Approximate Reasoning using Fuzzy Set Theory
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Lecture - 55
Functional (In) Equalities involving FLCs

Hello and welcome to the first of the lectures in this week 12 of the course titled Approximate Reasoning using Fuzzy Set Theory, a course offered over the NPTEL platform. In this lecture, we will look into some of the Functional Equations or Inequalities involving Fuzzy Logic Connectors.

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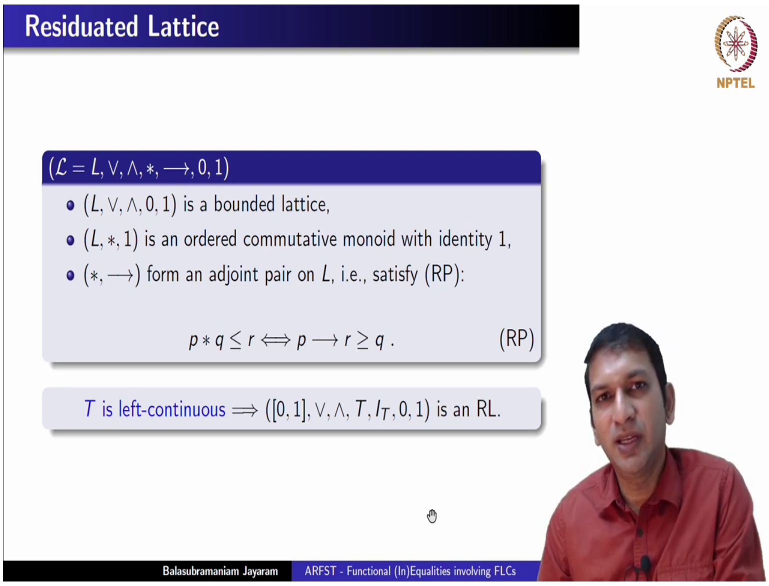

The motivation for this stems from the fact that we have always considered residuated lattice to suppliers with the operations that we were employing in the inference systems. So, quickly, what is a residuated lattice? As an ordered structure, it is a bounded lattice; as an algebraic structure, it is an ordered commutative monoid with identity 1, and the star and the arrow operations they form an adjoint pair; that means, they satisfy this particular residuation principle.

And, when we are considering fuzzy logic connectives which can lead up to a residual lattice structure on the [0,1] interval we saw that left continuous t norms were the ones that were giving rise to the residuated lattice structure on the [0,1] interval.

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Now, let us look at the role of such functional equations or inequalities.

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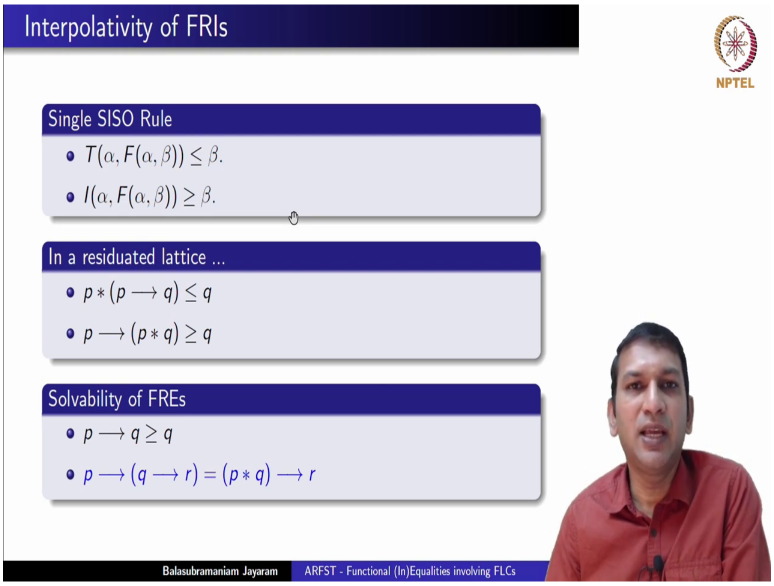

In the last few weeks, we have discussed many desirable properties of a fuzzy inference system, we began by looking at interpolating of FRIs. So, there again we started with a single SISO rule – Single Input Single Output rule, and we found that these were the two functional inequalities that were coming into play. What is a functional equation or an inequality?

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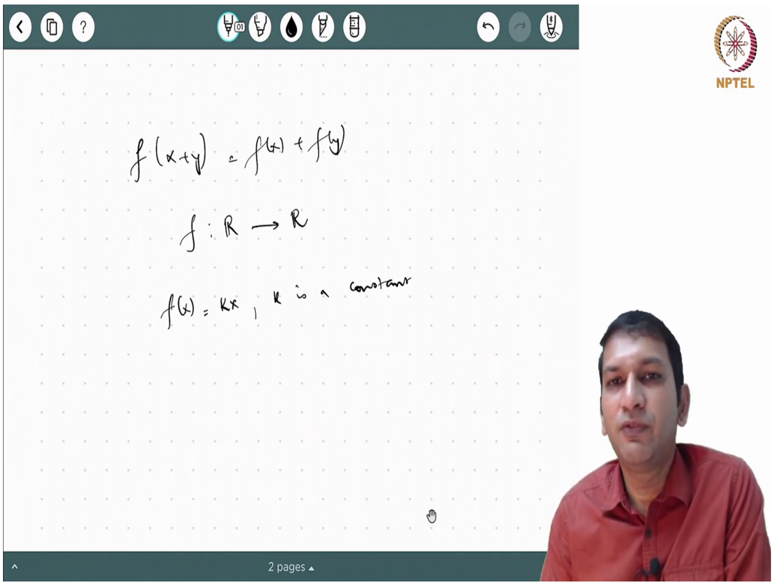

Here, let us look at such an equation, where f we know is from R to R. Now, here in this equation what we want to find is such a function f which will satisfy this equation. So, in the given equation the solutions of the equation we are searching for it in the space of functions.

So, what we are interested in is finding such f which will satisfy this equation. It can be easily shown. This is known as a Cauchy additive function equation that with some reasonable assumptions about continuity on f it can be shown that the only solutions are such functions f(x) = kx, where k is a constant.

So, functional equations essentially are equations involving functions where we are looking for functions which will satisfy the equation. So, in that sense if you are looking at it, these are two functional inequalities because it instead of equality we have an inequality and what we are doing is over the entire range of alpha, beta which in our case is the unit interval [0,1] we are looking for T and F functions T and F that will satisfy this inequality.

And, what we found was when we consider the residuated lattice structure these were two properties that were already available for us when we consider T to be star and F to be the corresponding residuated implication. When we discussed multiple SISO rules, we moved into looking at the inference process itself, the composition itself and looked at it as dealing with fuzzy relation equations and to discuss the solubility of it we found that these were some of the inequalities that our properties from residuated lattice structure that helped us in discussing the solvable equation.

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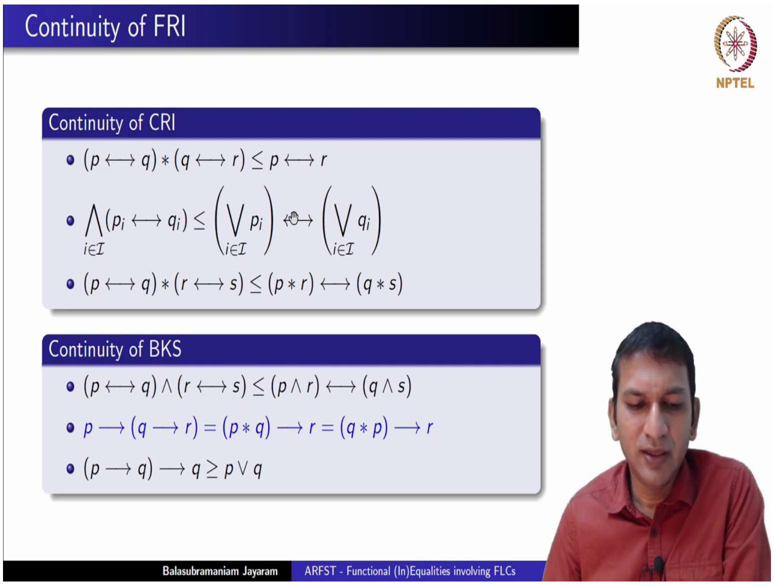

When we move to continuity when we first discussed continuity of CRI you may have noticed these are the properties that we employed in the proofs of the different research that we actually discussed, and these were properties that were coming from the residuated lattice structure.

In the case of continuity of BKS, when we discussed those results we found that there was another set of properties from the residuated lattice that we were actually using. You may have seen that there is one particular equation that is highlighted in blue, it is only to show that it keeps recurring again and again.

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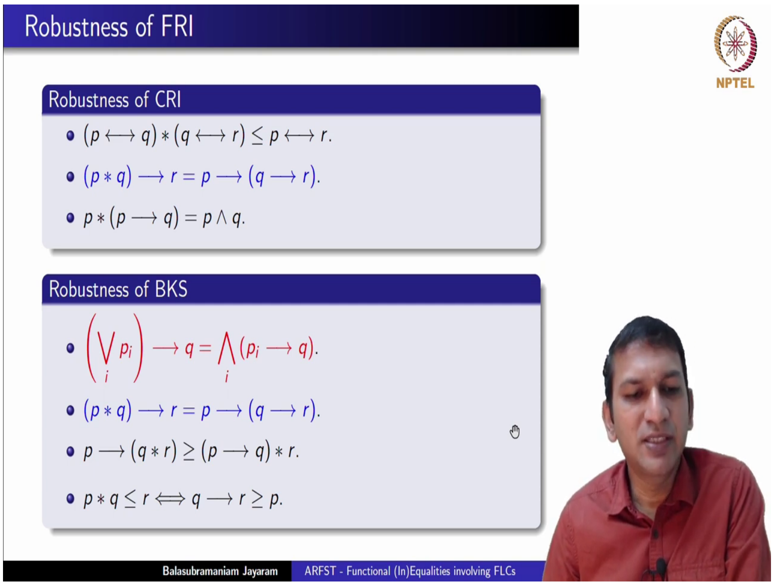

When we discussed the robustness of an FRI, again this equation came up. Of course, there were also many other properties that we made use of. There is another property interesting property which is highlighted here in red which also kept recurring. It is not to say this, other properties which are not colored are not useful. In the context that we have discussed, these are perhaps two important properties that have come up again and again and helped us in our research.

Now, these are some of the properties that we have used when we discussed robustness of CRI and BKS.

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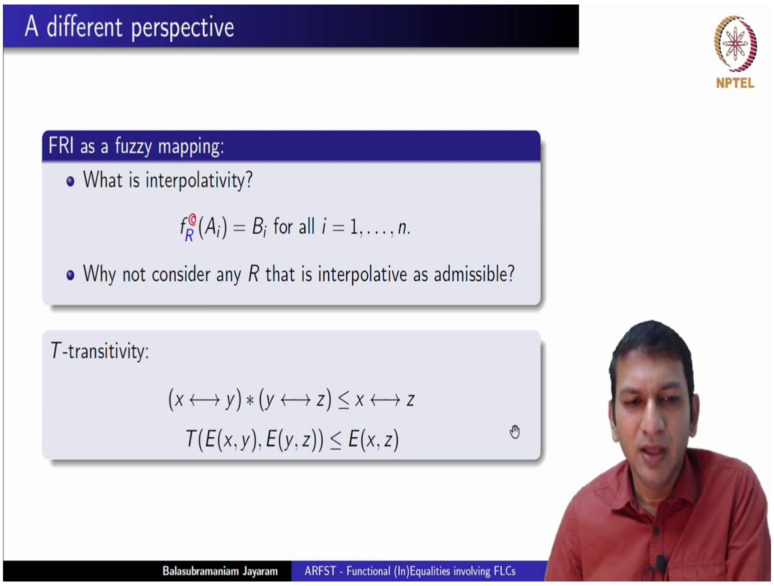

Then we took a different perspective and looking at FRI as a fuzzy mapping; mapping between F(X) and F(Y), we asked the question what is interpolativity and there we found we could just discuss about interpolativity not from the point of view of solubility of FRIs, fuzzy relational equations, but from the point of view of a mapping of A_i to B_i. And, in that sense all we needed was to look at R and the composition.

And, once you fix the composition, the question was why not consider any R that is admissible based on whether it is interpolativity or not. So, we were trying to break the cycle from of ourselves or not R check and R gap. And, we you may recall that at that point of time the by implication played an important role because we defined what is called a delta function there based on the by implication.

And, we showed that by implication is an equality or equivalence relation and this is the property that played a role there.

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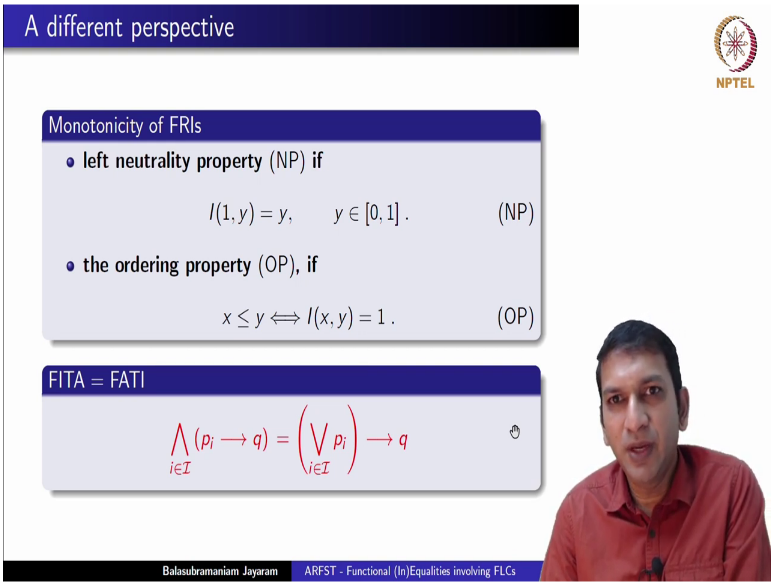

When we came to Monotonicity of FRIs we saw that the left neutrality property and ordering property played a role which obviously, we knew that residual lattice structure was able to supply us with implications having these two properties. And, when we discuss FITA and FATI, their equivalent once again we saw that it is this what we call the distributive equation that came into picture.

Now, you may have seen that under this different perspective what we wanted to do was, we were trying to do some kind of a generalization. We came out of residuated lattice structure in the case of monotonicity why not consider implications that have OP and NP. And, in the in the case of interpolativity we wanted to come out of solubility of FRIs and then we ask the question why not consider some R which is interpolative as admissible.

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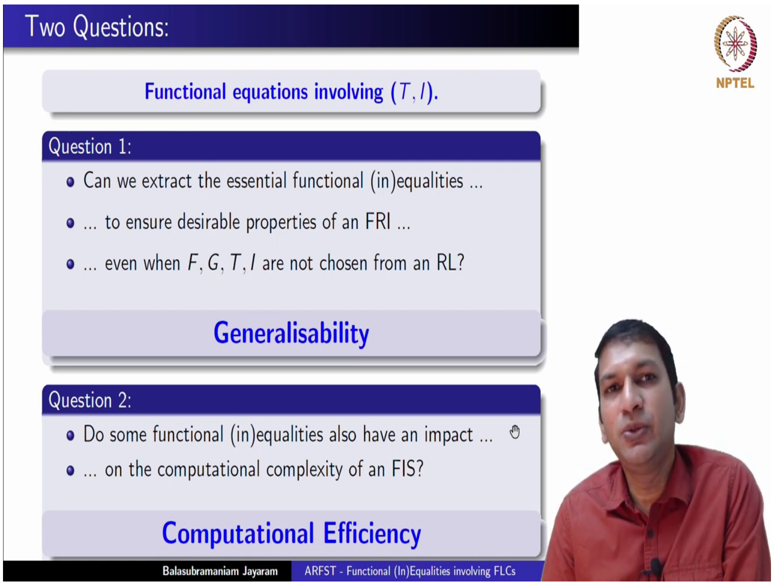

Now, taking cue from this we would like to ask two questions. We have seen among all the properties. Obviously, we are using properties or equations or inequalities typically involving the T norm and the implication because that is what we have always been using.

Now, it breaks two questions. The first question is, can we extract the essential properties in terms of functional inequalities or equations to ensure desirable properties of FRI, even when these operations that we consider F, G the aggregation operator, T or I they are not chosen from a residuated lattice structure. So, this was this is one question that we can ask. Essentially, we are discussing generalisability of the results that we have got.

The second question that we can ask is, do some functional inequalities and equations do they also have an impact on the computational complexity of a fuzzy inference system of a FIS? So, this is something that we have not seen so far. In this week of lectures, we will see this. So far we were interested in finding out whether they had some desirable properties like interpolativity, monotonicity. There you could think of them as really not related to the computational aspects of it more on the correctness aspects of it.

So, we would also like to look at whether some of these properties, if they are enjoyed by the operations employed in the FIS whether they lead to some computational efficiency. So, these are the two questions that we would like to address during this week of lectures.

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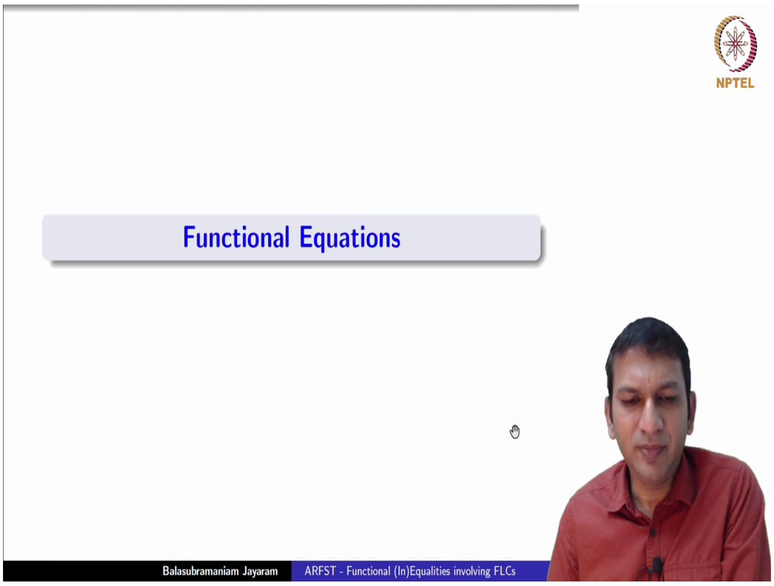

So, what are the functional equation we are going to consider?

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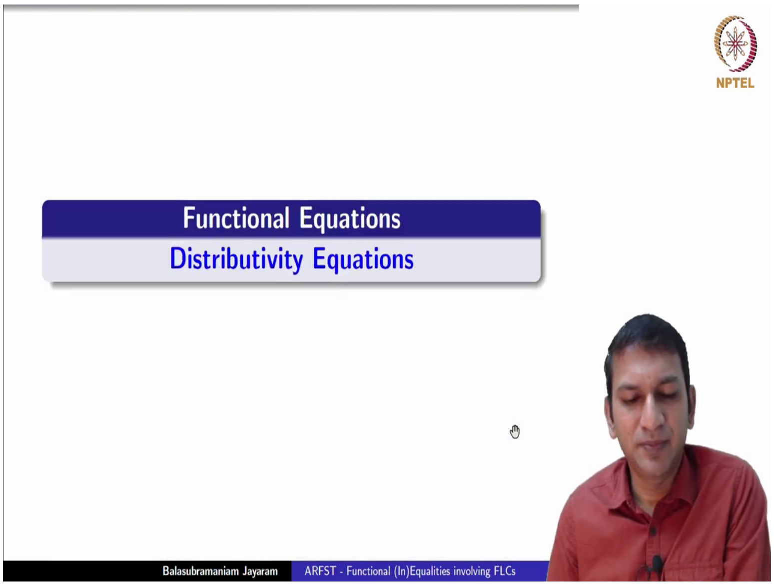

So, first is the distributivity equation.

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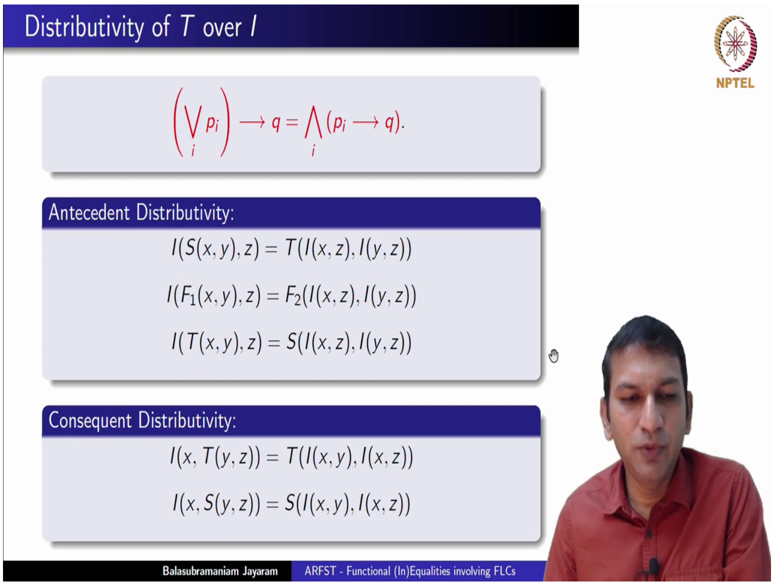

We have seen this. So, now, writing it in terms of fuzzy logic connectives, it would look like this. We could consider the distributivity either at the antecedent side or the consequence side. So, this is called the antecedent distributivity equation. So, now you see here instead of max we are taking a T conorm S, instead of min we are taking T norm T.

Now, the question is you see that on the left hand side you have one operation, on the right hand side you have another operation. So, is it warranted? Well, let us consider an equation of this type I_F. So, instead of S let us put F_1 and instead of T let us put F_2 and then try to fix what these operations could be.

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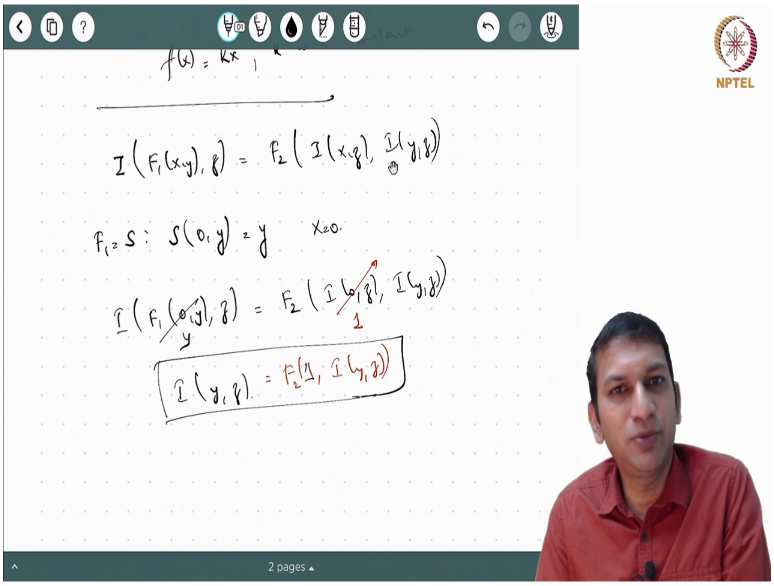

So, what we have here I of F comma of x, y, z is equal to F 2 of I(x,z) comma I(y,z). Now, we need to fix F 1 and F 2 be either t norms or t conorms. So, let us assume that F 1 is a T conorm; that means, what we know is 0 is in fact, a left neutral element, right neutral element because S is commutative. So, let us take x to be 0 in this equation. So, then that equation reduces to F of 0 comma y comma z this becomes F 2 of I of 0 comma z comma I of y z.

Now, we know that I of 0 comma z is 1 because I is an implication. So, that means, this is F 2 of 1 comma I of y, z on the left hand side. What we have is because this is y, what we have is I of y comma z. So, what we see from here is F 2 of 1 comma I y, z is in fact, equal to I of y, z and this should happen for any y, z arbitrary y, z which means 1 becomes the neutral element of F 2 which in our case since we are considering only operations coming from T norms and T conorms it is clear that F 2 is the T norm.

So, that is why we have taken when we take S here, it automatically translates it as fixing the right hand side operation F 2 S T. Similarly, easily it is it can be worked out that if you take T here, then it will become a T conorm on the right hand side. Well, this is antecedent distributivity. On the consequent distributivity, similarly it can be shown that the same operation can be retained because it is also clear because I in the first variable is non increasing. So, essentially it is going to flip it a decreasing function.

Whereas, in the second argument it is increasing; so, that means, it retains the same order. So, T remains a T and S remains a S. So, this is one equation that we would like to discuss now.

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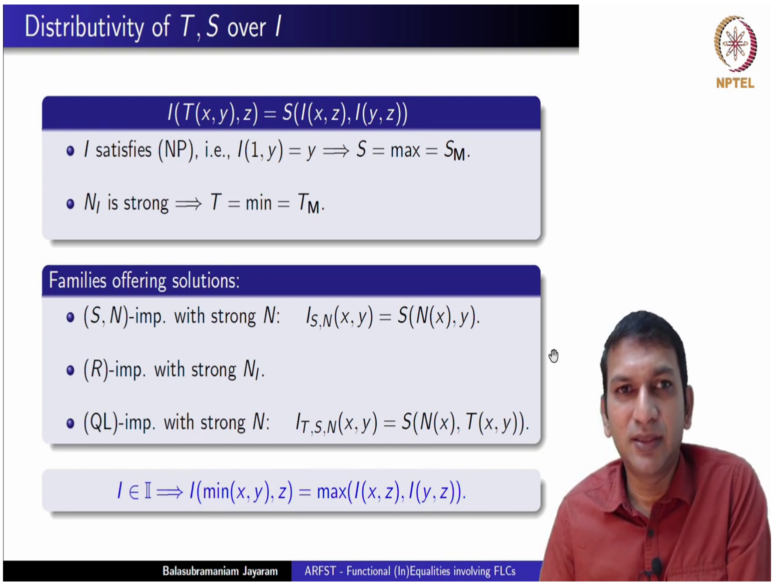

Now, so, let us look at given T and S and I, we need to discuss what kind of we want to fix t and S given I is an implication. So, now, how do you fix on this T and S? Can we arbitrarily use any T and any S? So, now we need to make use of the properties of the implication. We have only fixed I and based on that we want to know what T and S would be. So, to begin with let us assume that I, satisfies neutrality property; that means, I of 1 comma y is y.

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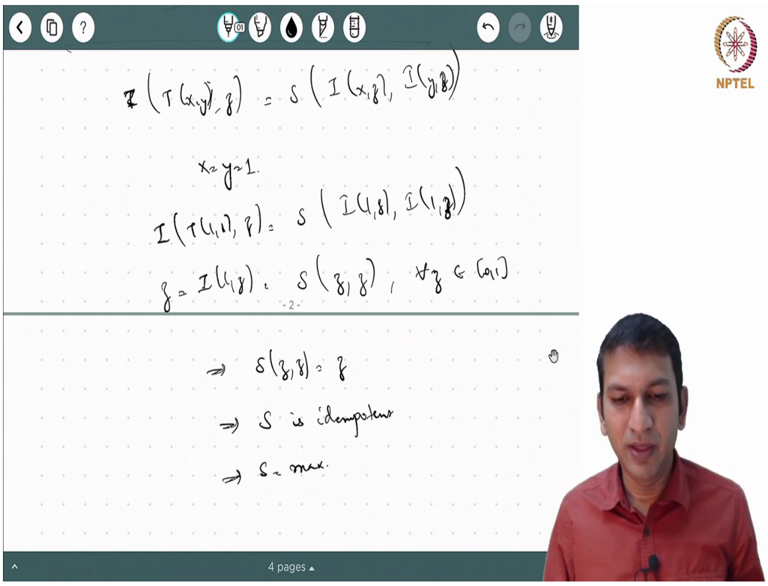

Now, let us take this equation I of T x, y comma z is equal to S of I x, z comma I of y, z. Now, if you take x is equal to y is equal to 1, then what we have here is I of T 1, 1 comma z is equal to S of I 1 comma z comma I of 1 comma z. Now, we know that T is a T norm, T of 1 minus 1. So, this is I of 1 comma z, but I of 1 comma z on this side I of 1 comma z is z and this is z and this happens for every z element of 0 1.

So, this implies S of z, z is equal to z implies S is idempotent which means S is in fact, the max T norm only because we know that the only idempotent T conorm is the max T conorm. So, if you fix I to satisfy an extra property that of neutrality property, we immediately see that S becomes the maximum T conorm well.

So, now this equation instead of S we substitute max and of course, I should have neutrality property. Now, let us take N i to be strong other than I having the neutrality property we insist the natural negation of I.

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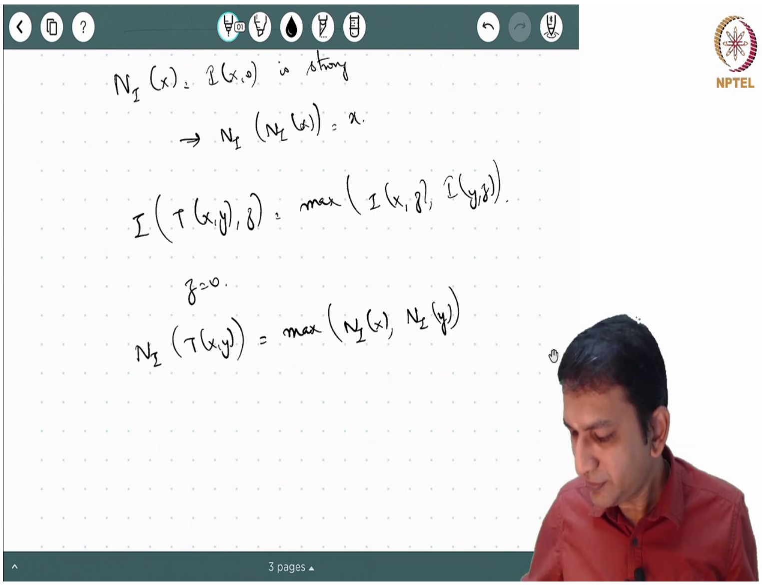

Recall N I of x is essentially I of x comma 0 this is strong; that means, it is (Refer Time: 14:38). It is N I N I of x is equal to x. So, now, consider this equation now. This is max of I of x comma z comma I of y comma z. Let us take z to be 0, then the left hand side becomes N I of T x, y this equal to max of N I of x comma N I of y.

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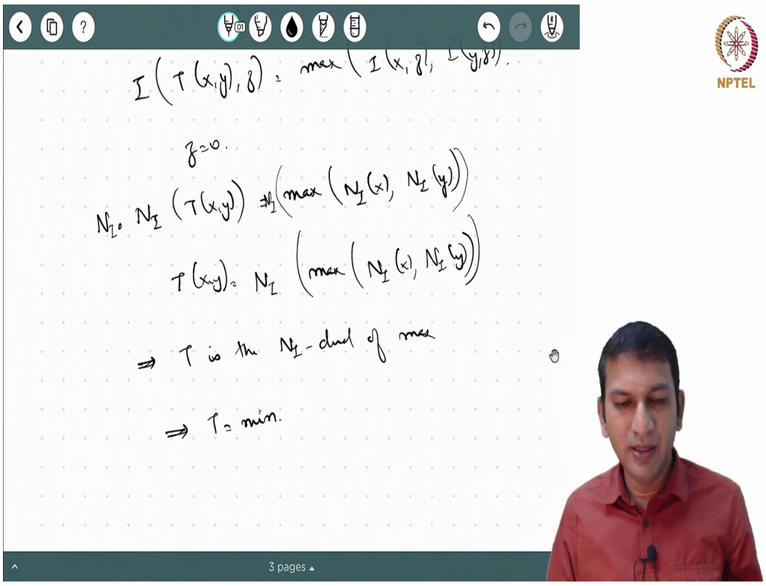

Clearly this means T of x, y is equal to since N I is strong we could apply N I on either side and this N I of max of N I of x comma N I of y implies T is the N I dual of max implies T is infact equal to min. So, just by imposing these two conditions on I that I has neutrality property and N I is strong we are able to reduce T and S to that of min and max. So, we have found solutions to this equation in terms of what T and S we can use so that this equality is valid. Of course, we have put some conditions on I.

Now, the question is what are those families which offer such implications; implications with I having implications with have the neutrality property and their natural negations are strong. Clearly, S N implications were strong N because we know that S N implications are given like this. If we know that the corresponding N I natural negations nothing, but the N that we consider and we know we know that S, N implications do have neutrality property.

So, if you consider S N implications with strong N, these are also called S implications in the literature. So, any S implication will fit the bill. Of course, we know that R implications with strong national negation with fit the bill. Also, Q L implications with strong negations, they will also satisfy this equality.

Now, what is interesting is in fact, if you fix T to be min and to be max it can be shown that any implication will satisfy this distributivity equation. This is nothing extraordinary, as was mentioned it is decreasing in the first variable. So, it just flips over there and min and max are operations available on or the lattice operations available on 0. So, this can be easily proven.

So, now, what it shows is if you are looking at using the distributivity property, then perhaps if you do not have to stick yourself only with operations coming from residuated lattices or R implication, but now you have a thorough of implication coming from different families where which also satisfy this distributivity equation. And, in some context where the distributive equation plays a role you could freely choose from implications coming from these families too.

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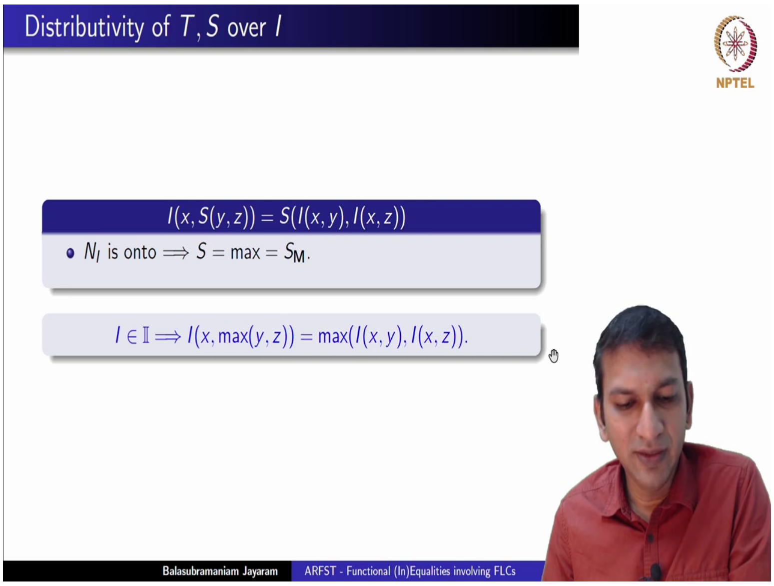

Similarly, if we consider the consequent distributive equation. Now, once again we argue like this.

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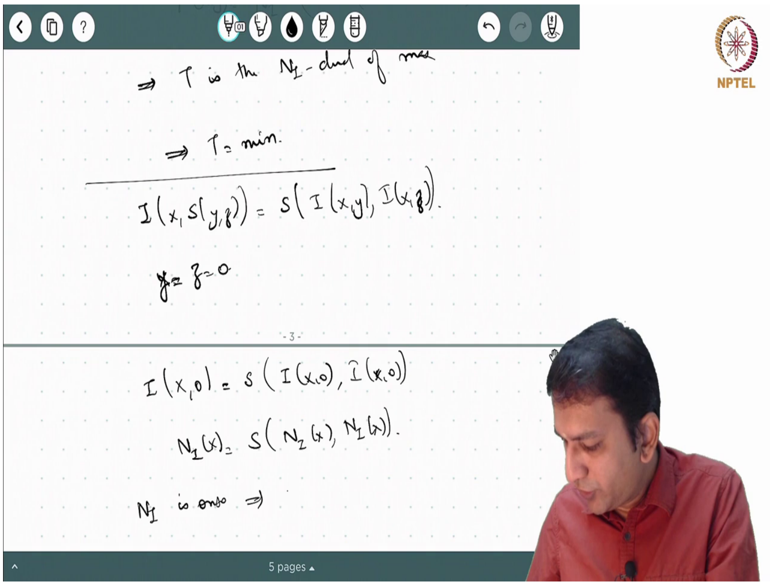

Let us assume that N I is on to this is the equation that we are consider we need to fix S now. So, now, we have N I is on to. Now, let us take z to be 0, then what we have is I of x comma y is equal to we take x, y is equal to z is equal to 0. Then what we have is I of x comma 0 is equal to S of x comma 0. Now, this is N I of x is equal to S of N I of x comma N I of x.

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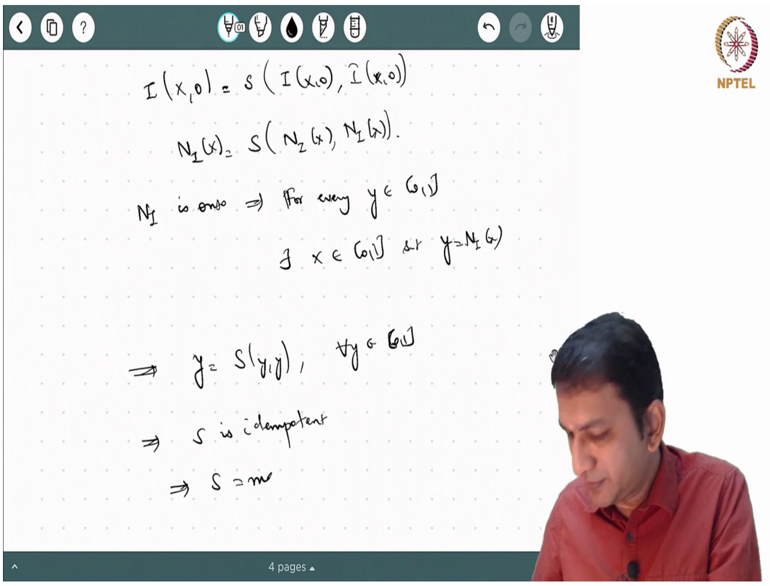

Since N I is on to this implies for every y in 0, 1 there x and x in 0, 1 such that y is equal to N I of x. So, now this implies y is equal to S of y comma y, for every y in 0, 1 this implies S is idempotent implies S is equal to max. So, we are just assuming that the implication that we are considering is such that it is natural negation is an onto negation, we can show that this S essentially bonds down to being only max T conorm.

In fact, it can be shown once again that any implication satisfies this equality when S is max. So, this is about the distributivity equation.

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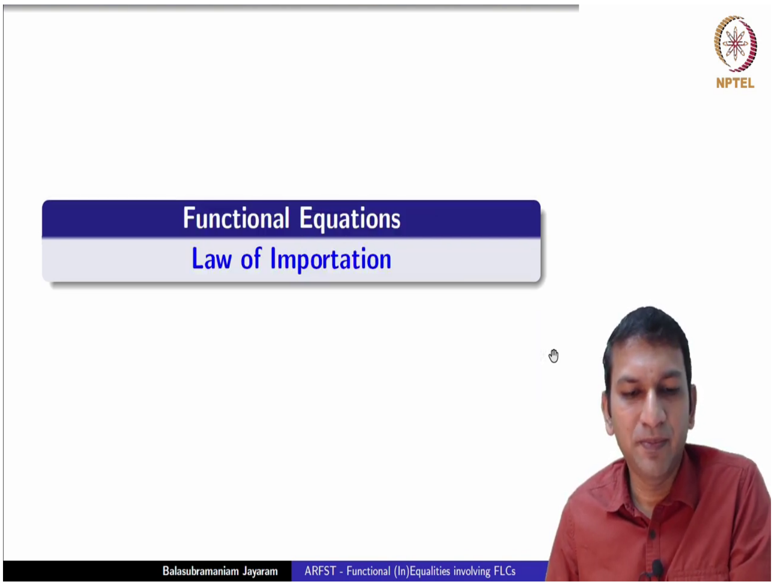

Now, what is the other equation that we had colored in blue it is called the law of importation we have referred to it like this many times in the earlier lectures too.

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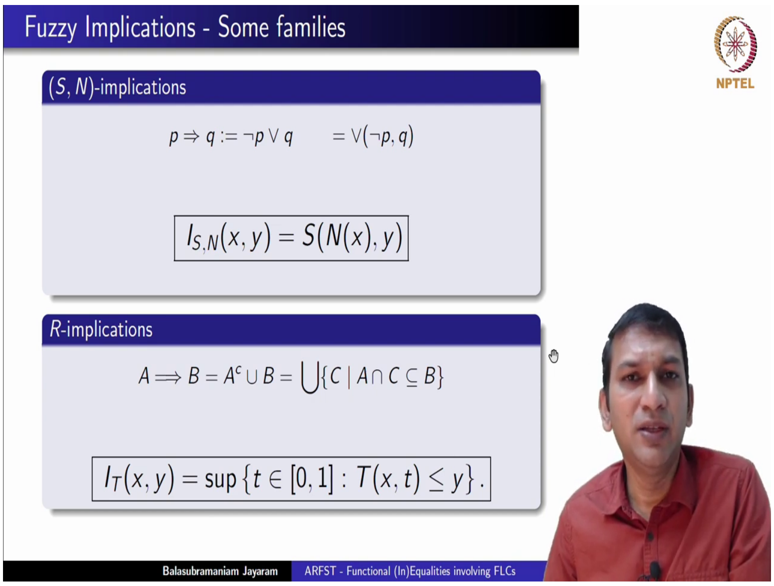

Let us look at some families of fuzzy implications and then discuss whether these families do satisfy the law of importation. So, let us start with the familiar ones we know that S N implications are obtained from the material implication essentially negation p or q that is what we generalized into S of N x comma y, S is a T conorm and N is a negation.

We have seen R implications one way to look at them as is from the set theoretic; equality of a complement union B and that is how we generated the corresponding formula for an R implication implications.

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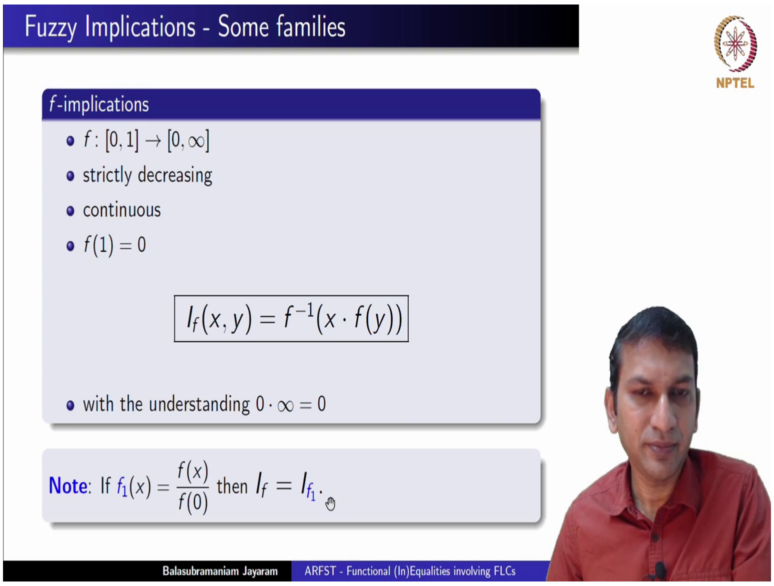

f implication were obtained from additive generators of T norms. So, we have a function f 0, 1 to 0 infinity which is strictly decreasing; continuous such that f of 1 is 0 and we have seen that if you define a function I f like this f inverse of x dot f of y with the understanding that 0 dot infinity is 0.

Then we have seen that this is an f e this is a fuzzy implication which we called an f implication. Now, we have also seen at that time that f of 0 may be infinity or it may be finite, either way if you take when there is finite f of 0 is finite, then instead of considering the f you could also consider f 1 which is the norm generator. So, essentially normalizes the range go domain to 0, 1 because now, it is f of 0 is finite.

And, we have also seen that whether you generate f implication from f 1 or I or f both are essentially the same.

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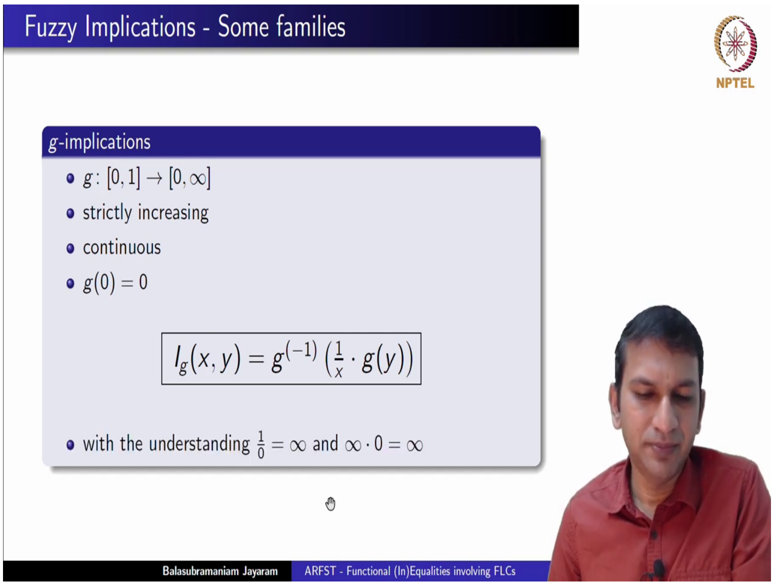

Similarly, we have seen g-implications which are obtained from the continuous additive generators of T conorms. So, here g is a strictly increasing continuous function g of 0 is 0 and the corresponding implication the g-implication is given by this formula and taking this convention.

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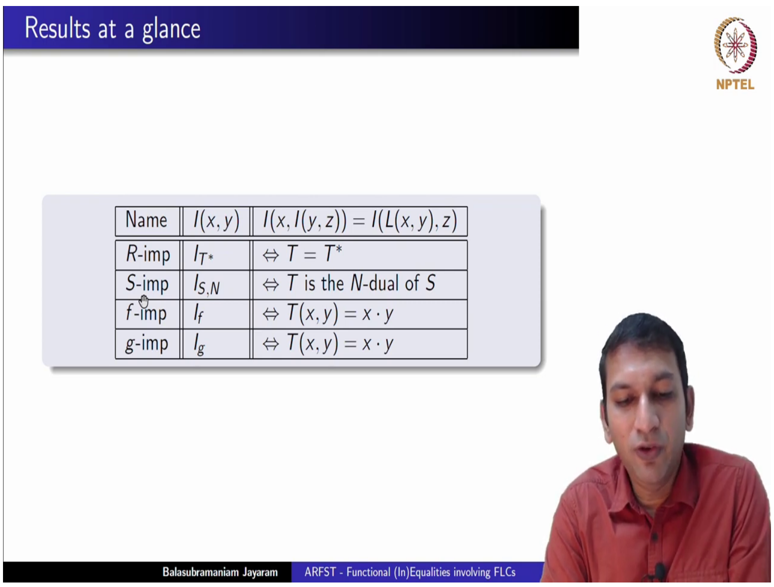

Now, let us look at the law of importation for these four families; that means, if you take I to b coming from one of these families what should be the T does it actually have a T such that this law of importation is valid and if yes, what should be the T norm T. Now, in the case of R implication, we have seen that T is T star.

So, there is it does satisfy law of imputation and T is equal to T star is the only solution that we get. Of course, here we are talking about R implications from the residuated lattice structure means the T is left continuous. Now, let us look at S N implication here, it can be shown that sorry.

Note that this S implication; that means, the N that we are considering is strong we can be shown that the T that we consider should be the N dual of S that can be easily seen.

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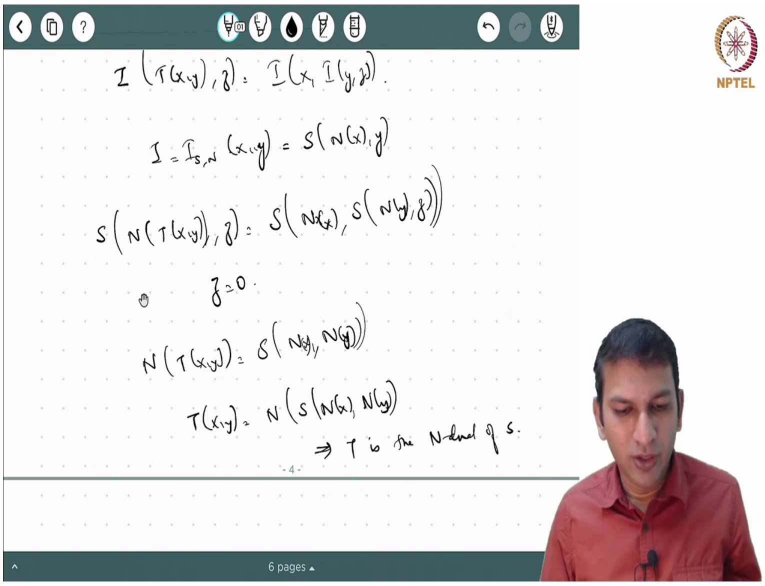

So, what we are looking at is I of T x, y comma z is equal to I of x comma I of y, z. Now, this I this I S, N which is given as S of N x comma y. So, if you substitute here, then LHS is S of N of T x, y comma z is equal to S of N x comma by the association associativity of (Refer Time: 23:51) remove it later on.

So, now let us take z to be 0 and clearly this means n of T x, y is equal to S of N x comma N y clearly; that means, since N is norm T of x, y is n of S of N x comma N y which means T is the N dual of S. So, not only S N and S implication satisfy the law of importation if they do satisfy then the T has to be the N dual of the corresponding S.

So, the moment you fix S and N, N to be strong then you know, that with respect to it is N dual T it will satisfy the law of importation.

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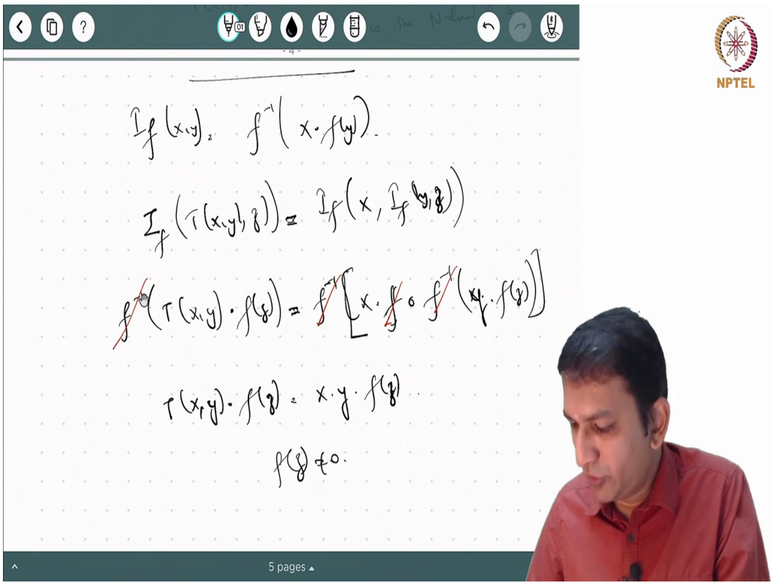

Now, what about f implications? Note that f implications are given like this I f of x, y is equal to f inverse of x dot f of y. So, now, the formula that we are looking at is I f of T x, y comma z is it equal to I f of x comma I f of y z. Now, let us assume that it is true, then we need to go and find out what is the T for which this is true.

So, now expanding this what we see is f inverse of T of x, y dot f of z is equal to f inverse of x into f of f circle f inverse of y dot f of z. Now, f we know is continuous and strictly decreasing, so, we could perhaps cancel it out. This f circle f inverse they will get cancelled. So, what we are left with is T of x comma y dot f of z is equal to x dot y into f of z.

Clearly, if you take the z such that f of z is not equal to 0; for instance, we know if in the case of clearly there will exist a z, such that f of z is 1.

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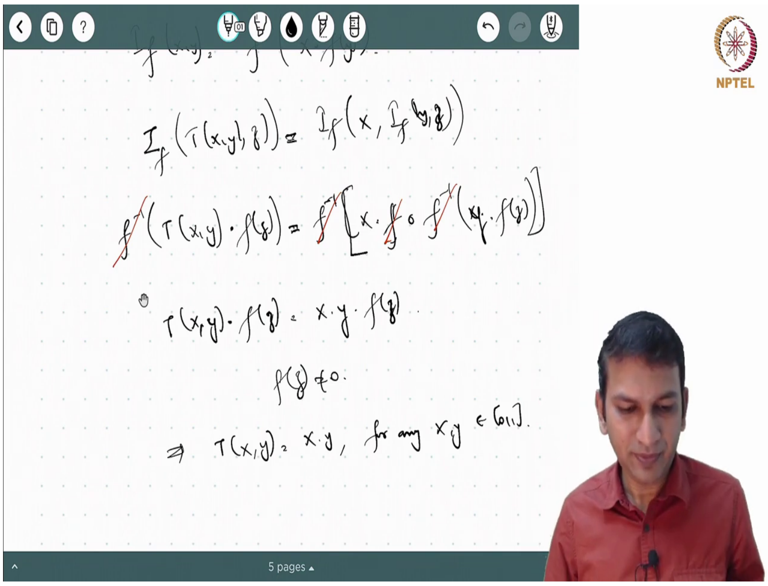

So, if you assume that for any nonzero value here we can cancel this and what we say is T of x, y is in fact, to x dot y for any arbitrary x, y, z we have considered which means T is in fact, a product T norm. Similarly, it can also be shown for the g-implication it does satisfy the law of importation and the T that we are considering there is in fact, the product T.

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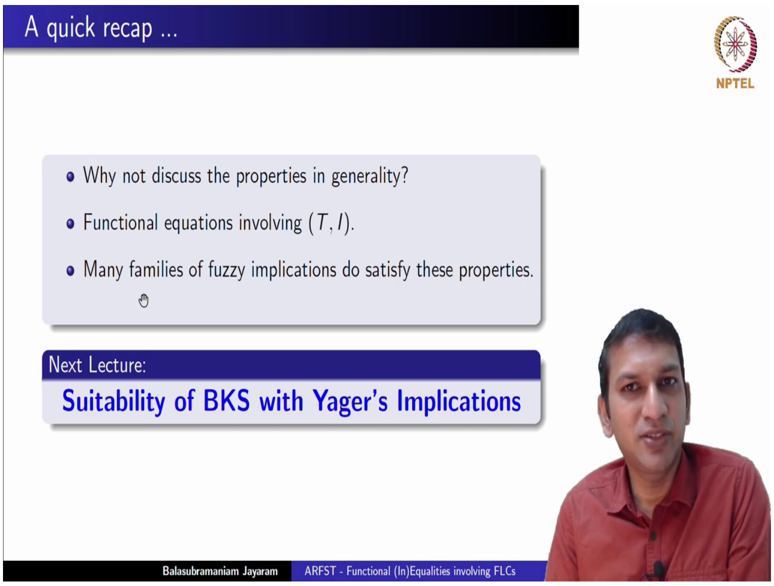

Now, a quick recap of what we have seen in this lecture – we asked ourselves a question why not discuss the properties that led to the FRIs and SBRs essentially the FIS to possess the desirable properties of an inference system that of monotonicity, interpolativity, robustness, continuity – why not discuss those properties in generality not only looking at it as being coming from the residuated lattice structure. And, it meant that we need to discuss functional equations involving T norm and the implication.

And, we have seen that not just the R implications obtained from left continuous T norm, but there exist other families of fuzzy implications which also suppliers with implications which satisfy some of these properties. So, where you would like to use implications having those properties it is not mandatory anymore to restrict ourselves to go only to the residuted lattice structure or R implications obtained from left continuous norms.

That is what we have seen in this lecture at least with respect to a couple of functional equations that we have discussed and our choice of choosing these two equations was also clear because they play a huge role in many of the properties that we have seen. Not only that, it can also be shown that these two properties lead to making our inferences computationally more efficient. We will see this and during the course of this lecture in this week.

In the next lecture, of course, we will try to answer the first question that of generalizing some of the results that we have seen for FRIs, where the operations came from residuated lattice structure two FRIs which employ implications coming from different families family other than the R implication family.

So, in that context in the next lecture, we will discuss the suitability of BKS inference mechanism where the implication is chosen to be an f or a g-implication this is what we will discuss in the next lecture. Glad you could join us for this lecture and hope to meet you soon in the next lecture.

Thank you.