Approximate Reasoning using Fuzzy Set Theory

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Lecture - 06
Operations on Fuzzy Sets

Hello and welcome to the first of the lectures in the second week of this course titled Approximate Reasoning using Fuzzy Set Theory. In the course of the lectures during the 1st week we have seen the theoretical and practical motivations for introducing fuzzy sets, we have also seen fuzzy sets as a natural generalization of classical sets. At that time we stated the representation that we have come up with for fuzzy sets was useful, appropriate and amenable for processing.

However not all of these adjectives would have become clear to you during the course of the lectures so far. In this lecture we will look at how Operations on Fuzzy Sets can be viewed and performed and hopefully now some of those adjectives that we used about the representation that we have come up with might make sense.

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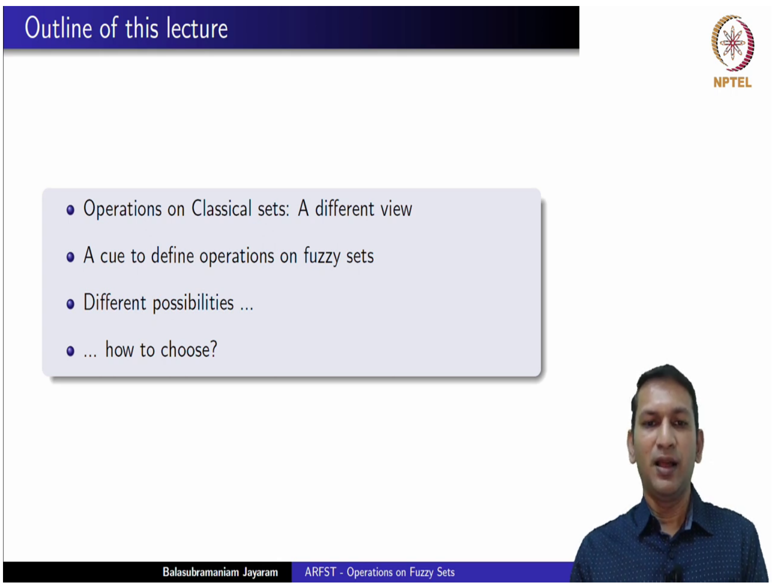

In this lecture we will look at operations on classical sets albeit in a different view and from here we will take a cue to define operations on fuzzy sets.

You will see there are very many possibilities and the question will boil down to how to choose among the many that we can see or the possibilities that exist.

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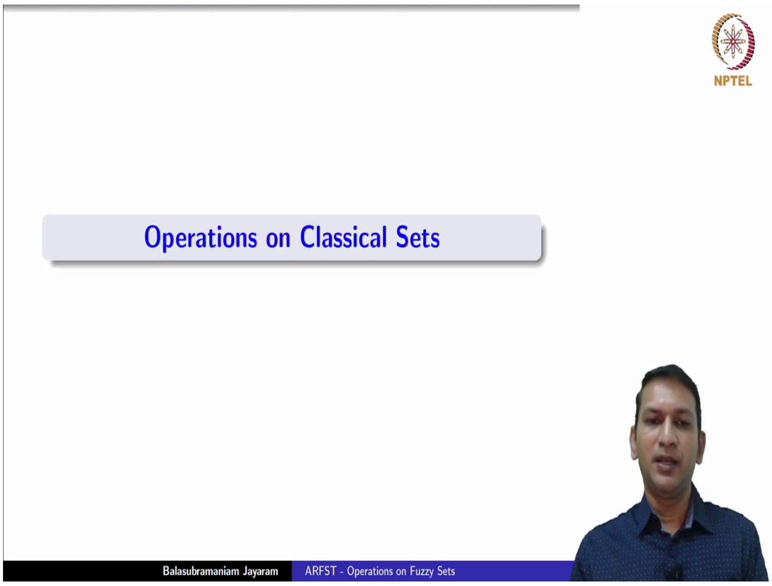

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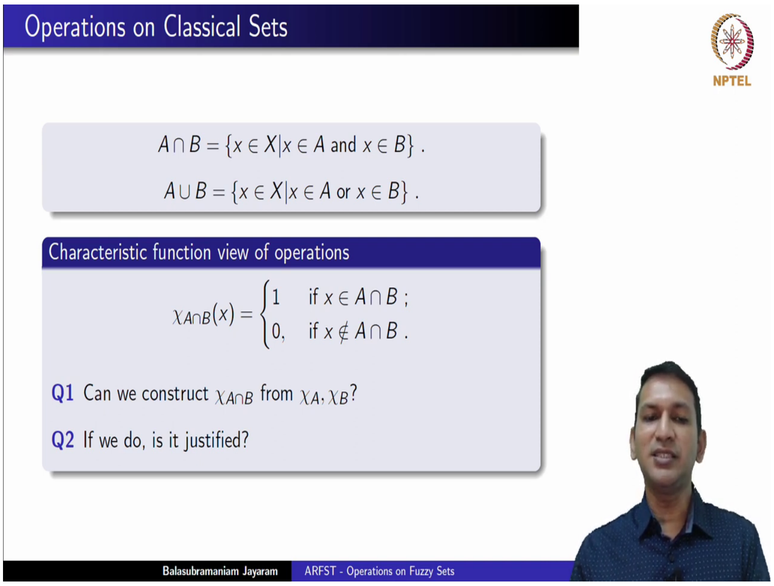

Let us look at operations on classical sets. We know that given a pair of sets classical sets we could discuss the intersection between them we know this is the usual definition of the intersection between any two sets.

A intersection B consists of all those elements in x, which belong both to A and B and if you are looking at the union or the disjunction of two sets then we say it consists of all those elements in x which either belong to A or to B or to both. We can also look at the intersection and union from the point of view of the corresponding characteristic functions let us look at this.

So, the characteristic function of A intersection B would look like chi of A intersection B of x is 1 if x belongs to A intersection B, 0 otherwise the question now is, can we construct the chi of A intersection B, the characteristic function of A intersection B given the characteristic functions of A and B? Second question is if we do is this kind of a construction justified?

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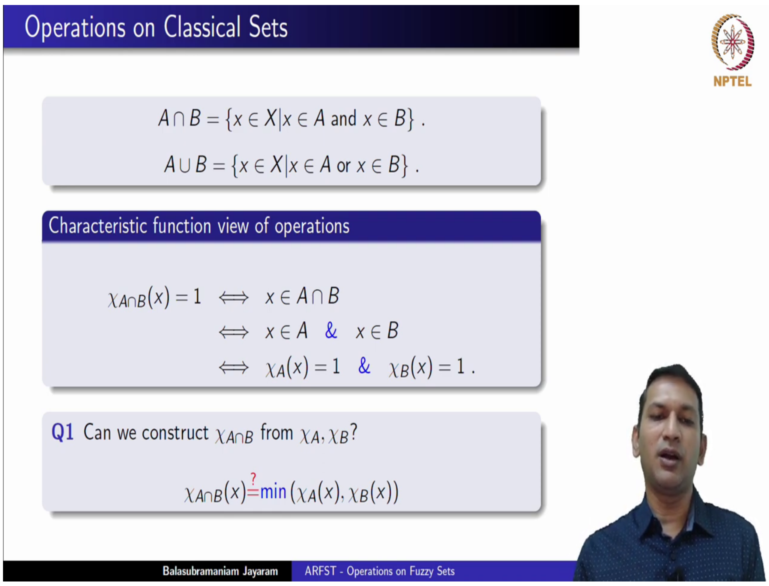

Now, let us look at each of these questions in this order. To understand this let us look at the characteristic function of A intersection B let us say at a particular x the characteristic function of A intersection B is 1 what does this mean? Chi of A intersection B of x is 1 this is will this will happen if and only if x belongs to A intersection B. Now, writing this again in terms of the corresponding characteristic functions what we find is either x belongs to A and x belongs to B.

Now, writing it in terms of the characteristic functions we see that chi A of x is 1 and chi B of x is 1. Remember the question that we asked can we construct the characteristic function of A intersection B from those of A and B? Looking at the values on the left hand side and the two values we have on the right hand side, it appears perhaps we could look at writing the characteristic function of A intersection B as a minimum of the corresponding characteristic functions of A and B.

If you substitute chi A of x as 1 and chi B of x as 1 minimum of 1 comma 1 is 1 and that is the answer that we are expecting for the characteristic function of A intersection B at x. However, this is valid only when x is takes the x takes the value 1 under the characteristic function of A intersection B.

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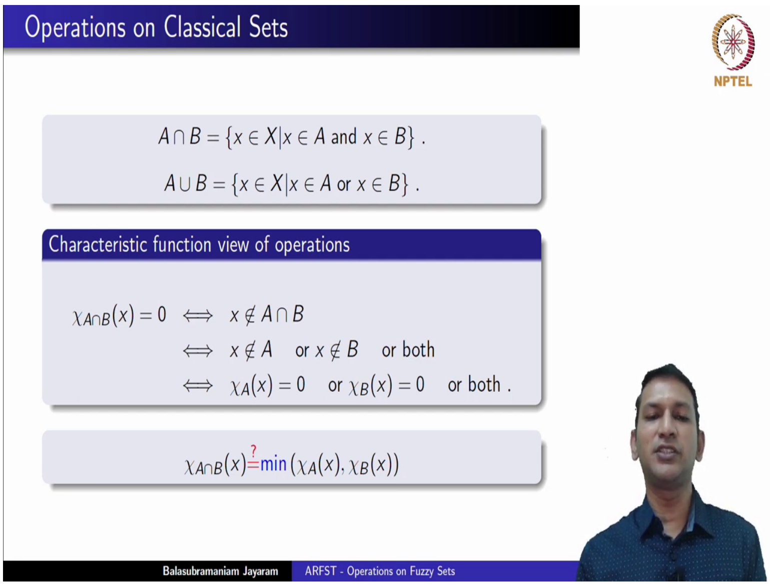

What about the case when an x takes the value 0 under the characteristic function of A intersection B?

This means that x actually does not belong to A intersection B what does this mean? This means either x does not belong to A or x does not belong to B or x does not belong to both the sets. In terms of the corresponding characteristic functions we see chi A of x is 0 or chi B of x is 0 or perhaps both. Now let us look at whether the formula that we wrote for chi of A intersection B in terms of chi A and chi B whether that formula is still valid.

Now, we wrote it as chi of A intersection B of x as min of chi A of x and chi B of x. Let us assume that both chi A of x and chi B of x are 0; that means, x does not belong to both of this sets; so, minimum of 0, 0 is 0. Now if it does not belong to only 1 of them; that means, it either belongs to x or to b for sure then we know that it does not belong to one of these that means either chi A of x is 0 or chi B of x is 0 and when you substitute those values in min we see that chi of A intersection B of x is 0. So, it appears that taking minimum of the corresponding characteristic functions of A and B does the trick.

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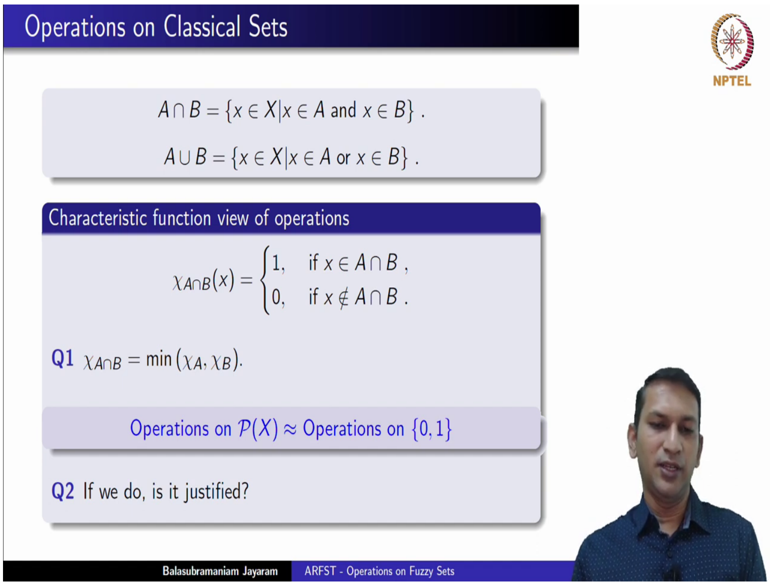

So, we have an answer for this first question that whether we can write the characteristic function of A intersection B in terms of characteristic functions of A and B this is one way we have come up with.

Now, immediately what it points to is a completely different way of looking at operations on classical sets. So far the operation that we have done on subsets of x as members of P of X. Now, we are seeing them as actually operations being performed on the set {0, 1}. Now the second question if we do such an operation if you operate on just only on the set {0 ,1} are we justified.

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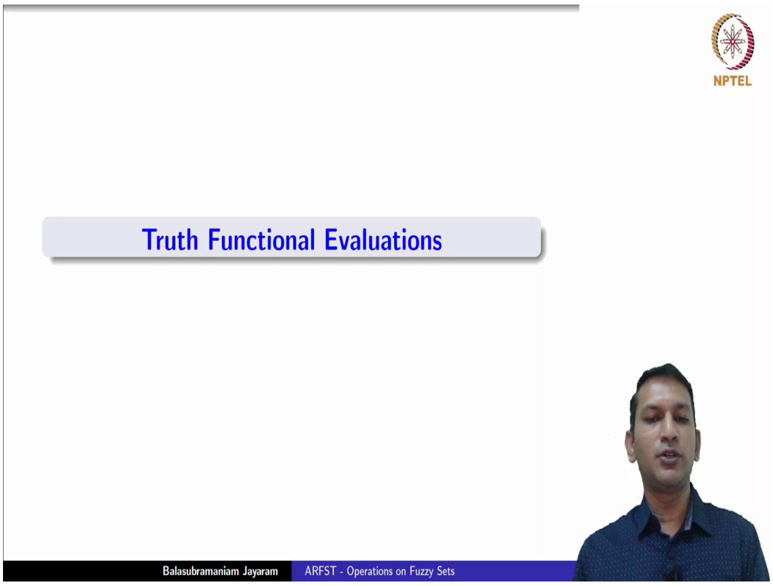

Well, let us look at it to understand this we need to look at truth functional evaluations what do we understand by truth functional evaluations?

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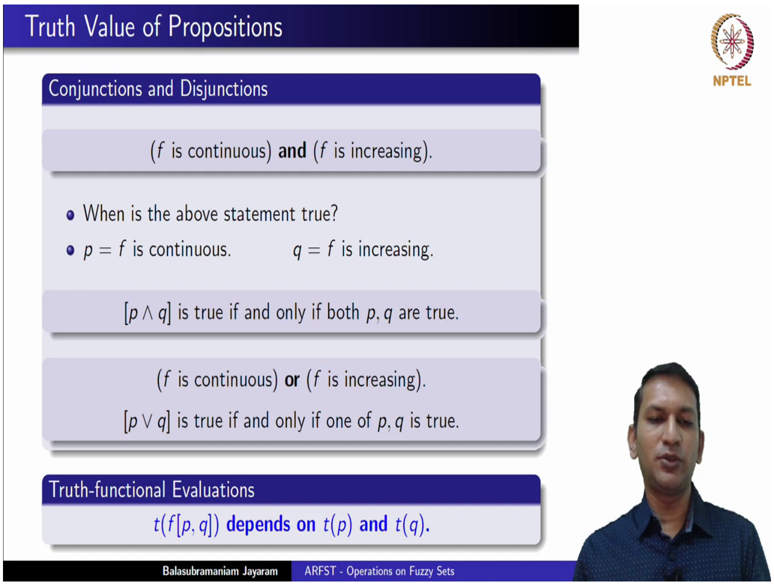

Consider the statement f is continuous and f is increasing. Now, we can ask the question when is the above statement true? Now it is a conjunction of two propositions, proposition p which is f is continuous and proposition q which is f is increasing.

Now, what is a proposition? Simply put a proposition is an assertion f is continuous f is increasing. Now to answer the question when is the statement f is continuous and f is increasing true. If you ask this question clearly we are asking this question p conjunction with q when is this true we know that this is true if and only if both p and q are true?

Similarly, we can ask for the disjunction f is continuous or f is increasing. So, this is a statement that we are making and if you ask when is this statement true it is equivalent to asking when is p disjunction q true, and we know that this is true if and only if one of p or q is true perhaps both also can be true, but at least one of them must be true. This is how we reason logic and because of the isomorphism the correspondences between classical set theory and classical logic.

The operations on classical set theory and the operations on classical logic two valued logic, we see that similar to how we interpreted intersection union we are interpreting both conjunction and disjunction of logical propositions. This is also known as truth functional evaluation wherein the truth value of a function of two variables actually depends only on the truth value of its constituents.

Please note we decided on the truth value of p and q based on the truth values of p and truth value of q.

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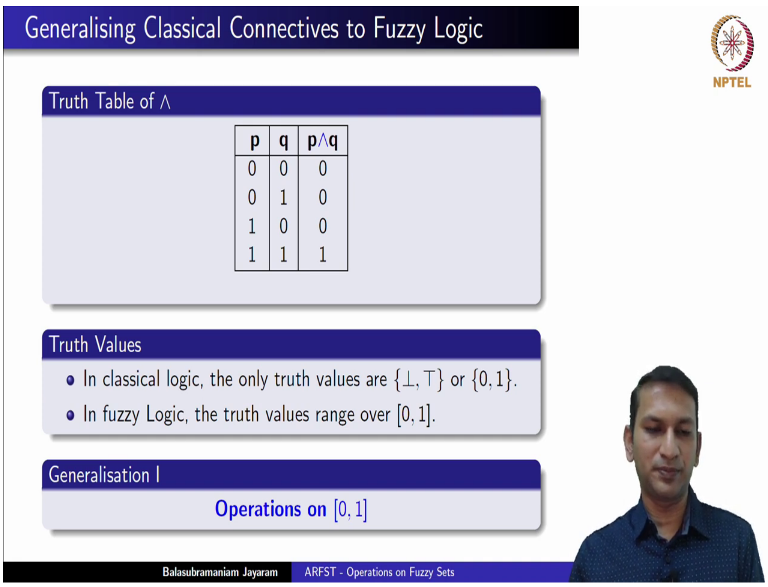

This also gives us a cue towards generalizing the classical connectives to fuzzy logic connectives and relate them to fuzzy set theoretic operations. Let us look at the truth table of conjunction in classical two valued logic where every variable can take one of two values 0 or 1 and conjunction is a binary operation; that means, it takes two variables.

So, when you look at them as p and q there are totally four possibilities and this is what we have seen before also, but in a tabular form you see clearly here that conjunctions of two propositions is true only if both of them are true. So, here you can interpret 0 as false and 1 as true. Now in classical logic only truth values are true or false or 0 or 1 essentially binary truth values.

However, in fuzzy logic the truth values range over the entire interval [0, 1]. So, now, here comes the first generalization that we can make looking at classical set theory and the correspondence to classical logic and how we can interpret operations on classical sets these are the classical logical connectives.

So, now just as how classical set theoretic operations can be looked at as operations on the set 0, 1; we can look at fuzzy set theoretic operations as operations on the entire unit interval 0, 1.

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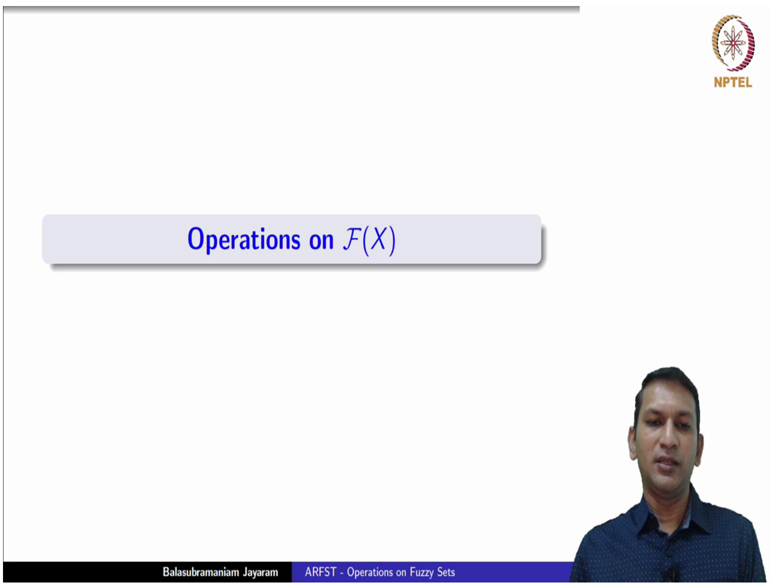

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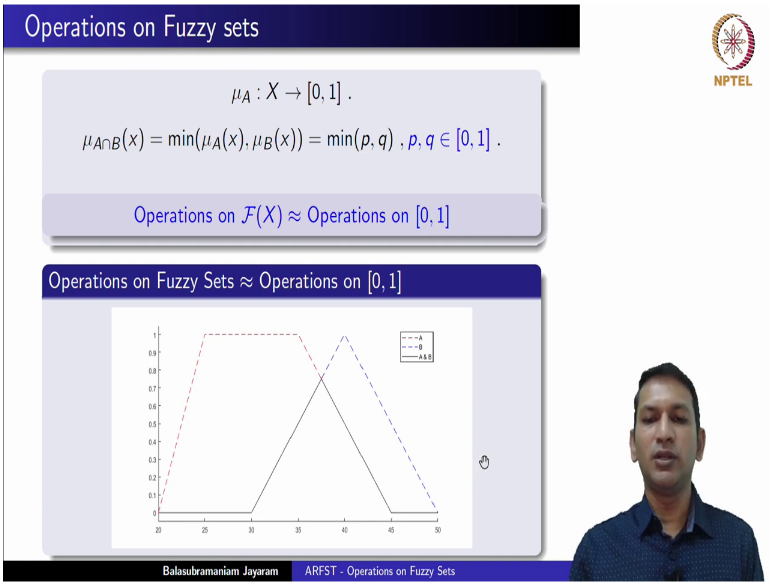

What does this mean? Let us look at them not as operations on 0 1, but as operations on the set of fuzzy sets. We know a fuzzy set is a function from the underlying domain x to 0 1.

If you are given two such fuzzy sets mu A and mu B we write by generalizing what we have seen a few slides earlier as the corresponding membership function to be mu of A intersection B of x as minimum of mu A x mu B x; that means, once again the membership function of intersection between two fuzzy sets, we are writing it as a minimum of the corresponding membership functions.

But note that where is the operation on 0 1 coming into picture, you will see that mu A of x is actually a value in 0, 1 mu B of x is a value of 0 1; that means, we can look at this simply as a function min of p comma q where p and q come from the 0, 1 interval. How does it look like on actual fuzzy sets? Remember fuzzy sets are functions on x to 0 1.

So, now this is what we meant by saying operations on set of fuzzy sets can actually be looked at as operations on the unit interval 0, 1. Let us for a moment take this fuzzy set A let us consider another fuzzy set B. Now what does it mean to talk about the conjunction of these two fuzzy sets? According to the formula given above we need to take the point wise minimum of both these fuzzy sets.

Essentially, it is the point wise minimum of both these functions defined on X to 0. It is easy to see graphically that if you consider the min of these two functions this is what you would get as the conjunction of A and B at every point here, all we are doing is for every point we are looking at corresponding membership value of that x in B and its on the x in A and we are taking the minimum of them both.

And you will see that since the support of B is only between 30 and 45, outside of the support of B the membership values are all 0 of these points so; obviously, when we take the conjunction using min they turn out to be 0. Similar is the case for the points between 45 and 50 while they belong to the support of B they do not belong to the support of A, which means their membership values in A is 0. And so, the membership value of them in the conjunction of these two fuzzy sets also remains 0.

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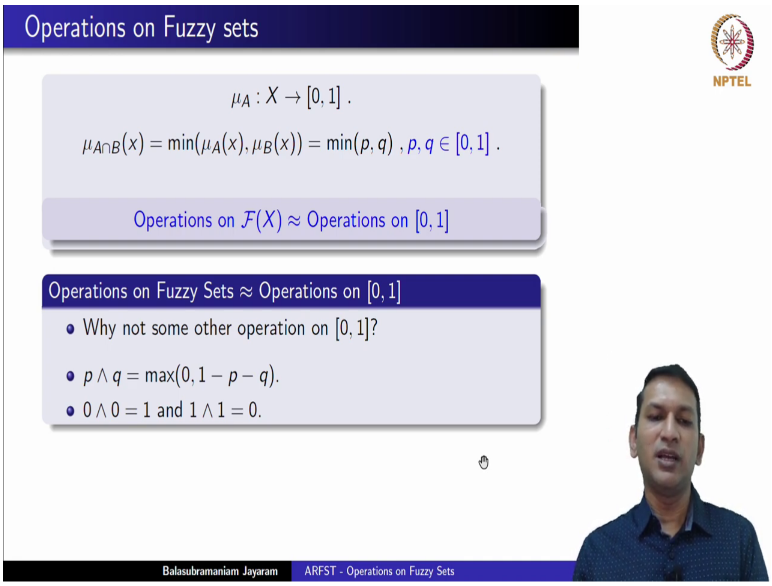

The question now is ok we tried with min we were successful why not some other operation on 0 1. Let us look at this operation let us try to say that the conjunction of two fuzzy sets will be interpreted like this.

Well there is no harm in assuming this but we would like to check if this operation is valid when applied to the classical sets; that means, instead of applying them on the membership functions, if you have to apply them on the characteristic functions will we obtained, will we get back our original intersection operation on classical sets vis a vis the characteristic functions. Let us look at that if p and q are 0; that means, those propositions p and q they are false.

Then what we expect is that the truth value of p and q or correspondingly when we look at the subset p and subset q p intersection q we know the element x in under consideration does not belong to p does not belong to q. So, it should not belong to p intersection q which means the corresponding characteristic function of p intersection q it should get a value of 0. So, let us assume that p and q are 0 and evaluate it.

If you take them to be 0 when you substitute here what we get is max of 0 comma 1 minus 0 minus 0 which is 1, which is essentially contradicting the idea of conjunction. What if you have an element which belongs to both p and q then we would want it to be in p intersection q? Now let us substitute the value 1 for p and q 1 for q which means that the truth value of p is 1 truth value of q is 1.

That means x belongs to the subset p x belongs to subset q when you up when you substitute these values 1 and 1 what you find is max of 0 comma 1 minus 1 minus 1 is max of 0 comma minus 1, which is actually 0. We see that somehow this does not capture the idea of conjunction when applied to classical sets ok.

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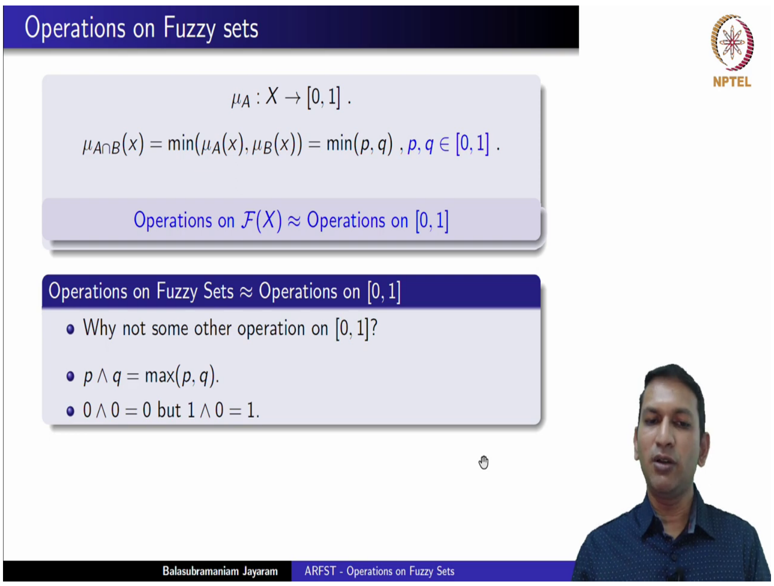

What about considering just the simple max operation? But if we consider 0 for p and 0 for q yes max of 0 0 is 0, but if 1 of them is 1 we get that it is 1 so; obviously, this does not capture the idea of conjunction on classical sets or nor does it capture the truth table that we have given for the conjunction.

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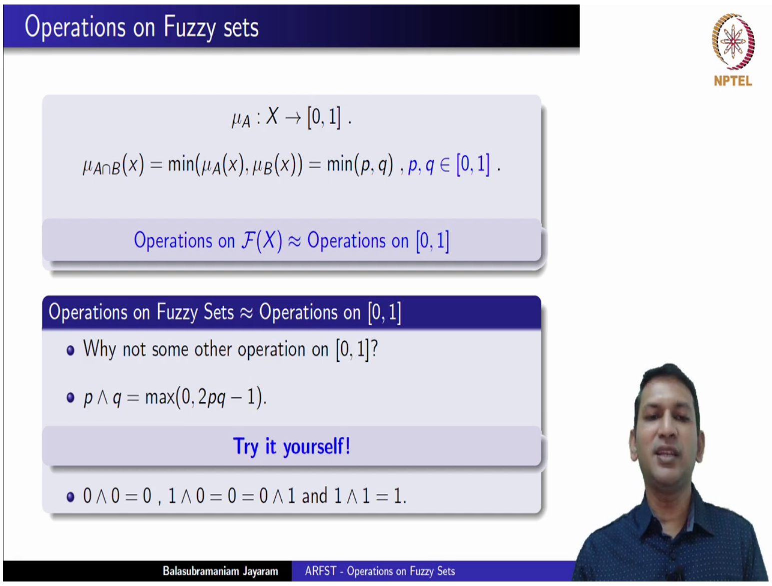

Let us try another operation what about interpreting p conjunction q as max of 0 comma 2pq minus 1.

And in this case I want you to try it yourself maybe quickly in 30 seconds or so, easily you can check when you substitute different values for p and q what would be the output for this particular operation? Please restrict yourself to giving binary values for p and q just 0 and 1 essentially trying to check if the truth table for the conjunction of binary valued logic whether it is being satisfied here or not.

Well, we hope you have had enough time to calculate the values for different p and q let us consider zeros for both p and q. When we substitute them here it turns out to be max of 0 comma 0 minus 1 which is 0 so far so good. Now let us give the value 1 to p and 0 to q and what we see is once again max of 0 comma minus 1 which is 0 and so, will be the case when q is 1 and p is 0 so far really good.

Let us look at what happens when we give the values 1 and 1. We see that it is max of 0 comma 2 minus 1 which is 1 which means this operation at least on the vertices that is when you restrict p and q to take values only from 0 and 1, we see that this function satisfies the truth table of the classical two valued conjunction.

Now you might then ask why not consider this function instead of the minimum function to interpret a fuzzy conjunction. That means, given two fuzzy sets why not consider conjunction of these two fuzzy sets using this formula well, that is an interesting thought and let us discuss that.

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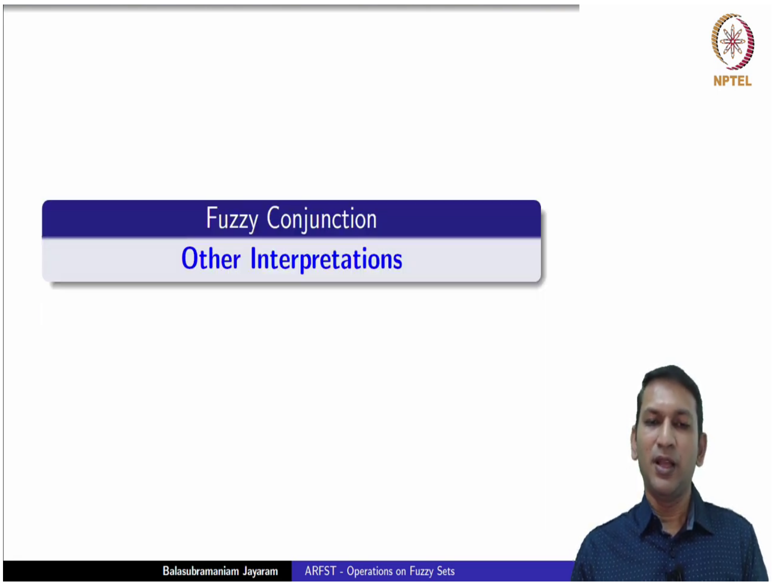

This example has also shown us that there is no one single way to interpret fuzzy conjunction and there are other interpretations also.

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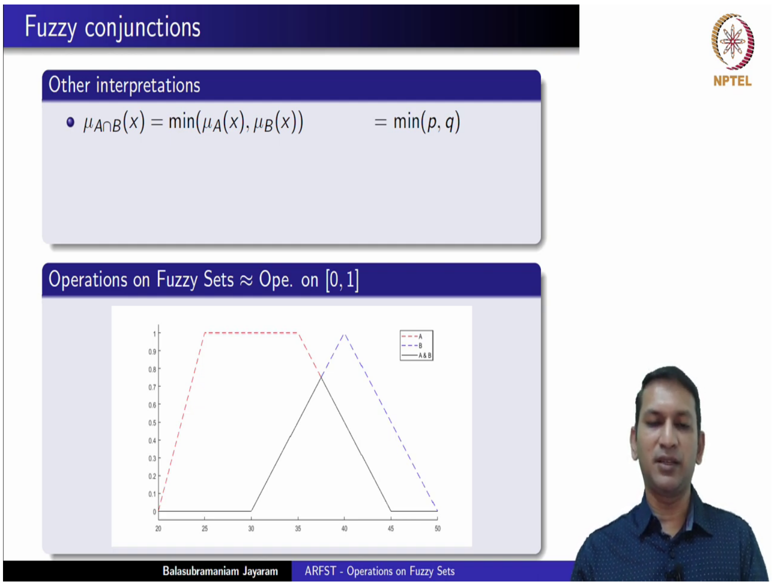

Let us look at a few of them and also look at how they operate on fuzzy sets so, as to understand and appreciate what are the properties that probably are being preserved by these functions. This is how we have interpreted in conjunction of conjunction between fuzzy sets as minimum between the corresponding membership functions. Note that we can write this simply as min of p comma q where p and q are values coming from [0, 1].

Remember we are going to interpret operations on fuzzy sets as actually operations on the corresponding co-domain, which is [0, 1] interval. Now we have seen that if A and B are fuzzy sets given by the red and blue curves, we know that interpreting conjunction using the minimum function we get the fuzzy set as given in the black color, why not interpret the conjunction using this operation?

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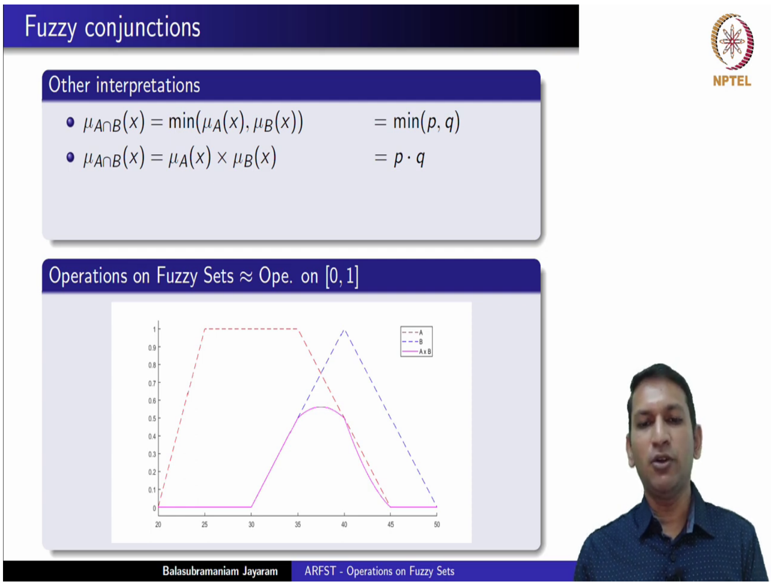

As you can readily recognize this is essentially the product operation p dot q where p and q are once again coming from the [0, 1] interval. It is clear that this operation will also satisfy the classical two valued truth table for conjunction and if you were wondering how would it look like this is how it would look like when it comes to applying product as the conjunction of these two fuzzy sets A and B.

And you will immediately see that while this function is continuous the function given in magenta is continuous it has altered the shape quite drastically compared to either A or B.

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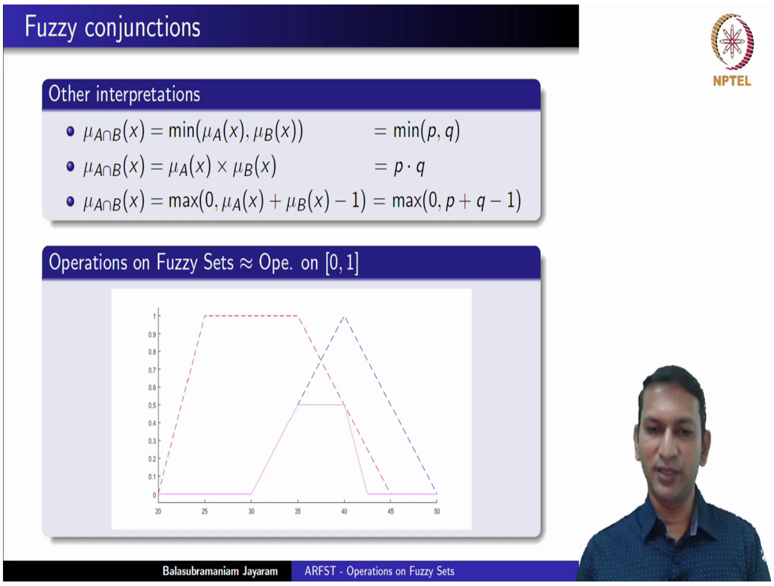

Consider this function which perhaps is easier to see if you were to write it as a function on the unit interval 0, 1 consider this function max of 0 comma p plus q minus 1.

This is much like the function we considered little while earlier max of 0 comma 2 p q minus 1 clearly if both p and q are 0 it is 0, if 1 of them is 1 the other is 0 it is still 0 and if both p and q are 1 then we obtain 1. So, once again this function also satisfies the classical two valued truth table for conjunction. Now, how does it look like when we apply this operation on the fuzzy set that we have considered so far?

Once again A and B given in red and blue are the fuzzy sets A and B and the conjunction of these two fuzzy sets using this last operation is what you see in terms of magenta. Once again the shape of the functions is kind of retained, but still there is quite a variance.

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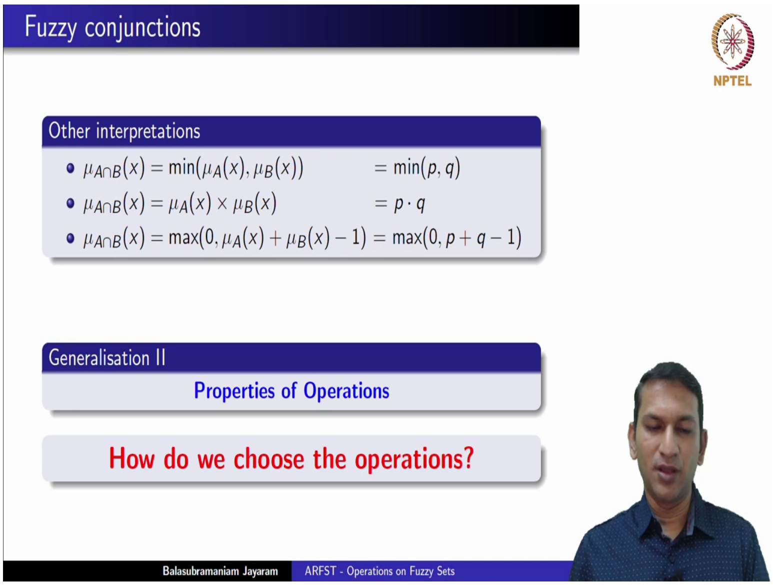

The most important lesson that we could draw from this is looking at the operations on fuzzy sets as operations on the entire unit interval 0, 1 we have been able to come up with many interpretations of such operations.

Now, this was made possible by the way we were able we have represented fuzzy sets themselves as functions from x to the entire unit interval 0, 1. Remember characteristic function was from x to just the set 0, 1, membership function the fuzzy sets are functions from x to the entire interval 0, 1. Now, operations on classical sets we have seen that they can be looked at as operations on 0, 1 the actual co domain of the characteristic function.

Similar is the case with fuzzy sets operations on fuzzy sets we are now able to see them as actually operations on the on its co domain which is the entire 0, 1 interval. Now this allowed us to come up with different interpretations of these operations. We have seen so far only for conjunction but this is true also for the other set theoretic or logical operations.

What is interesting is you will you would have noticed from the curves that we got by applying these different operations that somehow the properties of these functions differ and they are actually making a difference when you are when these operations are being applied to a pair of fuzzy sets. Now the question that automatically arises is, how do we choose the operations?

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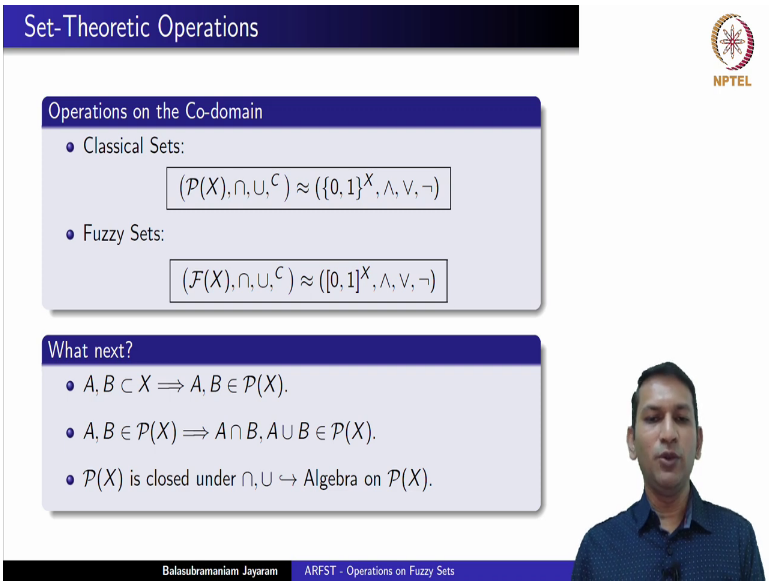

Of course, the context will play a role; however, there are some interesting correspondences that if you look into they are quite insightful for us. This correspondence must be immediate for us all because we are only relating a subset of x a classical subset of x in terms of its characteristic functions. And immediately you see that operations on classical sets can be seen as operations on the corresponding co-domain of this characteristic function which is 0.

And hence this correspondence with respect to operations on fuzzy sets is also immediate as operations on the corresponding membership functions or essentially as operations on the unit interval 0, 1. Now let us take the classical sets A and B subsets of X we know that they are also elements of P of X. Now, as elements of P of X if you apply the conjunction of the disjunction operation, we see that once again we get subsets of X and; that means, they actually become elements of P of X.

This essentially means that P of X is closed under this binary operation conjunction or disjunction. What this tells us is that we could now look at this entire study of operations of P of X with respect to conjunction and disjunction as a set with a binary operation which is closed on the set and this gives us motivation to look at some algebraic structures on P of X.

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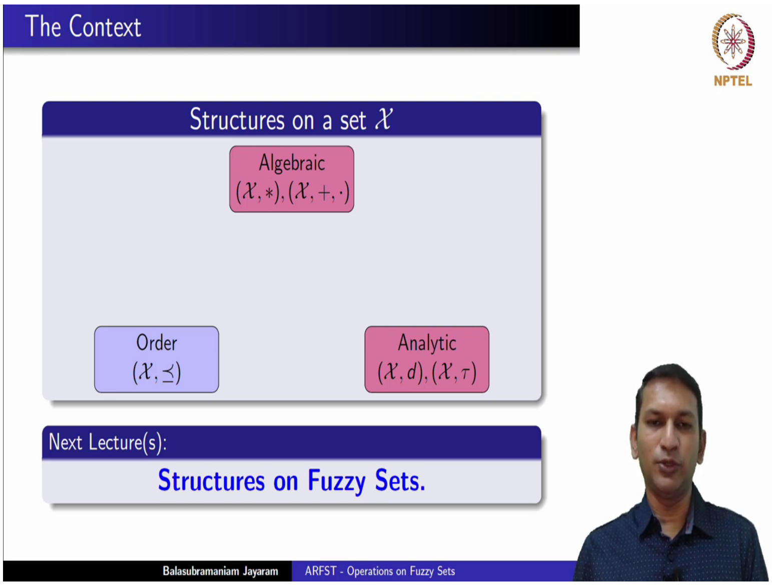

Now, not just the algebraic structure of P of X we will do little more than that if you look at what are the structures available on a set x. Of course, the first that springs to our mind is that of an algebraic structure you could have a set with one operation one binary operation you probably think are thinking of monoids groups. If you have a set with multiple operations let us say two operations, you could think of a vector space you could think of rings.

But this is there are many such algebraic structure that you can define, but these are the algebraic structures that you can define on an x. You can also define relational structures on a given set x for instance many of you may already know about partially ordered sets, which will be subject of discussion in one of the coming lectures in this week we could also have analytical structures on x.

For instance, a metric structure; that means, a set x on which a metric is defined or a topological structure set x with a topology, which is essentially a collection of subsets of x satisfying a set of axioms we call it a topology. And we can study the set x with respect to the set of subsets which we call topology this we do as an analytic study.

So, essentially on a given set x we can have algebraic relation or analytic structure. Now considering x as a set of all fuzzy sets f of x we could actually do the study and we will do the study in the rest of this week during our lectures. Once again we will do what is necessary for the course the way we have designed it, keeping in mind that the course is regarding approximate reasoning using fuzzy set theoretic tools.

So, keeping that in mind we only pick and choose what is relevant. There is so, much to be done outside of this, but given the context we will limit ourselves to what is relevant and appropriate and what is needed. So, in the rest of this week the next few lectures we will look at different structures on fuzzy sets and perhaps how they also immediately allow to generalize some concepts or introduce some new concepts on fuzzy sets.

Thank you once again for joining me during this lecture and hope to see you soon in the next lecture.