Approximate Reasoning using Fuzzy Set Theory

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Lecture - 56
Suitability of BKS with Yager's Implications

Hello and welcome to the second of the lectures, in this week 12 of the course titled Approximate Reasoning using Fuzzy Set Theory. The course offered over the NPTEL platform. In the previous lecture, we had questioned ourselves, if it were possible to extract the properties that led to the desirable properties of a fuzzy inference system. If you were able to extract those properties as functional equations or functional inequalities.

And if we knew that there are also implications from outside of the family of R implications which satisfied them can we really employ those implications also in an FRI or an SPR. And if we do, do they also enjoy the same desirable properties that we expect from the inference scheme that of monotonicity or interpolativity, continuity or robustness.

In this quest, in today's lecture, in this lecture, we will discuss the suitability of Bandler-Kohout subproduct inference scheme, but with a difference. We will use one of the Yager's families of implications, either an f-implication or g-implication in the place of residual implication obtained from a left continuous t-norm.

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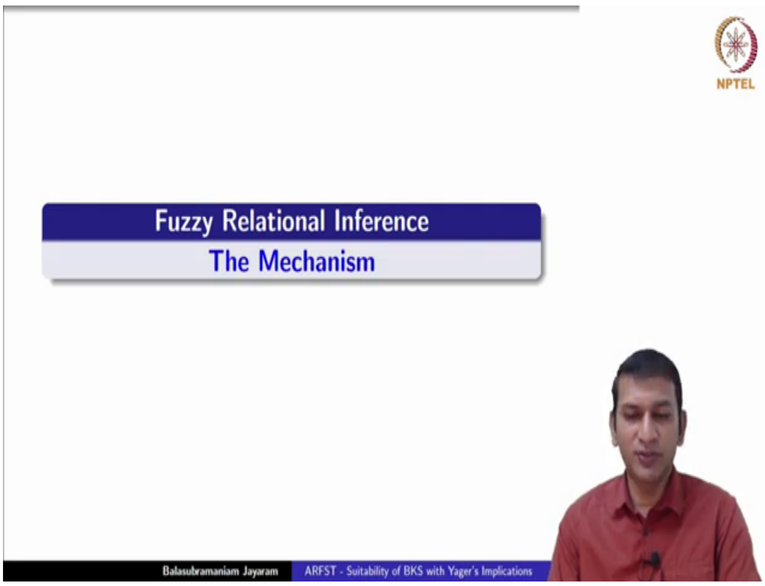

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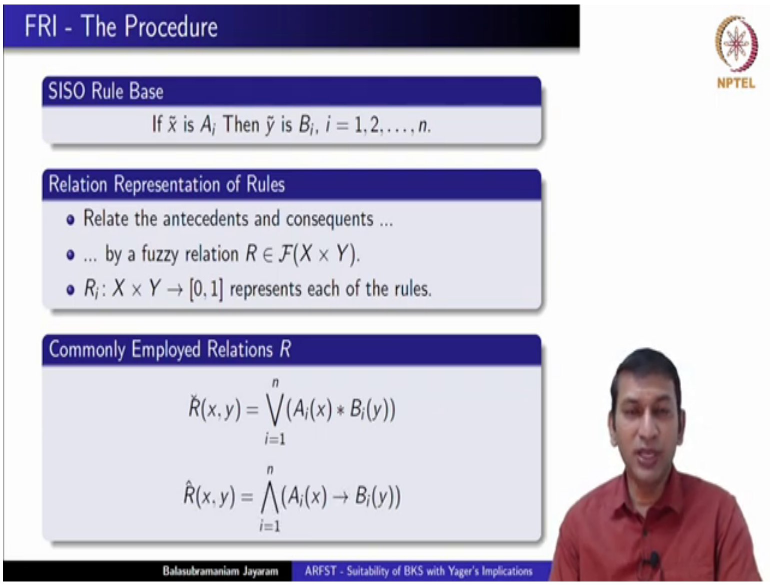

Let us have a quick recap of the mechanism of fuzzy relational inference. We are given a SISO rule base a single input single output rule base, n of them, such rules. What we do is we relate the antecedents and consequents by a fuzzy relation. So, for each rule we have a relation R_i.

Now, we know these are the commonly employed relations R check and R cap, where with R check you used to t-norm and in case of R cap we use an implication.

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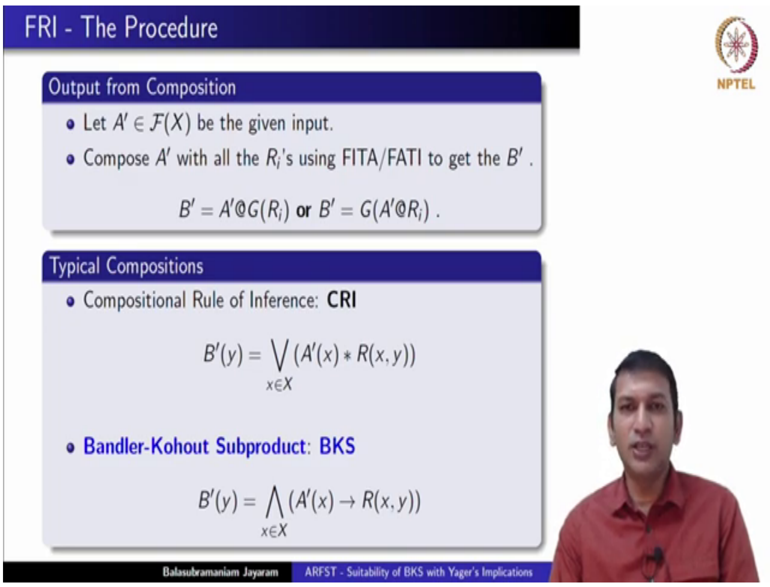

Now, given an input a dash, we obtain the output using a composition operator. And of course, we use in the case of multiple rules either the FITA or the FATI inference strategy, which essentially means if it is FATI what we do is we aggregate all the rules, the relations of the rules R_i using the aggregation function G. And then, compose the given A dash with the aggregated rules or we compose A dash with each of the rule, the relations of the rules and then finally, aggregate. So, this is how we obtain the B dash.

The typical compositions are the sup-T and the inf-I composition. So, we these two are packaged together under FRIs and one of them when we use sup-T composition we call it the compositional rule of inference. And in the case of BKS inference, we use the inf-I composition.

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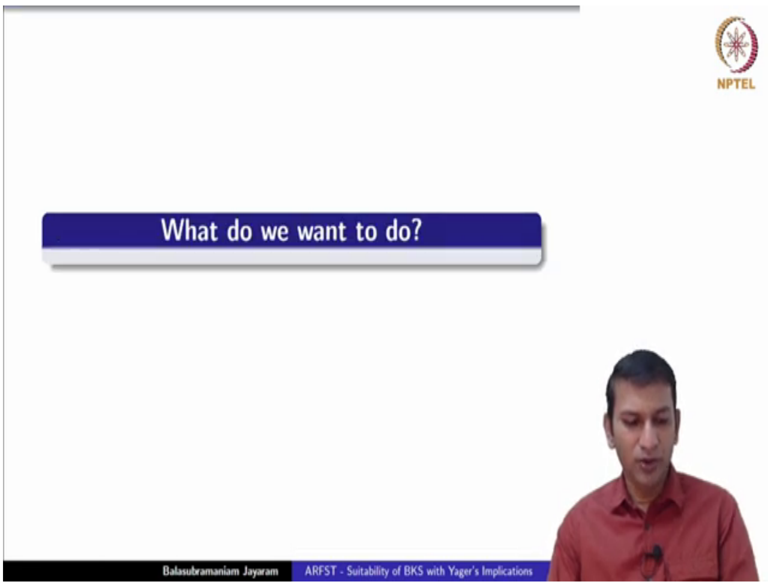

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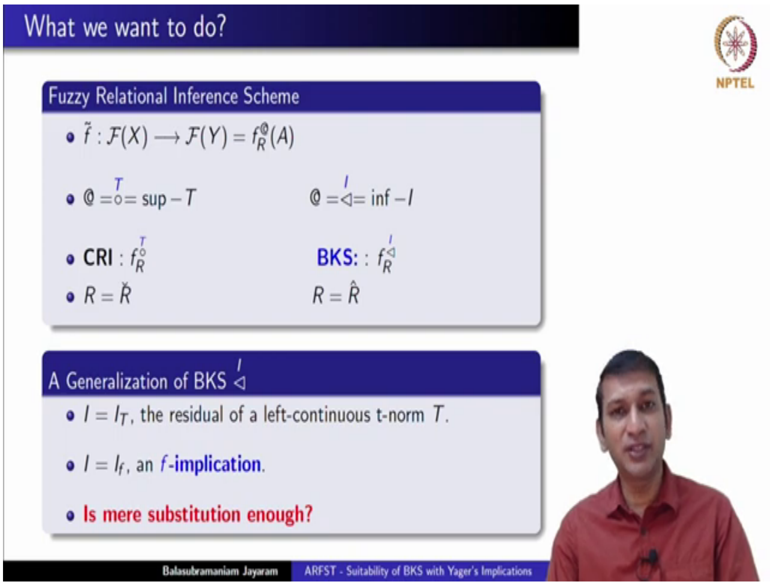

Now, what do we want to do? We want to take a fuzzy relational inference scheme, look at it as only a mapping between F(X) to F(Y). In that sense for a given input fuzzy set A, this inference scheme depends only on R, the relation, and the composition (Refer Time: 03:33).

So, if we consider this either the sup-T composition or the inf-I composition, we this is what we call the CRI or the BKS, we know that for R, both R check and R cap they play an important role, either in terms of the corresponding interpolativity of the corresponding FRIs or FITA being equal to FATI or being able to look at FRI in terms of SPI.

But now what we want to do is we want to generalize this, at least since we are using an implication, let us generalize the Bandler-Kohout sub product wherein we are considered using a residual implication instead now we will use an f-implication. Of course, since it is only inf-I composition, instead of using I_T we could use an I_f. But the question is mere substitution enough. As a composition it may be ok, but now what we are looking at is as an inference scheme.

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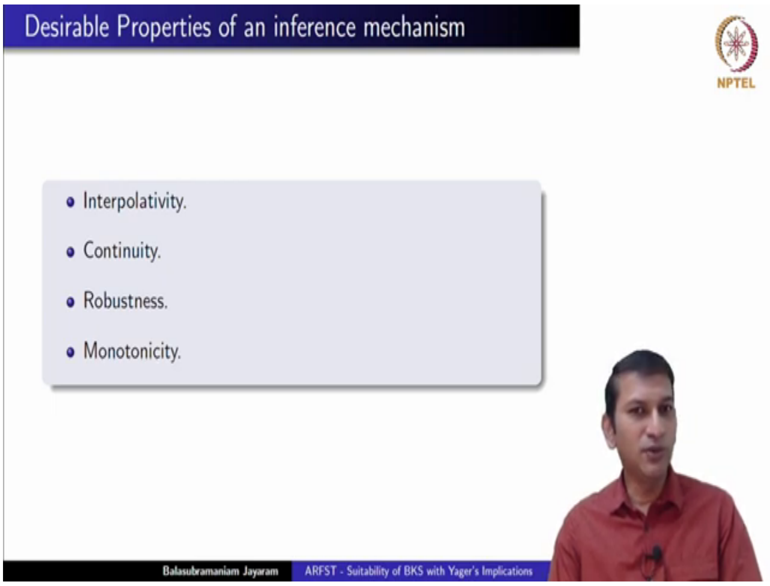

So, as an inference scheme we know that, we expect a few desirable properties of an inference scheme. Some of them that we have discussed over the last 4 weeks are these interpolativity, continuity, robustness, and monotonicity. So, instead of merely substituting and saying let us use it, we should also after substitution discuss these properties. And that is what we will do in the rest of this lecture.

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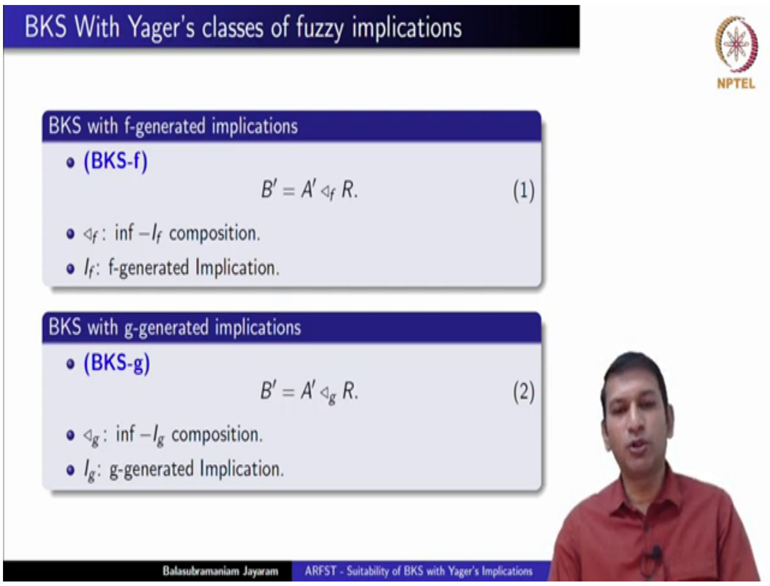

So, we will denote the Bandler-Kohout subproduct inference scheme with f-implication by using the symbol delta f. So, it is an inf I_f composition, where I_f is the f generated implication. Similarly, we can also discuss the inf I_g composition, where I_g is the g-generated implication.

We will see from the approach that we take in this lecture, that we will discuss BKS in depth. But it is clear to see as we progress that whatever properties that we have discussed with respect to f-implications, if you substitute a g-implication in its place the same properties will also hold.

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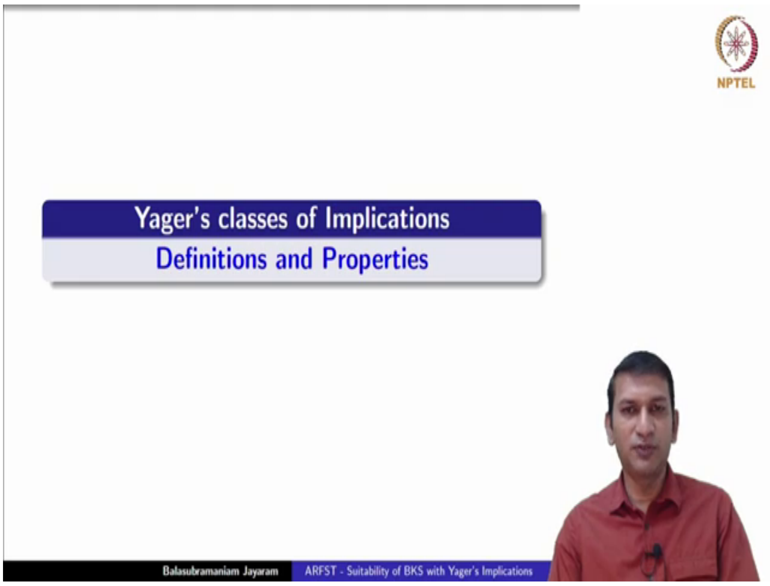

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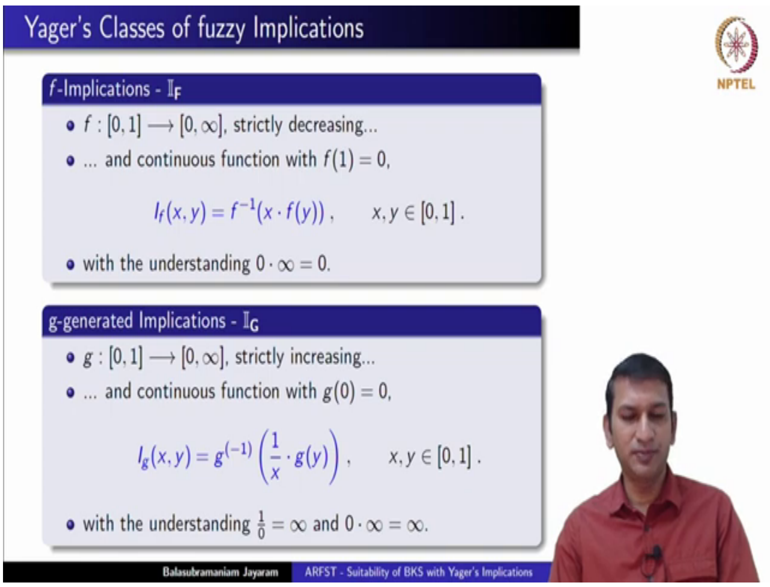

Let us recall the definitions of these two families of fuzzy implications. The f-implications which we will denote by f. These are obtained from the additive generators of t-norms; that means, these are functions from [0,1] to [0,infinity], which are strictly decreasing continuous with f(1) is 0. And this is the formula of an f-implication, f inverse of x into f(y) with this convention.

Similarly, for g-generated implication or a g-implication, it is obtained from the additive generators of t conorms means g is a function from [0,1] to [0,infinity], which is strictly increasing and continuous, with g(0) is 0. And then, g-implication itself is given by this formula. Note that we also employ this convention.

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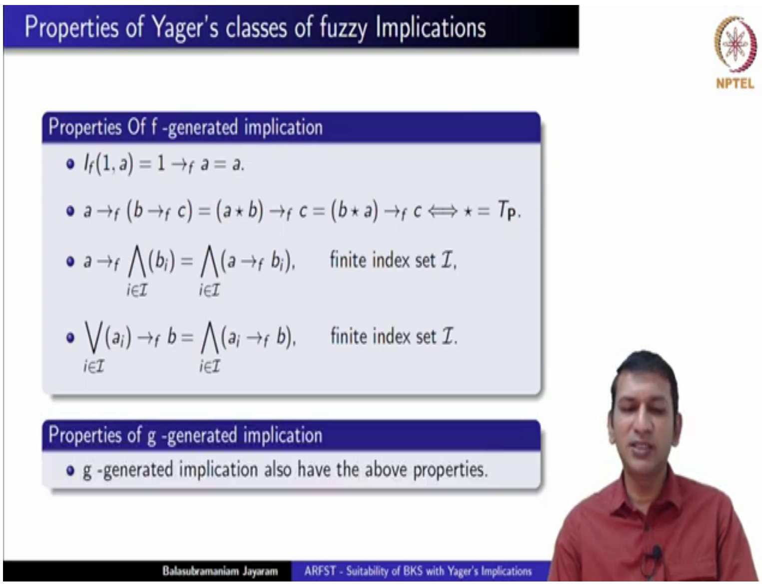

What are the properties of Yager's class of fuzzy implications? We know both of them are neutral. We will indicate it for the f-implication, but many of these immediately follow also for the g-generated implications. We know clearly it has neutrality property. In the previous lecture, we have seen that f and g-implication they do satisfy law of importation with respect to the product t-norm.

And we have seen that any implication as long as the index set is finite, also satisfies both these distributivity laws, the antecedent distributivity and the consequent distributivity. And as was mentioned these are properties satisfied by g-generated implications also.

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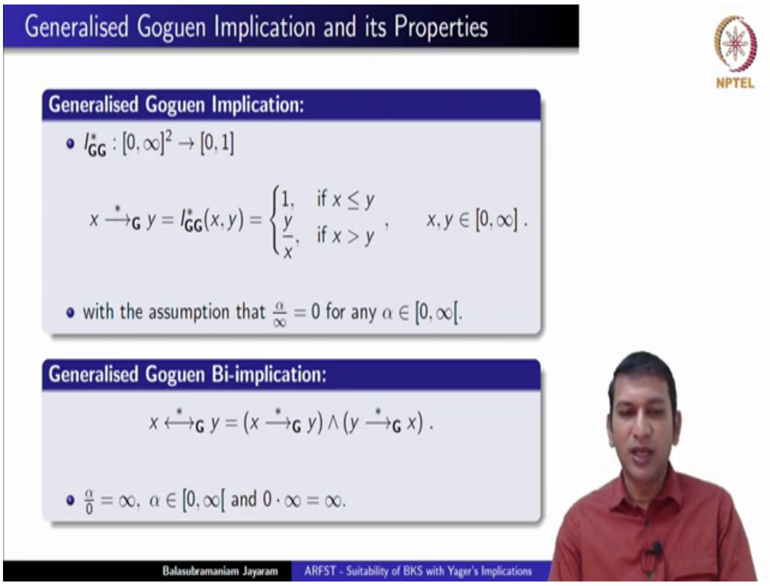

We will introduce what are called Generalized Goguen Implication. So, there is a functions from [0,infinity] square to [0,1]. Why is it called generalized Goguen implication? Because it generalizes the formula of Goguen implication which is the original Goguen implication. We have seen is from [0,1] square to [0,1], and here essentially we take the same formula, but only that we allow the x and y to vary from 0 infinity and of course, with this convention.

Now, once you have an implication you also can get a bi-implication, which is typically x bi-implication, y is x implies y and y implies x, where and is interpreted as a minimum operation. And you see that these are actually, while x and y are coming from 0 infinity, we know that the corresponding x implies y with respect to this Goguen implication generalized Goguen implication, they actually push it to the 0, 1 interval. So, it is essentially mean operating on values in 0. And of course, we will use this convention.

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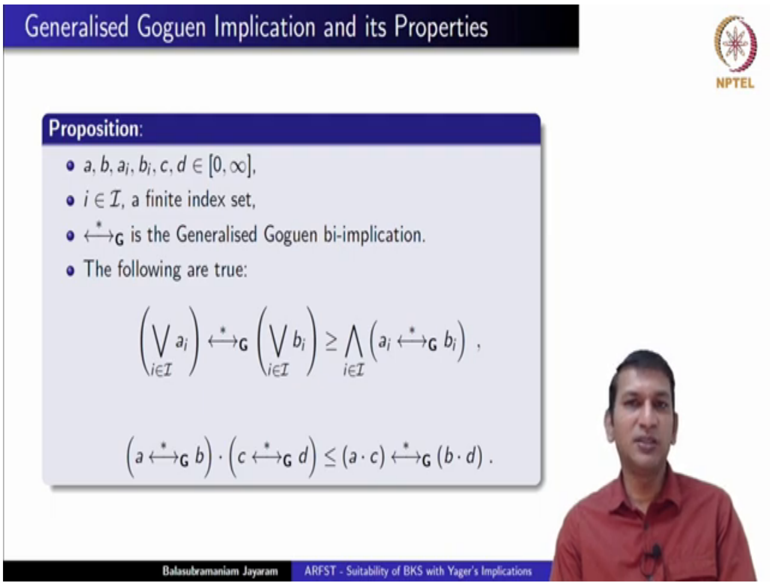

Now, if you take this generalized Goguen implication means corresponding bi-implication, if you pick these points a, b, and a is and b is from 0 infinity. Once again, where i comes from a script i which is a finite index here, and if we consider the generalized Goguen bi-implication, we can show the following properties are in fact true.

Now, please recall the second of the properties is something that we have made use of many times over. Here we are using the Goguen bi-implication and the product t-norm. Whereas, when we discussed from the residuated lattice structure, these were the bi-implications obtained from the R implication, the residual implication, and instead of product we were using the star, the t-norm, that gave rise to the residuated lattice structure.

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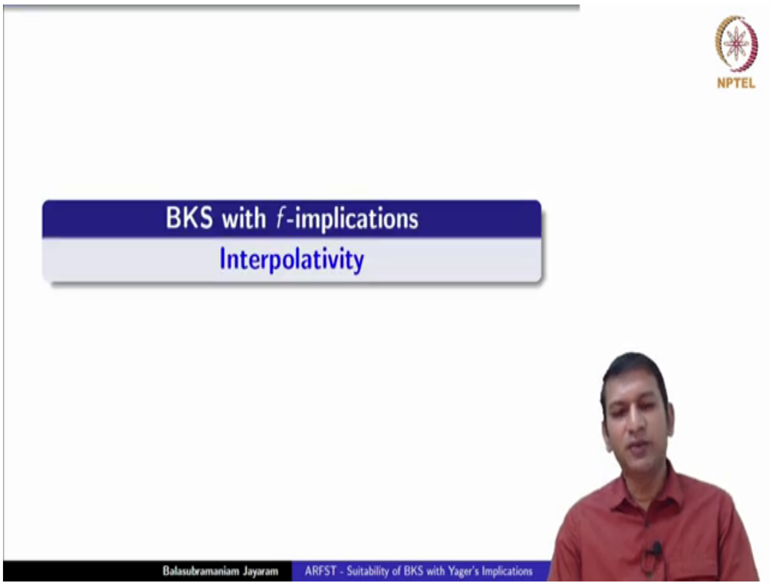

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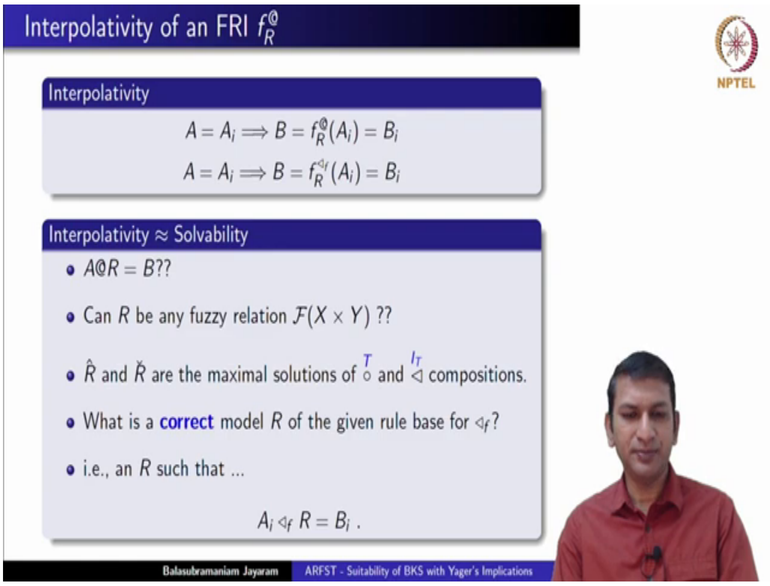

Now, let us begin by discussing one of the desirable properties and slowly we will move on to the others also. The first of them as we have seen is interpolativity. So, note that by interpolativity, we only mean that if the input matches one of the antecedents, then the corresponding output should be the corresponding consequent.

Now, we are looking at FRI as a mapping from F of X to F of Y, where R is some relation and we have a composition, some composition at the end. So, in that sense, we are going to use delta f that is BKS of BKS-f, which is BKS with f-implication for the implication in the composition. And we are now looking for an R, that is the generalized perspective that we have had for quite some time now.

Now, we have seen interpolativity is also related to solvability. So, we are questioning, ok when is A composed with R equal to B? When A is A and B is B? And can R be any fuzzy relation? We have seen in the case of sup-T and inf I t compositions, R cap and R check played an very major role. However, we also knew that we could consider any R which was giving us interpolativity, and such relations are we call them correct models.

So, now, the question is for this BKS-f inference, what is the correct model of R? That is can we find an R such that A i BKS-f composed with R is actually equal to B i for every i.

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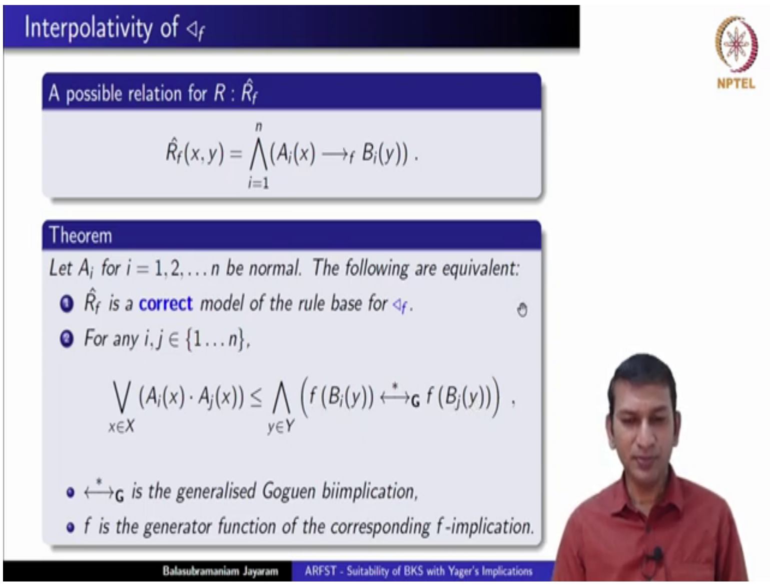

Now, what could be a possible relation for R? Now, taking q from R cap because we are using an implication, why not consider this? So, instead of using an R implication, why not use an f-implication, is the question. Well, let us go ahead and use this R f cap as the possible relation. We have a result which says that if the antecedents are all normal, then the following are equivalent.

This R f cap is a correct model of the rule base for BKS of composition. What does it mean? That means, using this R f cap we can ensure interpolativity. This is true if and only if, for any i j coming from 1 to n, the following inequality is in fact valid, here the bi-implication is the generalized Goguen bi-implication. And the f that we have here it is not simply just the output the consequent fuzzy set of the rules, but f of that, so transformed by f. So, this is the result that we have.

So, R f cap can be considered as an admissible relation. It will have interpolativity, it will give rise to interpolativity in a BKS-f composition provided this inequality as well.

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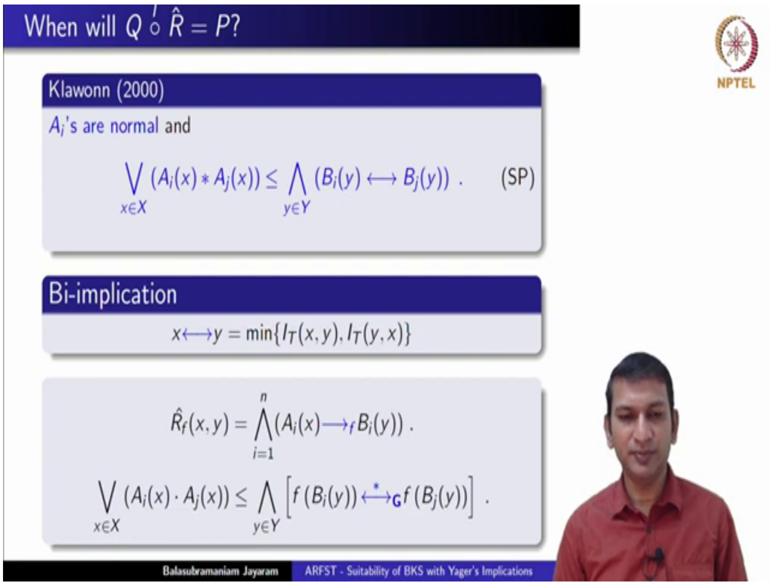

Now, it is quite easily similar to what we have seen earlier. Remember, when we are discussing sup-T composition with R cap, we have seen this result by Klawonn, which says that if A i's are normal and if this inequality what we called as the semi-partition inequality, if it is satisfied, then we know that that corresponding R is in fact an admissible relation in the inference scheme. Of course, the bi-implication there was obtained from the corresponding R implication.

In our case here, what we see is almost the same thing. We are using, we are obtaining the bi-implication from the Goguen implication, the extended Goguen implication, and instead of using an R implication we are using the f-implication here. And we have ensured under this condition that the system will be interpolated.

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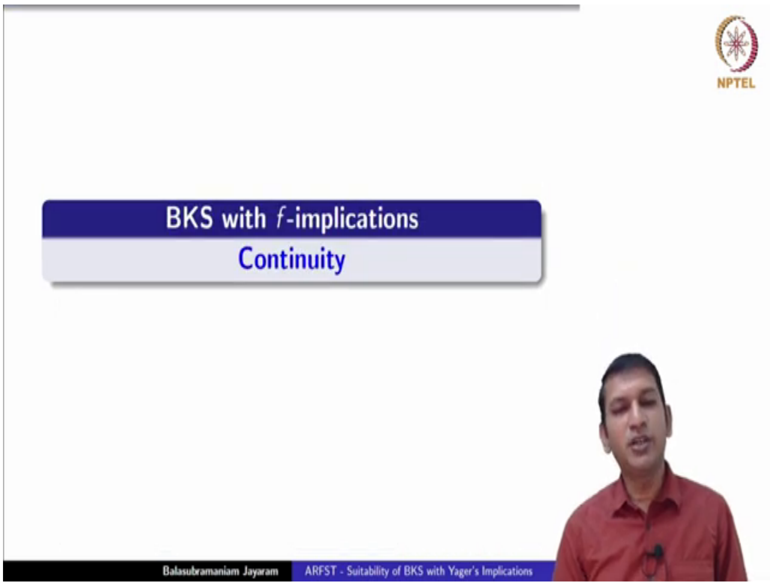

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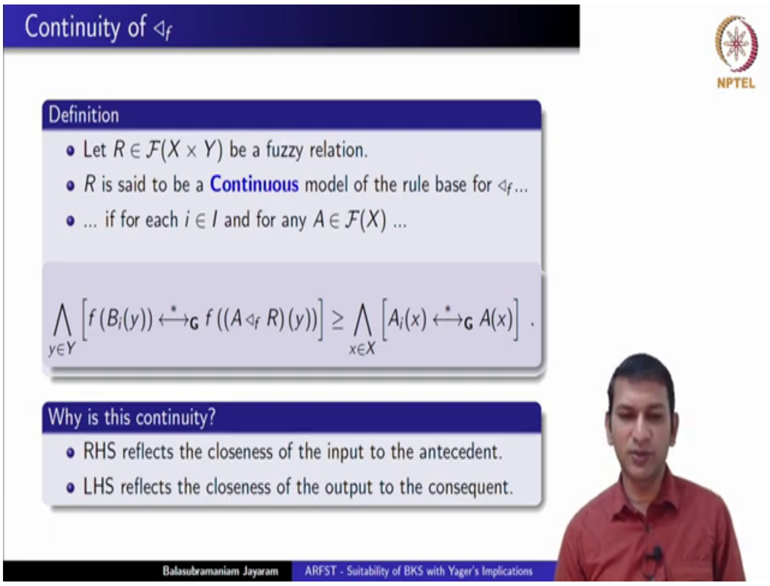

Now, what about continuity? Well, what is continuity? We have a relation R, we say it is continuous model for the rule base with respect to this composition, if for each i the following inequality is valid. Now, why is this continuity? We know that the bi-implication can also be looked at as an equality, an equivalence relation with respect to a particular t-norm.

So, in that sense, the right hand side it reflects how close the inputs are to the antecedent and in the left hand side reflects the closeness of the output to the consequent. So, essentially this is what it is trying to capture.

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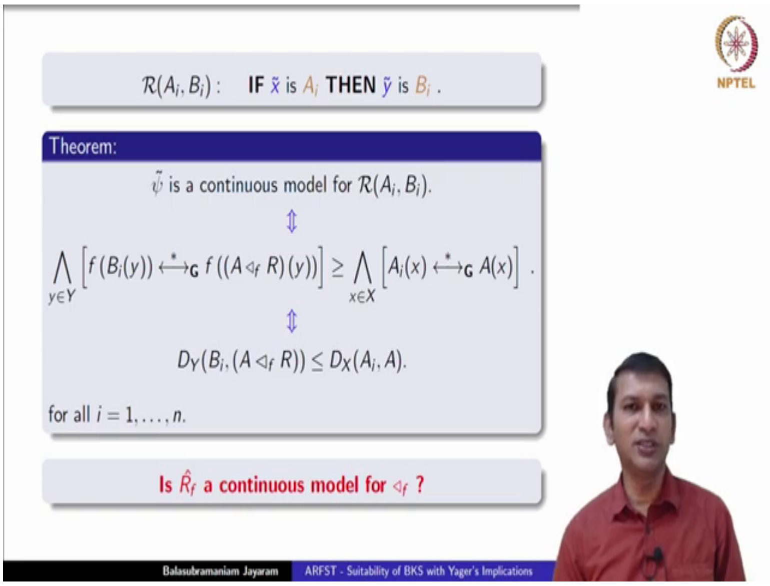

However, we can also give another interpretation. Recall, that when we talked about having psi tilde which is the system function of an f i s seen as a mapping from F X to F Y. We called it a continuous model for a given rule base, if and only if, this property was valid. And we were able to show this actually related to showing that there we created a metric D f on the input and output spaces.

And we showed it as the corresponding outputs from the inference obtained for a given A, we showed that as a mapping with these matrix on F X and F Y, they can be considered continuous. In fact, a similar kind of result can be shown when we are using BKS-f. This equation that we saw a couple of slides earlier, we called it the continuity equation.

We can show that psi tilde which is a system function obtained by from BKS-f for a given rule base. It is a continuous model if and only if this is valid this inequality is valid. And why is this called continuity? Because we can obtain we can construct matrix on Y F of Y and F of X, D Y and D X, such that once again as an inputs are closer to antecedents, we can show with respect to another matric, the corresponding outputs are also closer to the consequents.

So, that is why we could consider this inequality as an equation of continuity, and say the corresponding model is continuous. Now, the question is if R f cap continuous? Does it satisfy this inequality? Only, then we can say that yes, the corresponding system function is continuous.

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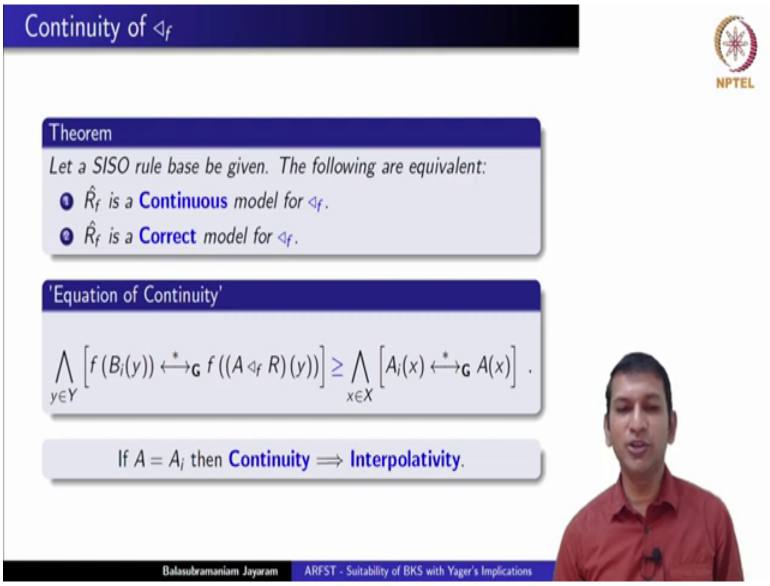

Well, let us start with the SISO rule base. And we can show the following are equivalent. That means, R f cap it is a continuous model for the BKS-f inference in a given rule base, if and only if, it is a correct model for us. So, that means, the equation of continuity and the earlier equation that we insisted for interpolativity, they must be clearly equivalent. That means, one should imply the other under the condition that we are considering for R cap f.

So, this is the condition that we thought of inequality we called it as equation of continuity. So, what it means is if you have continuity correctness is nothing, but interpolativity. So, we have also found the conditions, almost the similar conditions for f for you when you employ f-implications in BKS to ensure both interpolativity and continuity.

And as in the case with R implication, we have shown that these are in fact, equivalent. If it is a continuous model, it is a correct model and if it is a correct model that is interpolative, then it is also continuous.

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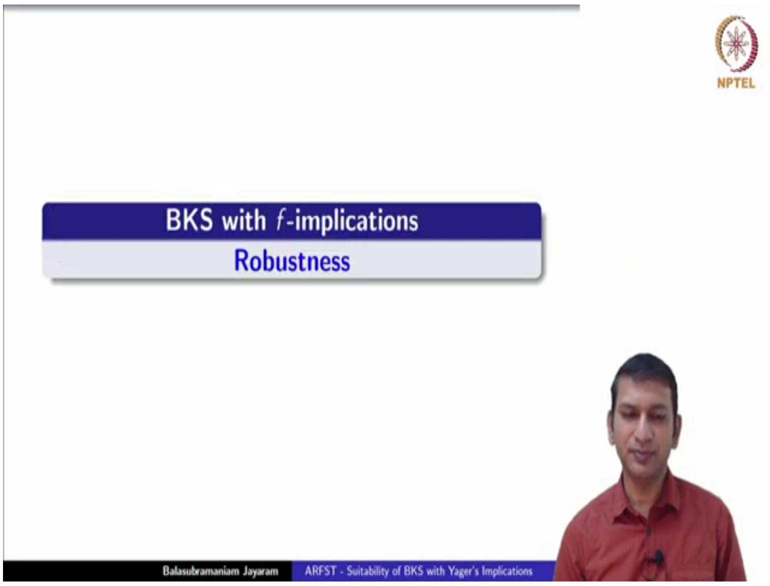

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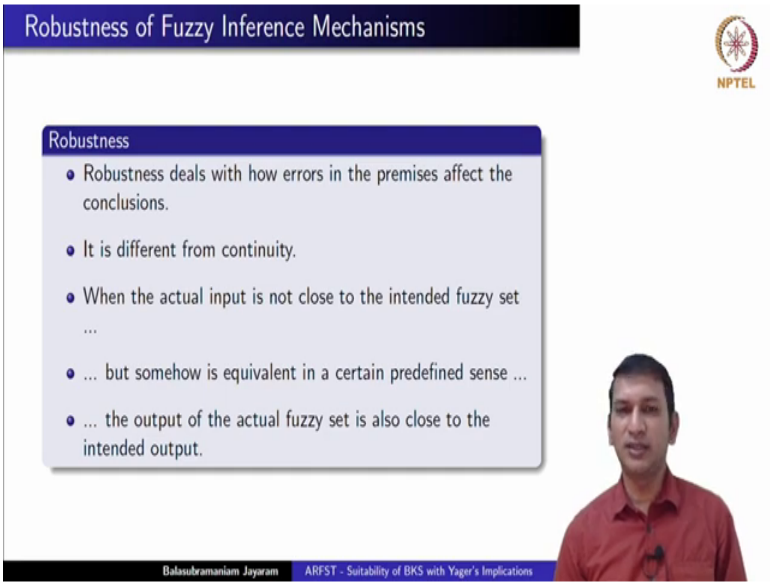

Now, what about robustness? Let us recall what is robustness. It deals with how errors in the premises affect the conclusions. It is different from continuity. In words, what does it say? It says that when the actual input is not exactly the intended fuzzy set, may be close enough based on the way we have captured the fuzzy set, even if it is not exactly the same, but close to the intent said, but somehow is equivalent in a certain predefined sense. So, there is an equality relation in A.

The output of the actual fuzzy set is also close to the intended output.

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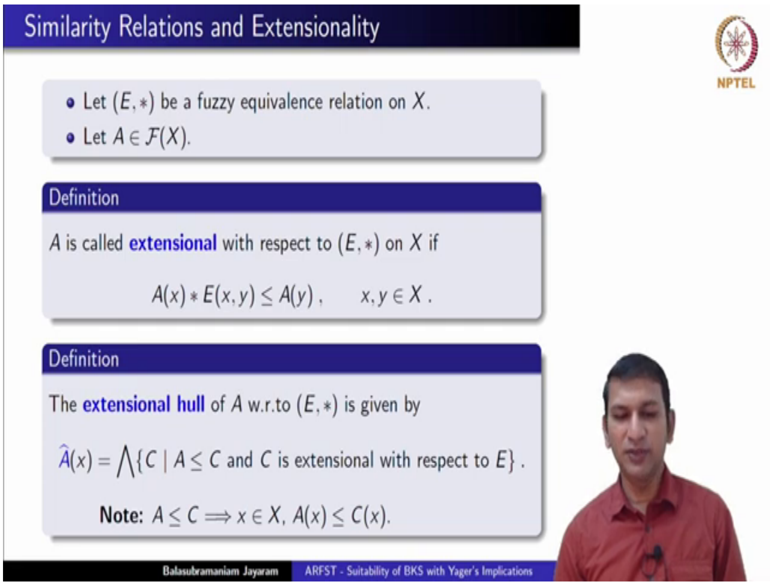

Towards this end, we introduced the concept of extensionality. So, we have began with an E which is an equivalence relation, fuzzy equivalence relation with respect to a given t-norm star. And if you take a fuzzy set a on X, we define what is when this A is said to be extensional with respect to this equivalence relation as follows, that A of x star E of x, y is less than or equal to A of y for every x, y in X.

And given an aim even if it is not extensional, we can make it extensional by constructing the extensional help of A with respect to E star. This is the formula that we have seen. It essentially means A cap is nothing, but the smallest extensional fuzzy set of A containing A. The smallest extensional fuzzy set with respect to E that also contains A. And of course, here we are using the point wise ordering between the fuzzy set A and C.

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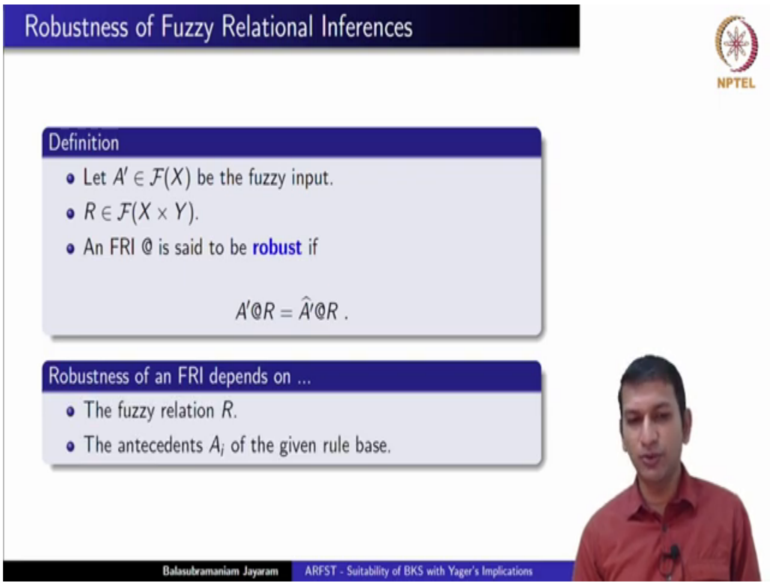

Well, what is robustness? To put it simply like this, given A dash as a fuzzy input and R. We see an FRI is robust, if A dash composed with R is the same as composing the extensional of hull of A dash with R. Well, it is clear that robustness of an FRI depends on fuzzy relation R and the antecedents A of the given rule base, because we have fixed the composition.

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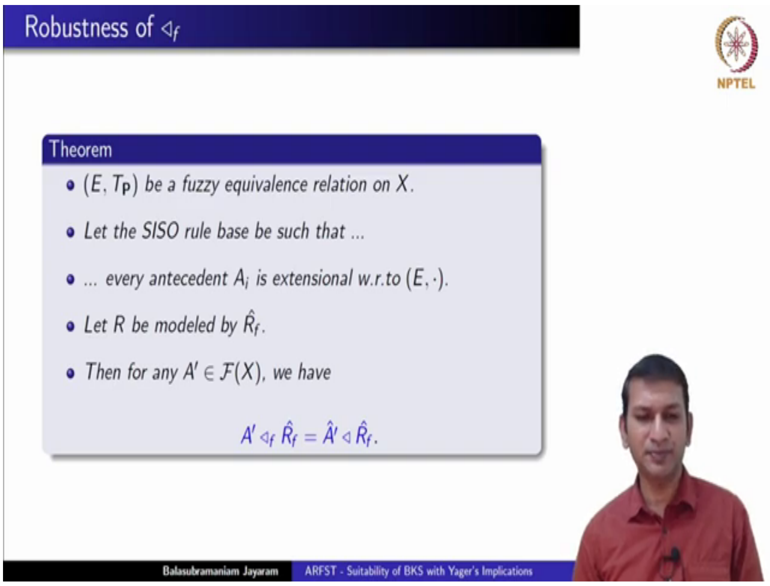

Once again it can be shown that if you take a fuzzy equivalence relation with respect to the product t-norm, and let the SISO rule base be such that every antecedent A is extensional with respect to E and this dot represents the product t-norm. And if we model the R using R f cap, same R f cap, then it can be shown that for any A dash, A dash BKS-f composed with R cap f is the same as the corresponding extensional hull of A dash being BKS of composed with R f cap.

So, you see here, the role that was being played by the t-norm, left continuous t-norm that gave rise to the residuated lattice. A similar role is played by the product t-norm here with respect to the f-implication here.

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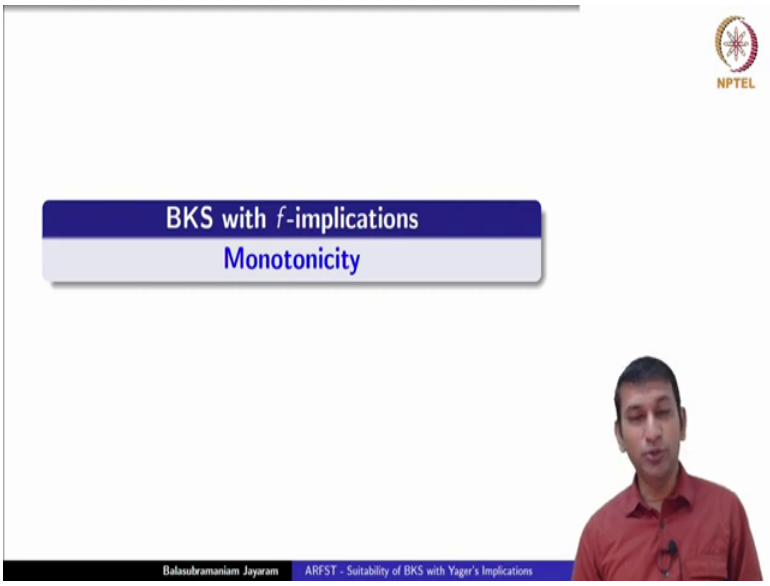

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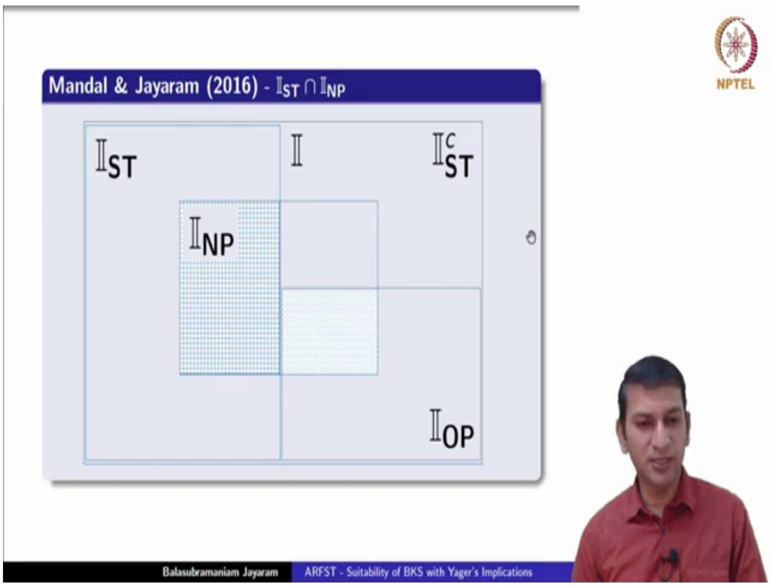

Finally, what about monotonicity? If we use BKS-f composition inference scheme will it have monotonicity? Well, notice it that when we discuss monotonicity, we already started generalizing from the class of R implications to implications fuzzy implication will satisfy both OP and NP.

So, we were able to show that anybody that fell here is an intersection of NP and OP, we were able to show that incorporating them in the BKS-f, BKS in inference scheme, instead of I T taking that implication high having both OP and NP we were able to ensure monotonicity.

We also mentioned that similar result is available, when we considered implications which are both strict and satisfying neutrality property. They can also be shown to have monotonicity when employed in the BKS inference mechanism.

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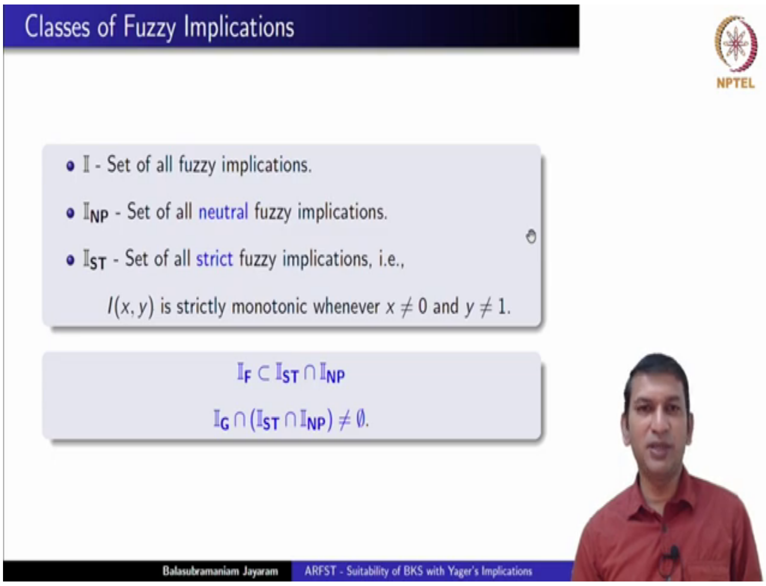

Now, quickly recall by I if we denote set of for fuzzy implication. By I NP, we denote set of a neutral implication. And a strict fuzzy implication is one such that it is strictly monotonic everywhere except on the left and top boundaries; that means, when x is equal to 0 or y is equal to 1.

Clearly, from the formula we see that all the f-implications are strict implications. They are constant only on x is equal to 0, y is equal to 1. If you consider the set of all g-implications, not all of them are strict. For example, we do know that the Goguen implication is the g-implication, but it has OP, it is not a strict fuzzy implication.

However, there are many g-implications which are in fact, strict and satisfy NP. For example, the (Refer Time: 21:02) path implication. So, we see that since I f is contained in this and I g intersects with this set, and any implication coming from this set does enjoy monotonicity when employed with BKS inference, we see that when you use f-implications or g-implications from this set, they continue to enjoy monotonicity.

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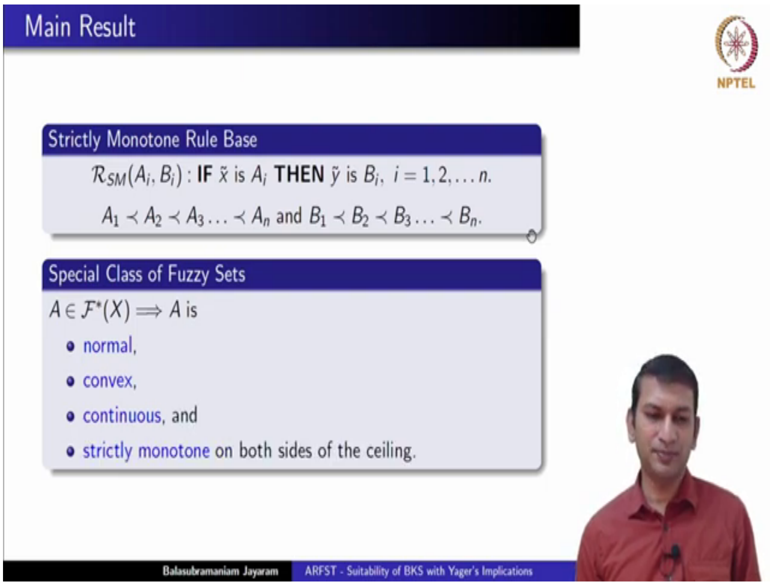

Let us recall the main result. So, we are given a strictly monotone rule base; that means, the antecedents are ordered in one fashion and the corresponding consequents are ordered in the same fashion and we are using the alpha cut based ordering here. And we consider the special class of fuzzy sets F star of X, where if A belongs there it is a normal, convex, continuous fuzzy set which is strictly monotone on both sides of the kernel.

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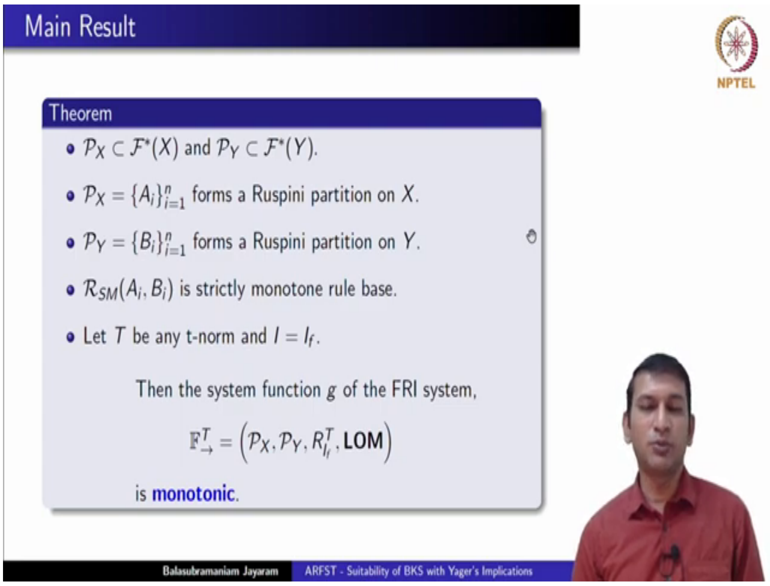

If you construct the partition P X and P Y from this class of fuzzy sets. And if these the antecedents which are part of P X, if they form Ruspini partition on X and the consequents form a Ruspini partition on Y. We have a strictly monotone rule base. And T is any t-norm and I is equal to I f, then it can be shown the system function g of the FRI system where we use I f instead of I T is also monotonic.

A similar result can be proven also for g-implications which are coming from, which are both strict and satisfying NP.

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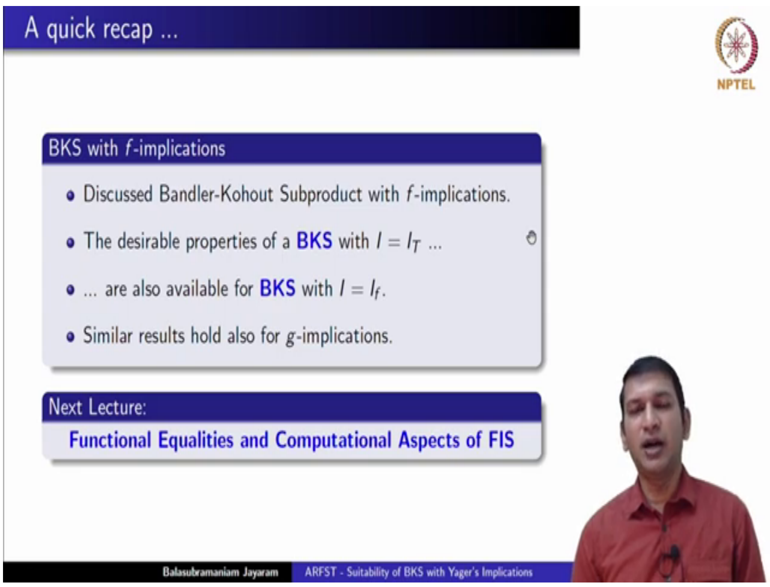

A quick recap of what we have seen in this lecture. We looked at BKS-f with f-implications. Essentially, considering I f instead of I T, and what we are able to show is that all the desirable properties of a BKS that you could have with I being a residual implication open from left continuous t-norm, these properties are also available for BKS when you consider substituting I with an f-implication. Clearly, such results also hold for g-implications.

In the next lecture, we will look at some functional inequalities and the corresponding computational aspects, how they impact the computational aspects of a fuzzy inference scheme. Specifically, in the next lecture, we will pick up the functional equality the law of importation and how it is, and see how it allows us to modify C R, right, in the case of multiple input rules to obtain a more efficient computationally more efficient scheme which is also equivalent to the original scheme that he would have considered.

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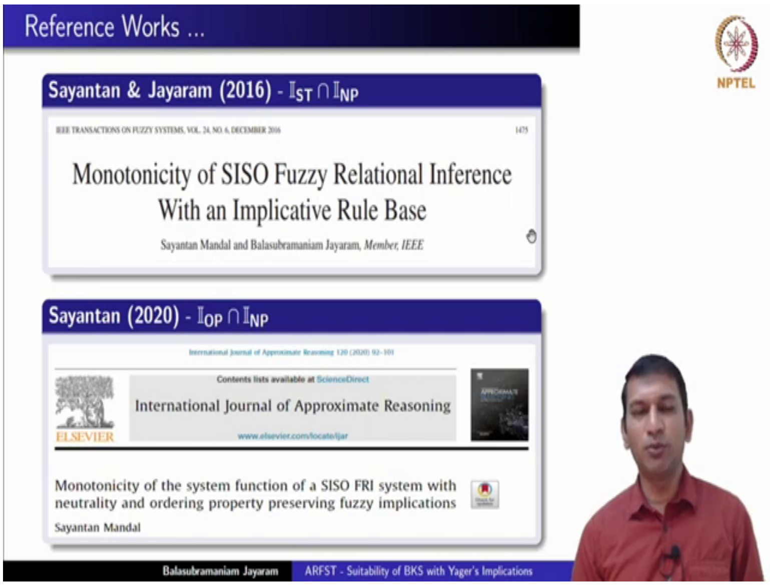

Note that this is the work of Mandal and Jayaram, in which they proved monotonicity of an FRI, especially BKS, when you employ fuzzy implication which is both strict and satisfies neutrality property. And another recent paper from Sayantan Mandal on showing the same, when the fuzzy implication is not strict, but satisfies copying and neutrality. Glad, you could join us for this lecture. Hope to see you soon again in the next lecture.

Thank you again.