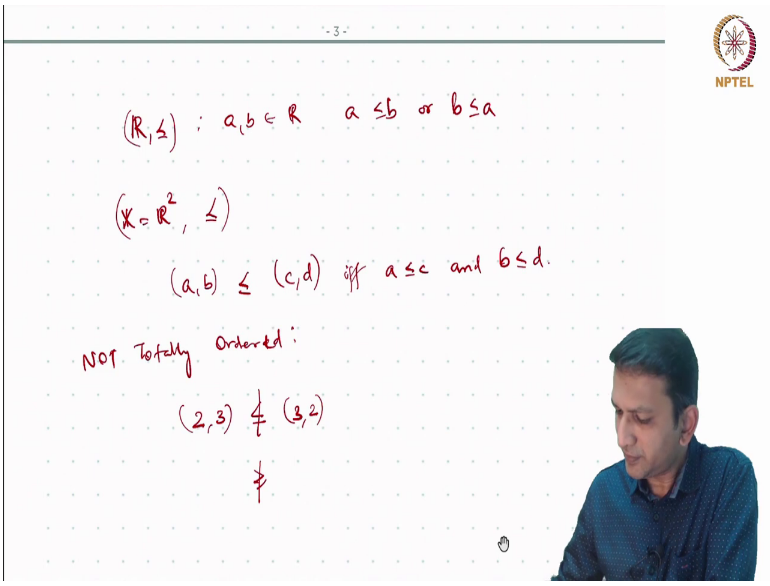

If I take R with the usual ordering, we know that any given any a, b element of R, we know that either a is less than or equal to b or b is less than or equal to a. And similar kind of ordering exist also on N.

Now, let us consider X to be R^2. You can give the usual ordering, the component wise ordering of (Refer Time: 12:08). How is this defined? If you take a pair of couples from R^2, you can define the ordering like this that a, b is less than or equal to c, d if and only if a less than or equal to c and b less than or equal to d.

It can be clearly seen that there is relation is reflexive, anti-symmetric and also transitive. However, it is not a totally ordered relation, not totally ordered relation. Why do we say this? Consider these two pairs of elements from R^2. Now, neither can we say this while 2 is less than 3, 3 is not less than 2, nor can we say this while 3 is greater than 2, 2 is not greater than 3. So, you have at least a pair of elements, which are not orderable under this relation.

Can we come up with another relation which will help us in this? Consider this relation.

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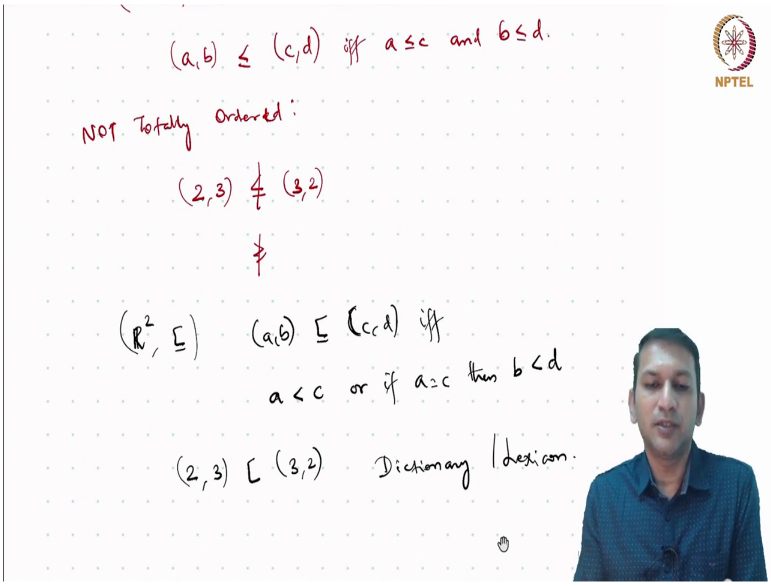

Let X be R^2 and we define another relation like this; a, b is related to c, d if and only if either a is less than c or if a is equal to c then b should be less than d. Now, using this relation it can be easily verified that this is not only a partial order; that means, its reflexive, symmetric and transitive, but it is also a total order; that means, any two elements can be related under this one. For instance, if you look at this 2, 3 and 3, 2, going by this definition the moment we see 2 is less than 3, we say that this is smaller than this. Thus we get a total order. This ordering on R^2 is often known as the dictionary order or the lexicon order.

So, having a different order would mean having more properties some special types of posets. We also will look at another important property of poset. A poset is said to be bounded above if there exist an element which we indicate by 1, such that every element is smaller than that every element is ordered below that.

We say that it is bounded below if there is an element below which no other element exists. Essentially, bounded above means there exists an element such that there is no element from the set which can dominated it is bigger than that. Bounded below means there is an element in the set P, such that no element of P is smaller than that. If it is both bounded above and below we call such a poset a bounded poset.

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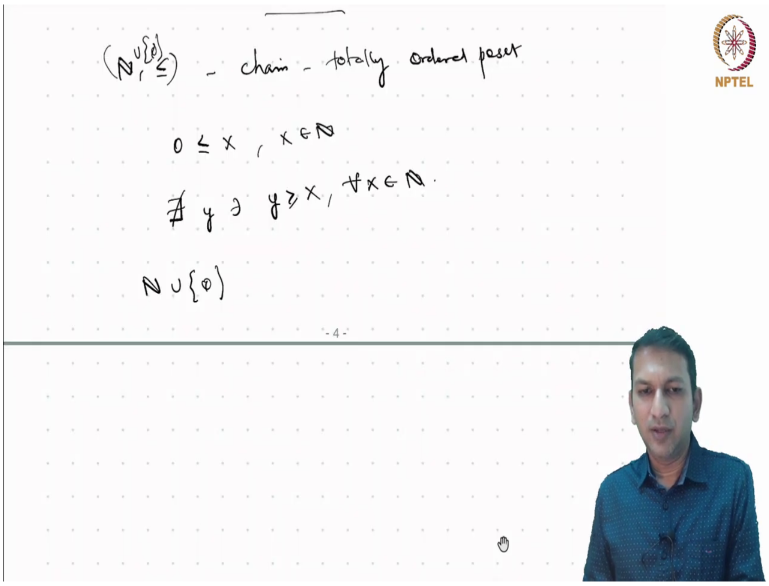

Consider the set N with the usual order while this is a chain or totally ordered poset. We know that this is not bounded above. Yes, 0 is smaller than any x, it is bounded below, but there does not exist any y, such that y is greater than or equal to x for every x in N. Clearly, this is not bounded.

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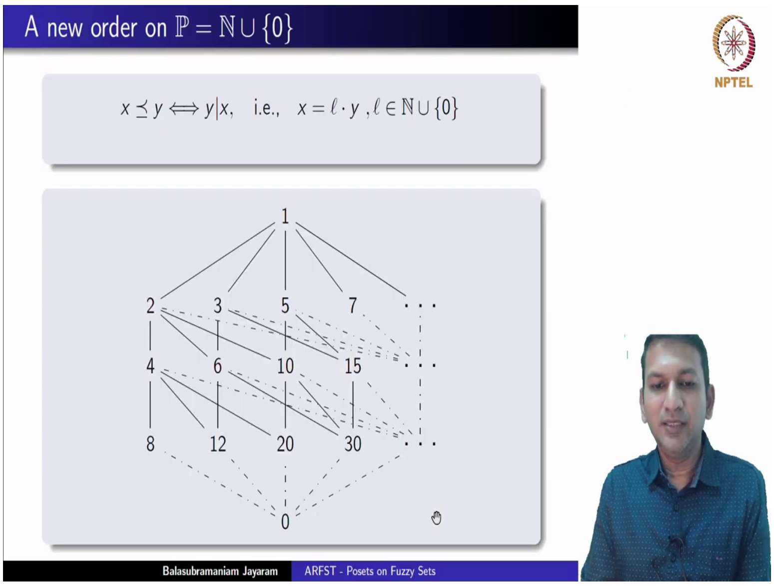

Now, let us see whether we can introduce a new order on the set. So, consider the set of natural numbers including 0. Now, this is the order that we are going to define on it; that x is less than or equal to y if and only if y divides x. Remember, this is the dual of what we considered right at the beginning. What do you mean by this? You mean that x can be thus written as a product of y and some other number l, from this set N union of 0.

Now, let us understand this relation. So, allow me to write this once again.

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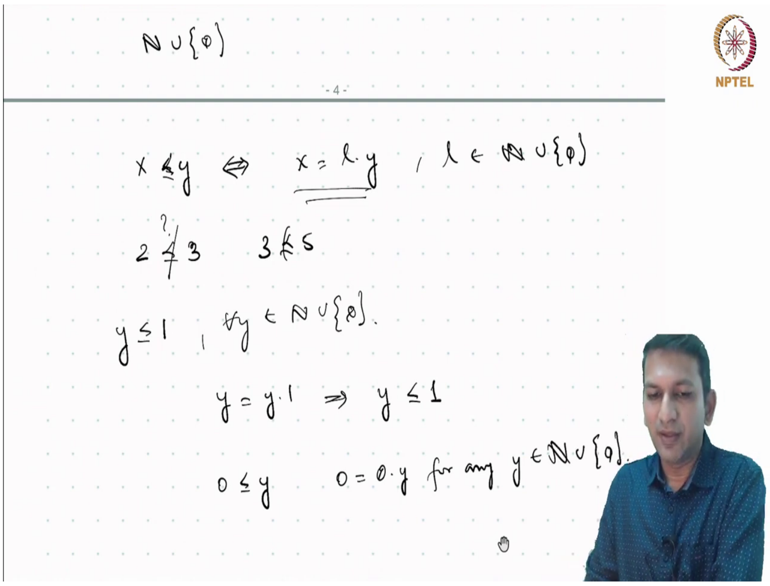

We say x is less than or equal to y if and only if x can be written as l y, where l comes from N union of 0.

Now, if you ask the question again, is 2 related to 3 under this order. That clearly 2 and 3 are prime numbers they are not factorizable. So, we see that under this new ordering neither 2 and 3 are not related to each other. And that would happen with any other prime number also.

So, the prime numbers are not suddenly relatable. What was once a chain has now suddenly started giving away; given two elements we are not able to order. But then why did we enforce this order? Do we stand to gain something out of this? But interestingly, if you look at it, consider 1, if you see 1 is actually greater than every y for all y element of N union of 0. Why this so? You can write y as y times one which according to this definition means y is less than or equal to 1. That means, 1 has become the upper bound for the set. What about 0? We will see that 0 is still less than or equal to y for every y, since 0 can be written as 0 times y for any y any N union of 0.

So, if you actually work this out, this is the kind of Hasse diagram you would get for the ordering, new ordering that we have proposed on the set N including 0. Now, you will see that while we have lost the total ordered property of N, that we used to get with natural ordering N. What we have instead got is a bounded above poset on N?

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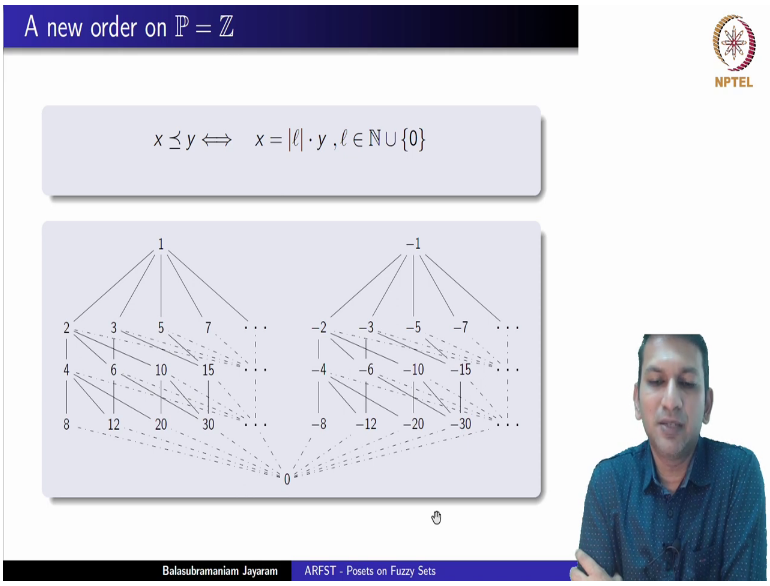

Now, if you consider the set Z, the set of all integers, while it is a chain it is neither bounded below nor bounded above. However, you can easily verify that if you define the order like this, then what we would get is a poset of this type where we have introduced a new bound below which is 0. Of course, it is still not bounded above, but there is no one unique element of Z, which is not dominated by anybody else. So, we, what we call as maximal elements of that, but it is still not bounded above, ok.

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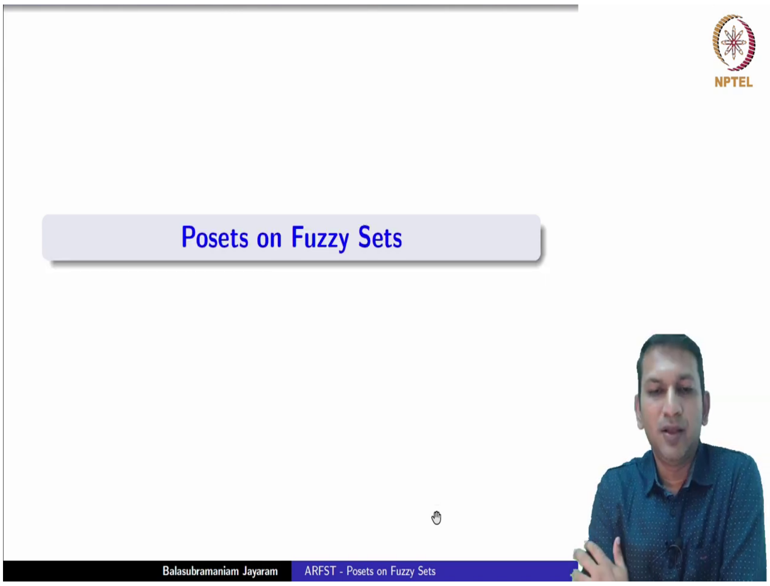

Let us come to discussing some order theoretic structures on fuzzy sets.

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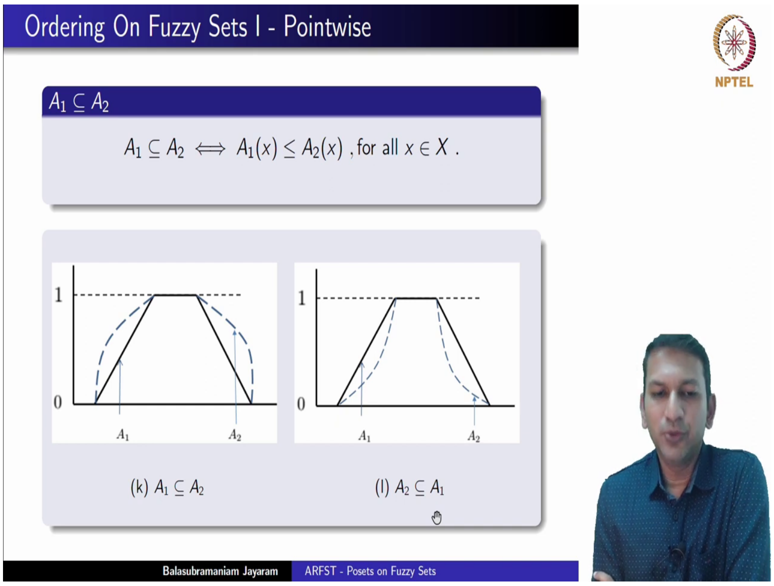

We have seen that on the set of fuzzy sets we can give a point wise ordering or not. So, we say that A_1 is related to A_2 if and only if the point wise membership values are less. That means, A_1 of x is less than or equal to A_2 of x. This is the usual point wise ordering of function. And we have already seen that when A_1 and A_2 are classic characteristic functions, then this actually coincide with the notion of subset could on classical sets.

These figures must be familiar to you by now.

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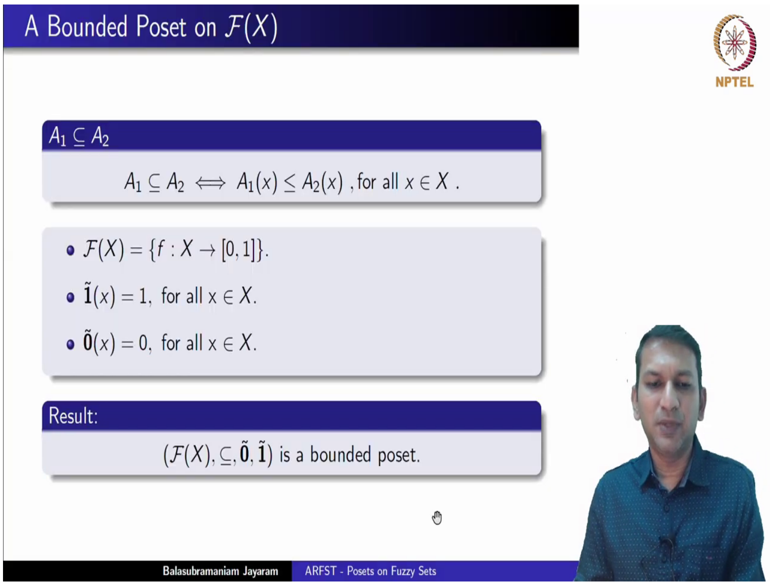

Now, the question now is this an ordering on the set of fuzzy sets. Well, consider F of X to be the set of fuzzy sets on X. But 1 tilde we denote the function that takes the value 1, the constant value 1 rho of X. By 0 tilde we denote the function which takes the value 0 on the entire X.

What we can immediately prove is this nice result which says that set of all fuzzy sets on X with a point wise ordering is a bounded poset. So, that means, this point wise ordering is actually a partial order relation on the set of fuzzy sets, and the 0 tilde and 1 tilde they act as the lower and upper bounds.

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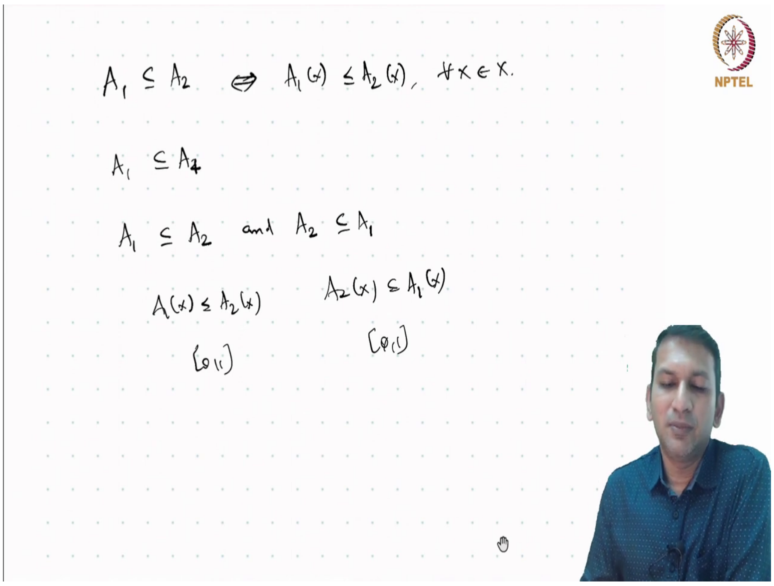

Now, if you would like to see why this is in order, remember we said that A_1 subset of A_2 if and only if A_1 of x is less than or equal to A_2 of x, for all x in X. Clearly, we see that it is reflexive. Thus A_1 less than is related to itself.

If A_1 is really contained on another A_2 ordered with respect to into this way, and A_2 is also related to A_1, clearly this means A_1 of X is less than or equal to A_2 of x and A_2 of x is less than or equal to A_1 of x. And we know these are numbers from the 0, 1, interval which means using the natural order on the [0, 1], interval we know that A_1 of X should be equal to A_2 of x, and this happens for every x in X.

Clearly, we are borrowing the poset property that is available on 0, 1, with respect to the natural order. So, it means it is also transitive. Clearly, this becomes a order on the set of all fuzzy sets and we get a bounded poset.

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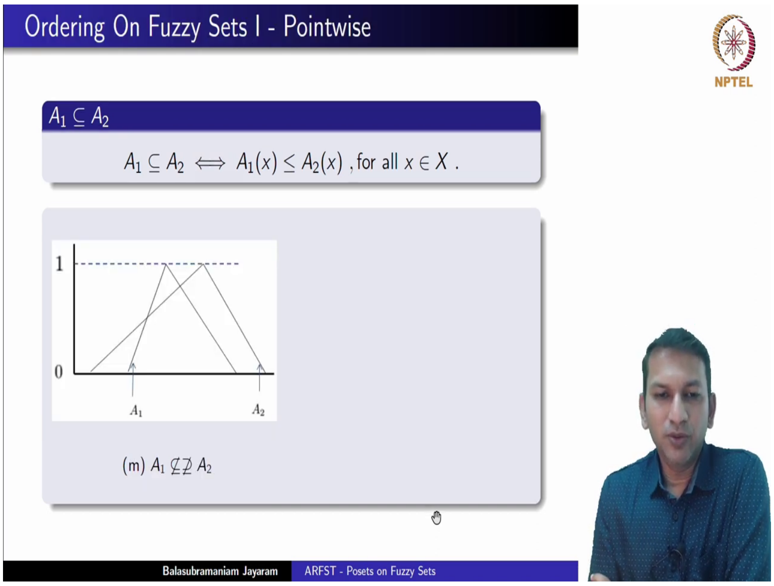

You might remember that we also were looking at an some other pairs of fuzzy sets which are not orderable under this particular relation that we have defined. For instance, if you look at these two fuzzy sets with respect to this ordering they are not orderable, nor are these two fuzzy sets.

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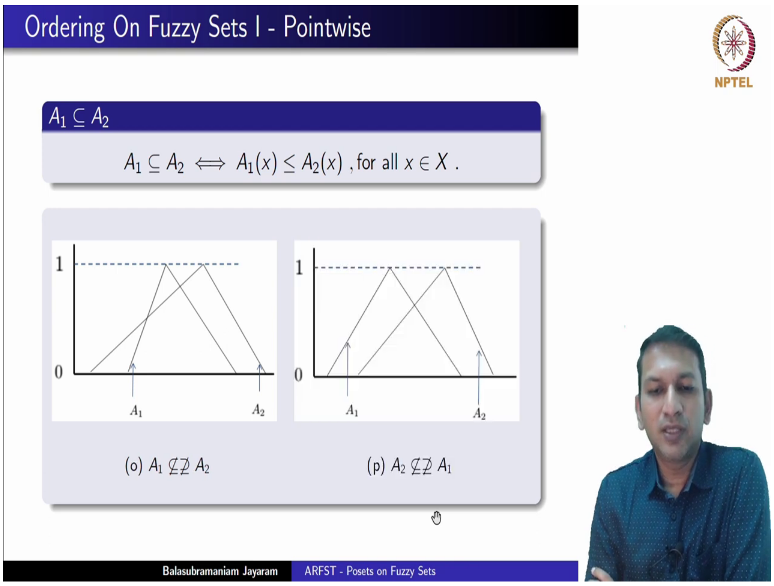

However, we felt there is something regular about the pair of fuzzy sets that you see on the right side of your screen, and hence we introduced a new order based on the level sets the alpha cuts.

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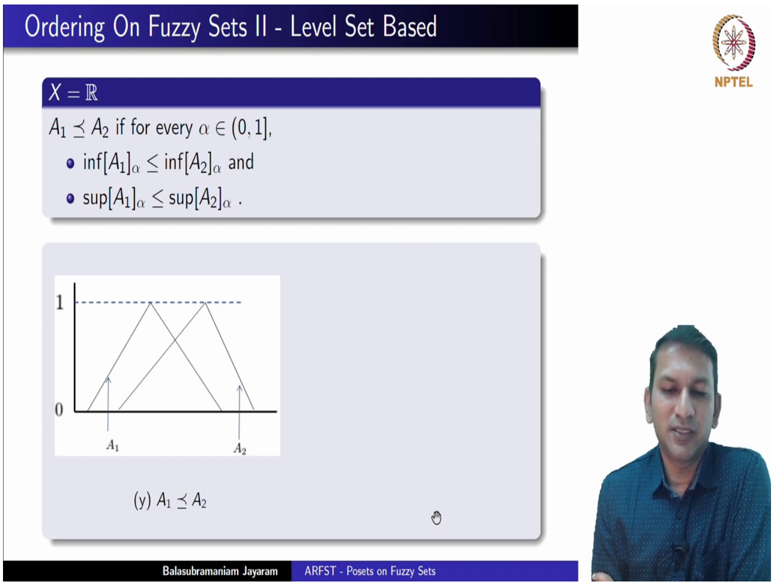

So, we say that if X this is the real line, we define this order on all those fuzzy sets where, which are defined on the real line as follows. A_1 is less than or equal to A_2 for every alpha in [0, 1]. These two inequalities are there. So, according to this, we have seen that this pair of fuzzy sets is orderable, whereas still this pair of fuzzy sets it is not orderable.

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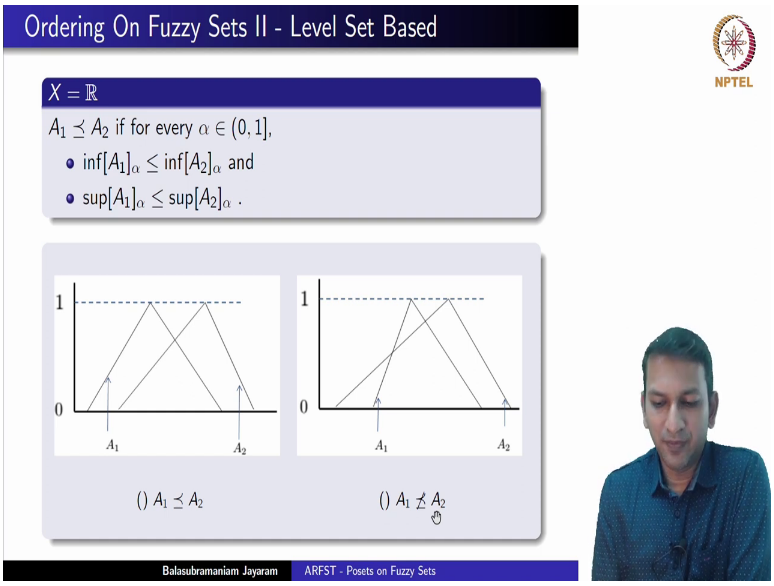

Now, the question is this relation even though we are calling them as order relation, we use the term that A_1 is less than or equal to A_2 is this actually partial order.

Now, interestingly it is immediate to see that this relation is in fact reflexive because A_1 less than or equal to A_1, once again I should make a special mention that here we could look at inf as in some sense the left end point of the alpha cut and sup as the right endpoint of the alpha cut. A justification for this will be provided soon enough in one of the upcoming lectures, ok. So, this relation is actually reflexive.

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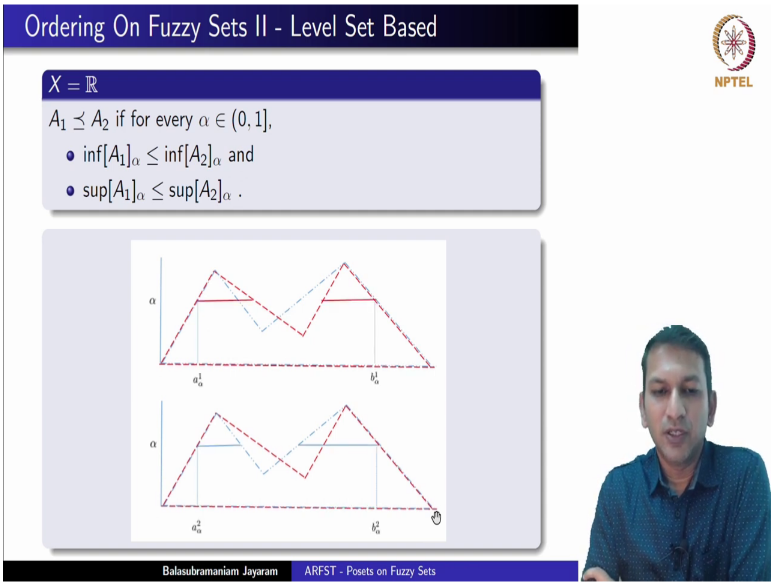

What about anti-symmetry and transitivity? To discuss that let us look at these two fuzzy sets, call them A_1 and A_2. The red one is A_1, and the blue, fuzzy set in blue is the A_2 fuzzy set. Now, so as to be able to graphically see what to see the point that we are trying to illustrate let us try to superimpose these two sets. But please notice that these are two distinct fuzzy sets. Let us superimpose them and take a particular alpha, any alpha.

If you consider the alpha cut of this alpha for the fuzzy sets A_1 and A_2, this is what you would get. So, the fuzzy set A_1, the alpha cut for that is indicated by the red line. So, essentially you have to take the inverse and look at the points on the X axis whose membership values are above that alpha.

So, you will see that the alpha cut in this particular case is a union of intervals and also the same with the alpha cut of the fuzzy set A_2 that is there on the bottom. While the alpha cuts themselves are different, what you would see is the infimum of the alpha cuts and the supremum of these alpha cuts are essentially the same. As was mentioned, we will interpret the infimum and the supremum of these alpha cuts at the left and right end points of the interval along the union of intervals.

Now, it is immediately clear that the left end point of the alpha cut, the set now we it is a union of intervals, so we call it a set, the left end point of this union of intervals of both the alpha cuts are identical, so are the right endpoints, which means we could say with respect to this relation A_1 is less than or equal to A_2. So, it is the case where A_2 less than or equal to A_1. However, we see that it is not anti-symmetric simply because A_1 is not equal to A_2. What this means is even though we called it an ordering relation; we see that this is not an ordering.

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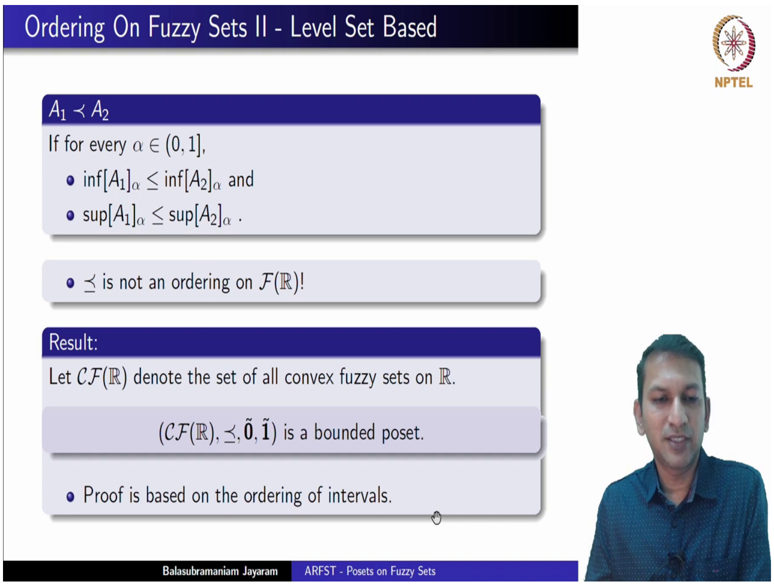

As we understand is a partial order on the set of fuzzy sets on R. However, not all is lost. So, if you instead consider a set of all convex fuzzy sets on R. Let us denote it by CF of R, then it can be easily shown that on this set of fuzzy sets, convex fuzzy sets on R this ordering is in fact, the partial order and with the same upper and lower bounds as we have considered earlier it becomes a bounded poset. The proof actually is based on the ordering of intervals.

Clearly, a quick proof would look like this.

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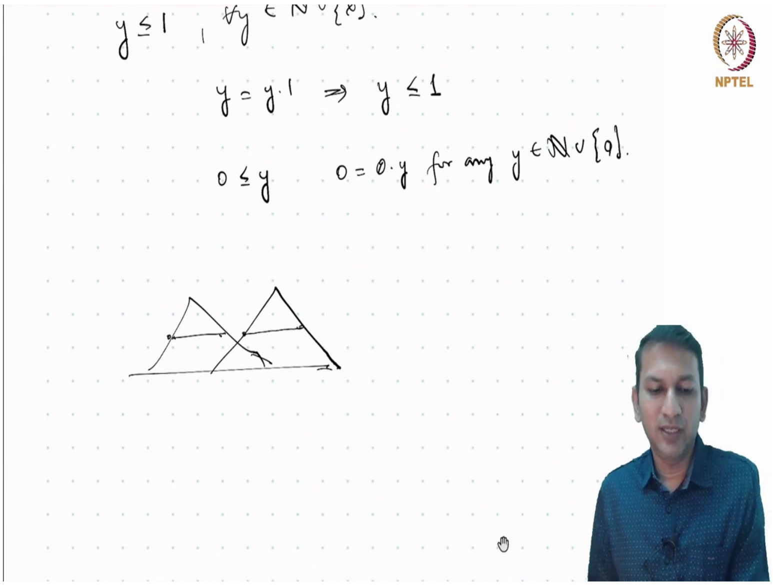

If you consider two convex fuzzy sets, let us for moment think of it like this and take any alpha cut. We know that the alpha types of convex sets are intervals, alpha types of convex fuzzy sets defined on R, their intervals, and essentially the definition here is about the ordering of intervals. So, based on that it is easy to see that this will become a partially ordered set with respect to this order.

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Well, a quick recap of today's lecture, of this lecture. We looked at binary relations especially partial order relations. We have seen that at times there is need for different orders as we have seen also in the case of fuzzy sets, we have two different orders; and because they often lead to some special posets.

And based on what we have seen on order sets we have been able to look at the ordering that we have introduced earlier, in one of the earlier lectures has actually been partial orders either on the entire generals power set of fuzzy sets or on a specific class of fuzzy sets, namely that of convex fuzzy sets defined on R. Now, the question is what lies ahead. In the next lecture, we will look at yet another special poset which we call the lattice.

We look at the lattice of fuzzy sets and hopefully this will allow us to discuss one of the questions that we have raised in a previous lecture, where we discussed that there are many possibilities to interpret operations on fuzzy sets, how do we choose among them. This line of exploration will clearly show us what each operation brings to the table. Next lecture we will look at the lattice of fuzzy sets.

Thank you once again for joining me. Hope to see you soon in the next lecture.

Thank you.