Approximate Reasoning using Fuzzy Set Theory
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Lecture - 53
Monotonicity of an FRI

Hello and welcome to the second of the lectures. In this week 11 of the course titled Approximate Reasoning using Fuzzy Set Theory, a course offered over the NPTEL platform. In this lecture we will discuss the monotonicity of a FRI, whether it is the compositional rule of inference or the BKS, we will see that they reduce to the same kind of a structure, where the composition does not play a role in the when the input is a singleton and when we use singleton fuzzification.

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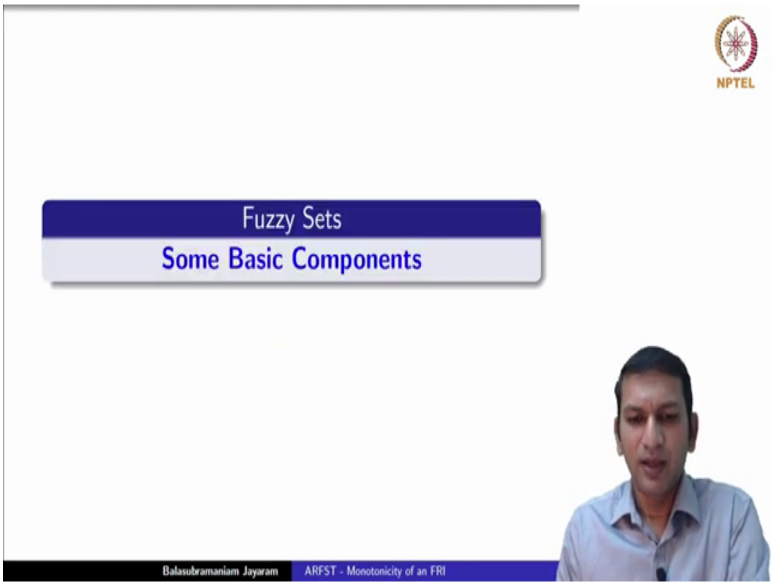

Let us begin by looking at some basics that are required through in this discussion.

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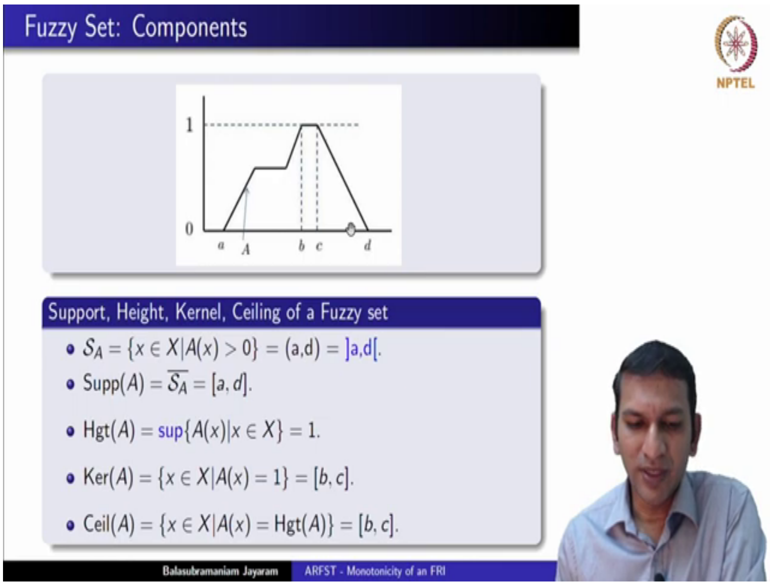

Given a fuzzy set we know what a support, height, kernel and ceiling of a fuzzy set are. Let us quickly revise them. So, if we consider the set of all elements, which belong to a degree greater than 0, let us denote it by S A for a given fuzzy set A, in this case we see that it is essentially the open interval (a,d). The support is the closure of this S A. So that means, it is the closed interval [a,d].

Height of A is nothing but the maximum membership value attained by A, in this case it is 1. The kernel of A is essentially the support of the height; that means, when it is actually normal. So, kernel of A is the set of all elements at which the fuzzy set attains the value 1. So, in this case we see that it is b, c. Ceiling of A is essentially the support of height even when height is not equal to 1.

So, if it is normal ceiling and kernel are same. If it is not normal then kernel is empty and ceiling will still exist. In this case both of them are same because it is a normal fuzzy set.

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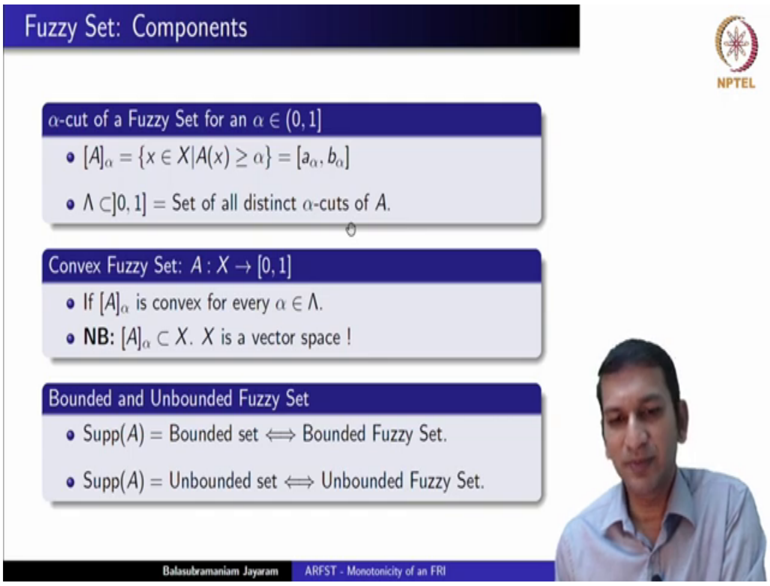

What is an alpha cut of a fuzzy set denoted by this symbol? It is set of all those x for which the membership value is greater than or equal to alpha. Typically, it is an interval, but it could also be an union of intervals and by the level set capital lambda we denote the set of all distinct alpha cuts; that means, alphas alpha values in 0, 1 which lead to distinct alpha cuts of A.

We call a fuzzy set convex if for every alpha cut, the alpha cut is convex subset of X. Note that which means that X also should have some kind of a linear space structure a vector space structure. We know what a bounded fuzzy set is, the support of A if it is bounded we call it bounded, otherwise we call it an unbounded fuzzy set.

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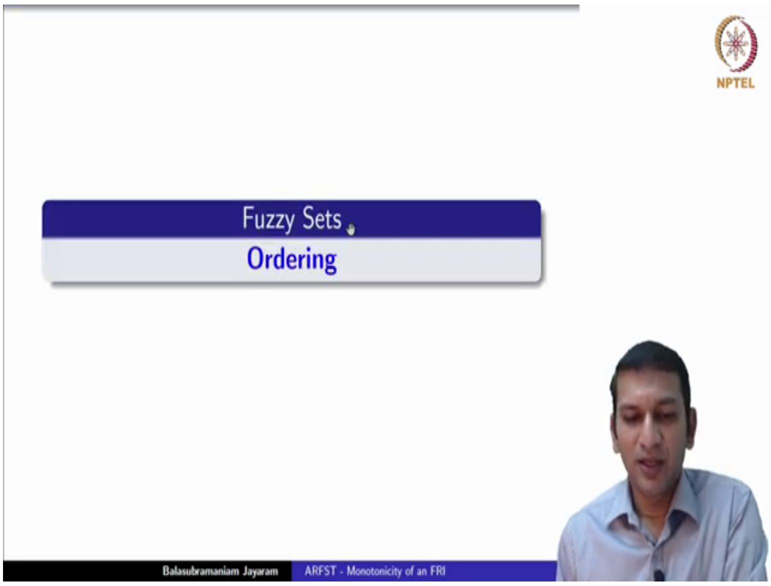

Let us revisit the ordering that we saw in the last lecture.

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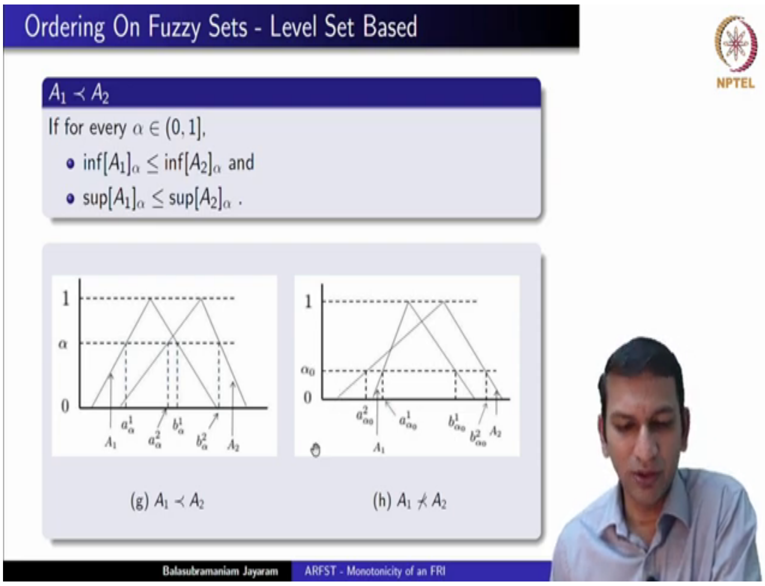

The ordering that we are going to consider especially with respect to monotonicity is that of level set based ordering. What does it say? It says that two fuzzy sets are ordered with respect to this order, if the corresponding alpha cuts for every alpha cut, they are orderable in the sense of interval order. That means, when we take two intervals, if the infimum of both the intervals they are comparable, the infimum of the 1st interval is smaller than infimum of 2nd interval.

And similarly, the supremum of the 1st interval is smaller than that of the supremum of the 2nd interval we say that that is the interval ordering and what we want is for every alpha cut between these two fuzzy sets if the corresponding alpha cuts are orderable with respect to this interval ordering, then we say that A_1 is less than A_2 with respect to this level set based ordering.

Now, we have seen these examples before, here we see that for every alpha cut the corresponding alpha cuts are orderable. Whereas, here it is not for not true for every alpha cut at least for this alpha cut, we can see that it is not orderable with respect to this given order.

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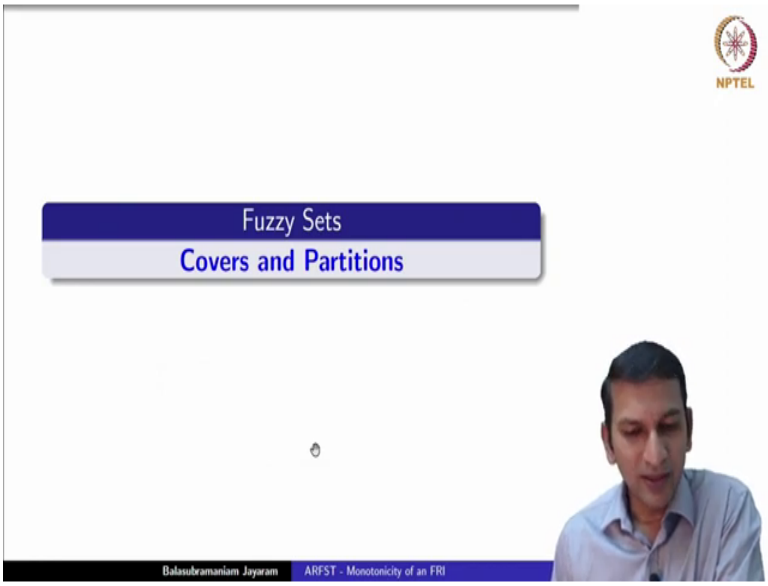

Now, quick recap of the covers and partition that we have discussed.

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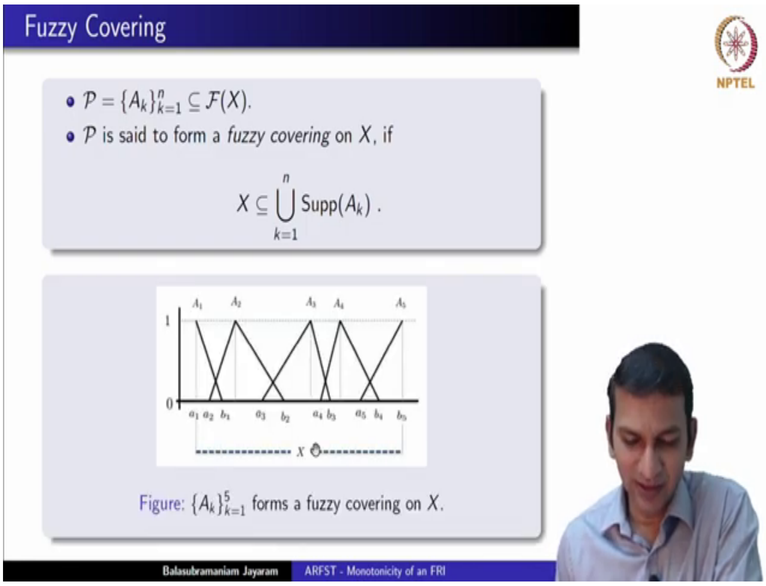

So, by fuzzy covering we mean a collection of fuzzy sets whose the union of whose support is actually containing X or equal to X.

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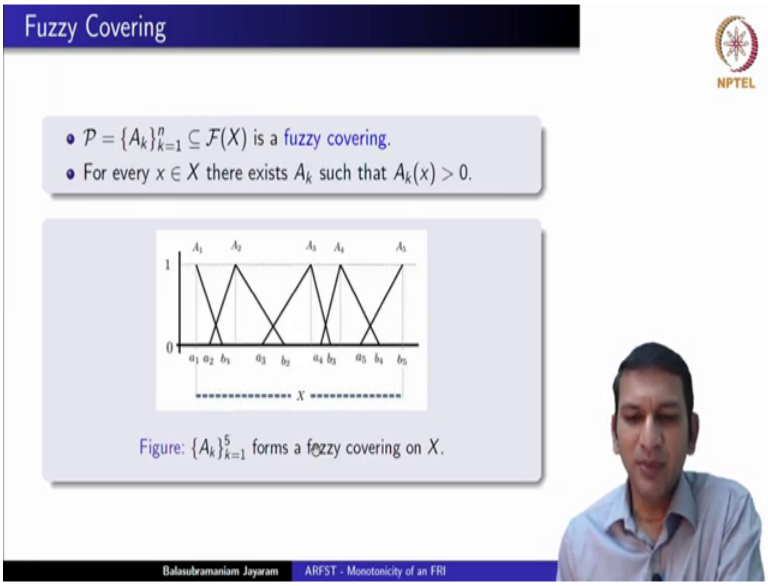

So essentially, we want that for every x in X, there must exist a fuzzy set in this collection to whom this x belongs to non-zero membership value.

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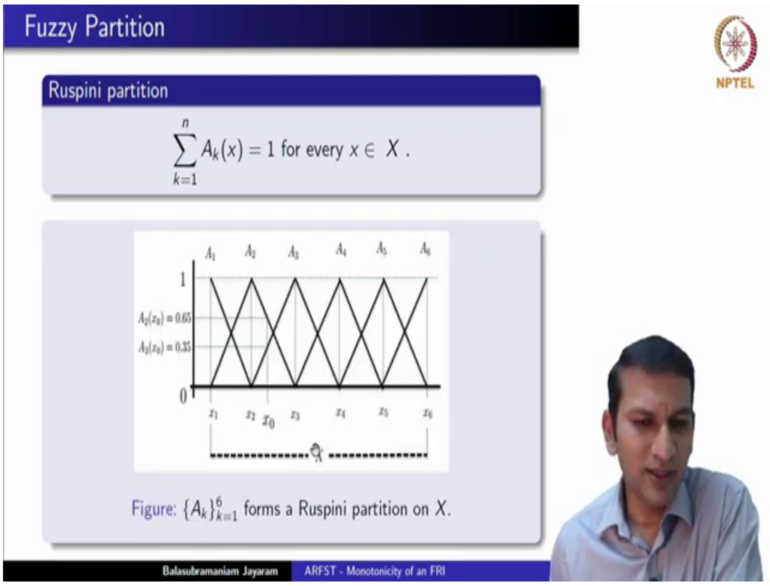

A partition on the other hand is a covering and there are some specific partitions, we call it a Ruspini partition, if the sum of all the membership values that an X can belong to in the covering that we have considered, if it adds up to 1 we say it is a Ruspini partition. That means if you take any x with respect to that covering the sum of membership values of this X, the different pieces of the partition. If they add up to 1 we call it the Ruspini partition. This is one such example that we have seen.

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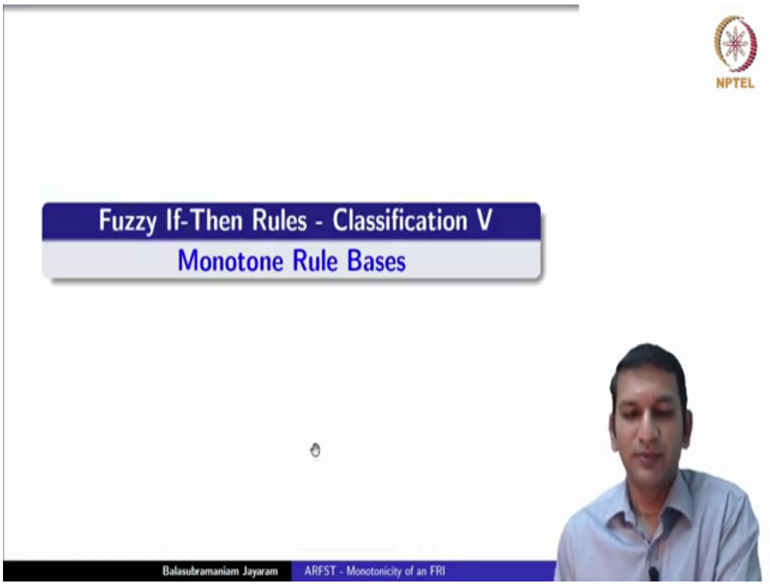

Well, we have seen many ways of classifying fuzzy if then rules. In the last lecture we saw one particular way of classifying it called the monotone rule bases.

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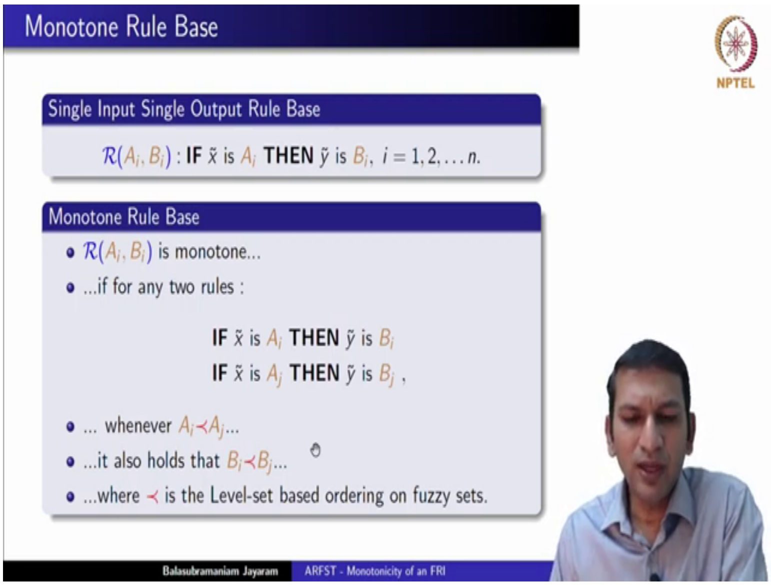

What is a monotone rule base? Let us consider a single input single output rule base. If x is A_i, then y is B_i, in such rules. We see this rule base is monotone if for any two rules, pick these two rules x is A_i then y is B_j, if x is A_j then y is B_j. We say it is this rule base is monotone, if whenever we pick any two rules if you see that A_i is related to A_j, with respect to the level set based ordering, then we expect that the corresponding consequence are also ordered with respect to the level set based ordering.

And this should happen for any pair of rules that we pick from this.

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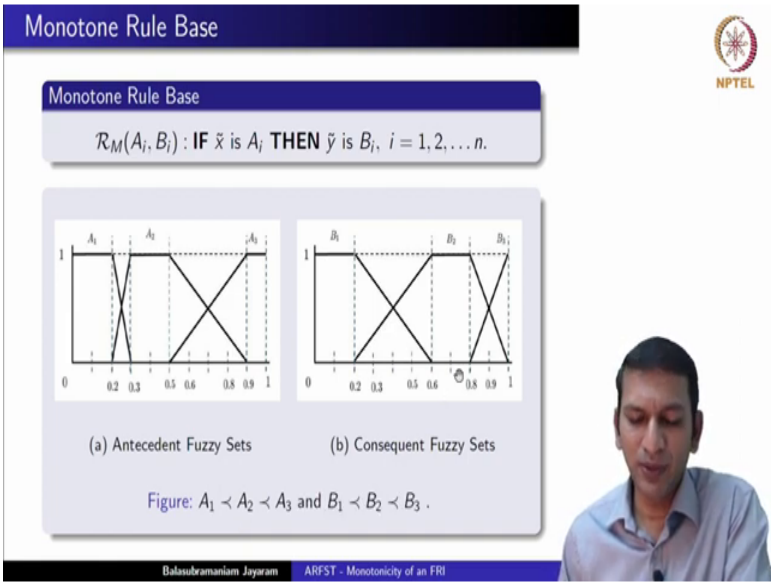

Now, let us look at some examples. If we consider these three fuzzy sets as the antecedent fuzzy sets and these three fuzzy sets are the consequent fuzzy sets, it is clear that these three fuzzy sets are orderable with respect to the level set based ordering and so are these fuzzy sets B 1, B 2, B 3. So, if we consider them as antecedents and these are the corresponding consequence, we see that we can construct a monotone rule base.

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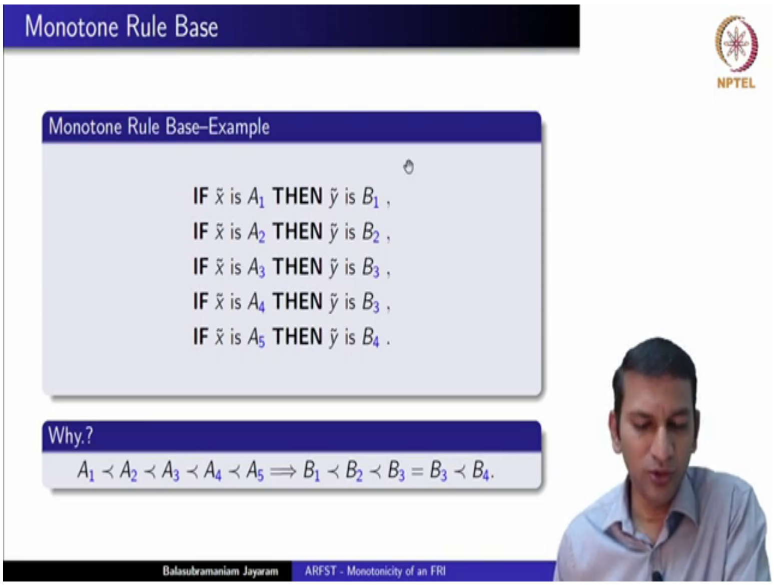

So, if we have A_1, A_2 ordered based on the indices and if these B 1, B 2, B 3 before order based on the indices then such a rule base is a monotone rule base.

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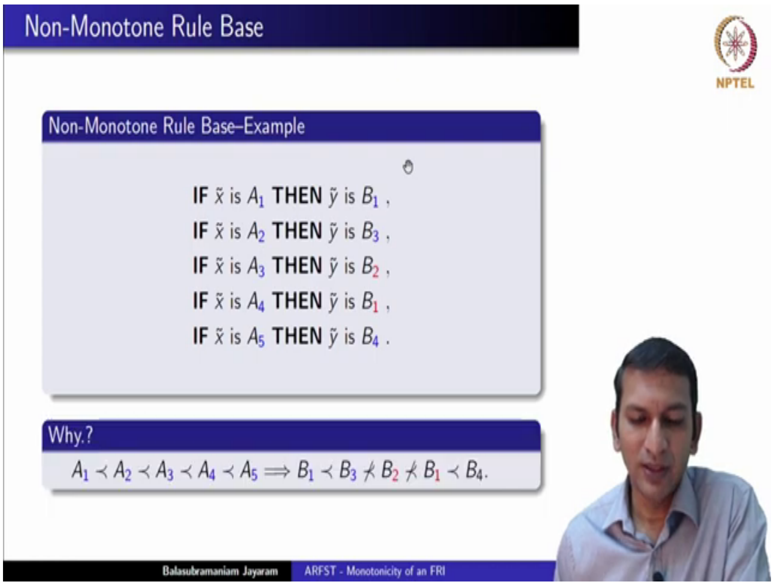

But if there is a shift in the ordering here, you see here that B 1 is less than B 3, when A_1 is less than A_2, B 1 is less than B 3. We see A_2 is less than A_3, but the corresponding consequence are B 3 and B 2. Clearly B 3 is not less than B 2, which means this rule based is not monotone.

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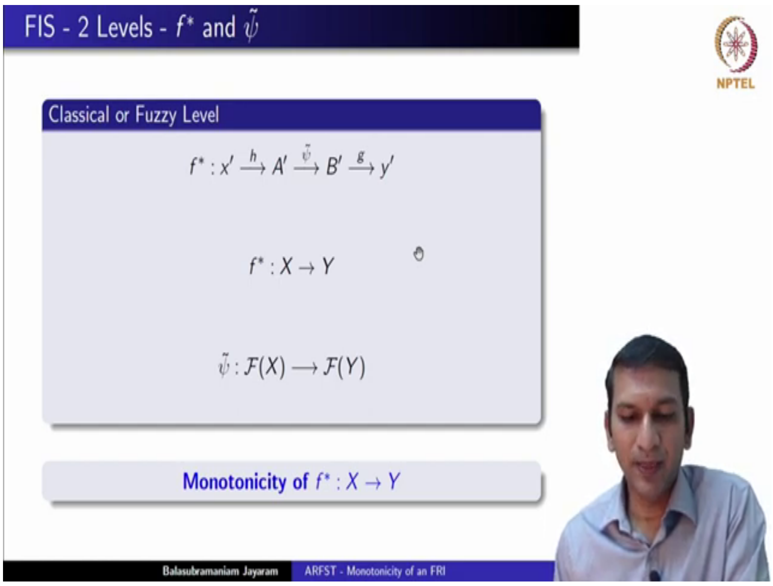

Well, we have seen that the system function of a fuzzy inference system can be seen at two different levels. One is at the classical level which is we can look at it as an f star, where we give an x dash which is the input coming from the domain x, we fuzzified to A dash give it to the corresponding psi tilde which is a function from f of x to f of y, we get A B dash and we use a defuzzifier to obtain a y dash. So, this is the function from X to Y.

So, this is one way of looking at the fuzzy inference system, the system function of an FIS, when we have both the fuzzifier and the defuzzifier or we could also look at it as just simply psi tilde which is a mapping from F of X to F of Y. Typically, monotonicity is discussed at the level of the function f star, the classical level as a mapping from X to Y.

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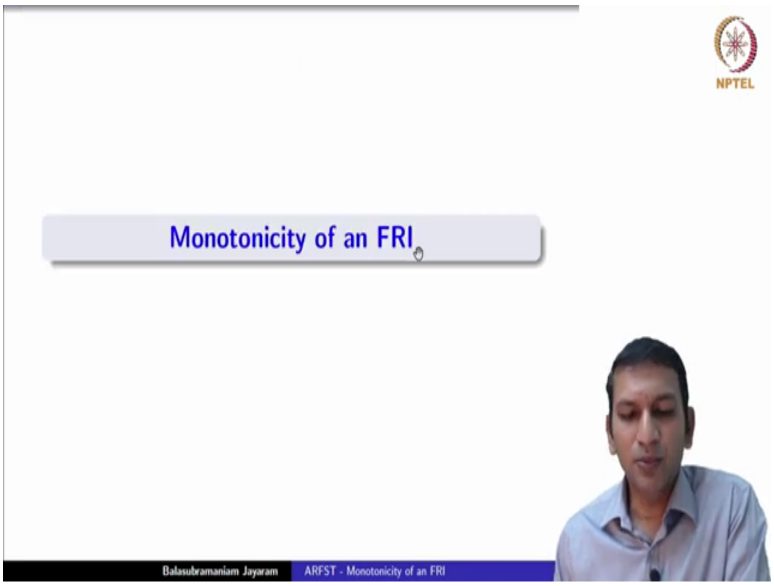

Let us discuss the monotonicity of an FRI.

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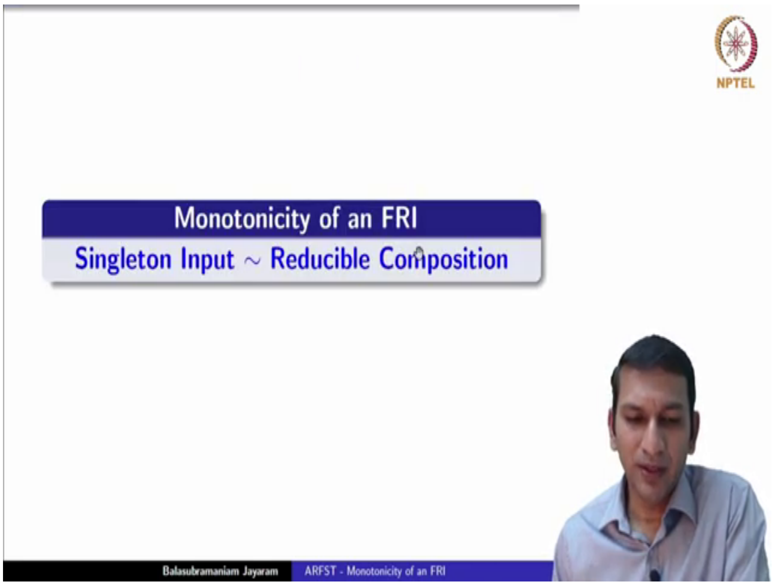

We will see that in the case of a singleton input when we use a singleton fuzzifier, then the composition really does not play a role.

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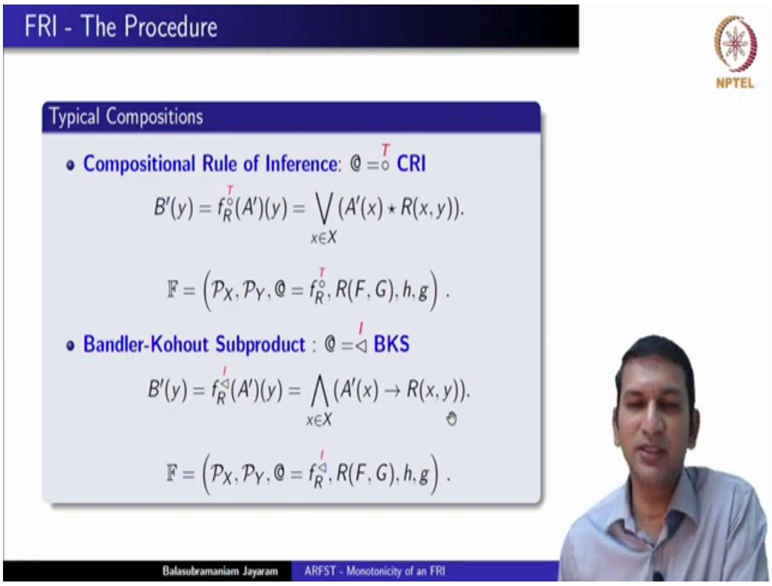

Now, there are two composition that we have seen; the CRI and the BKS, the Bandler-Kohout subproduct. We see here in this case we are looking at using the infi composition and in the case of CRI we have looking at star composition. And to get the rules we use F which relates the antecedents to the consequence and G to aggregate it. h and g are the fuzzifiers and defuzzifiers.

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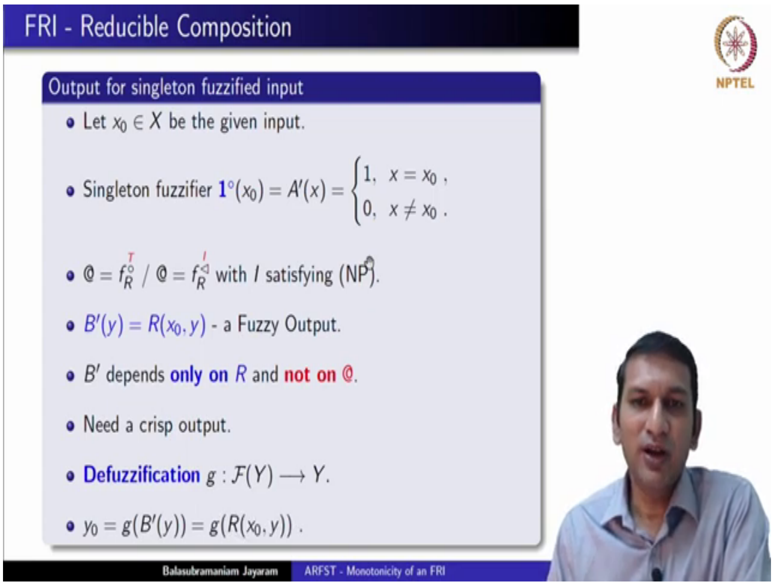

Well, let us look at the case when we are given a singleton input and we are using a singleton fuzzifier. So, we have some x naught, which is given as input and we use the singleton fuzzifier. We know that it is defined like this as an A dash, which takes the value 1 at x is equal to x naught and elsewhere it is 0.

Now, it does not matter if you use the CRI composition; that means, sup-t composition or the infi-composition with I satisfying NP what we find is the B dash fuzzy output is essentially R of x naught comma y.

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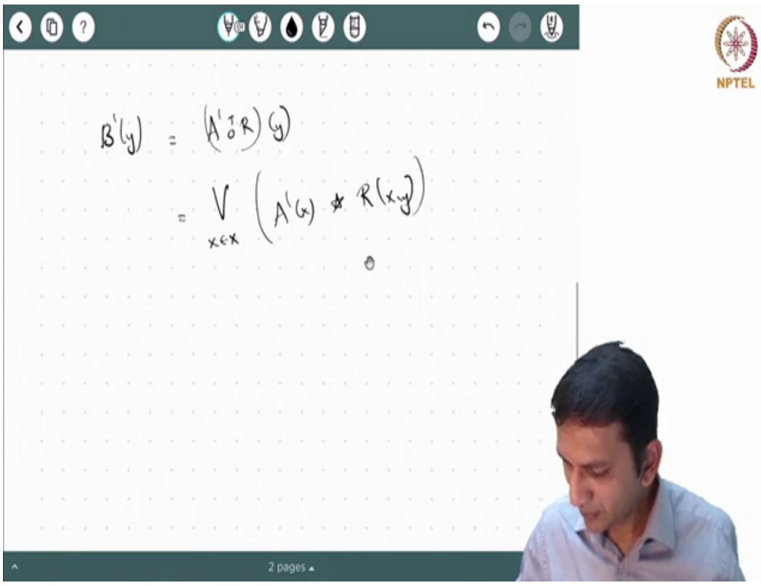

It is quite easy to see. Look at this, if you have B dash of y this is nothing but A dash circle R and y. In the case of sup-t composition we know that supremum over x A dash of x, let us use any fixed notation R of x comma y.

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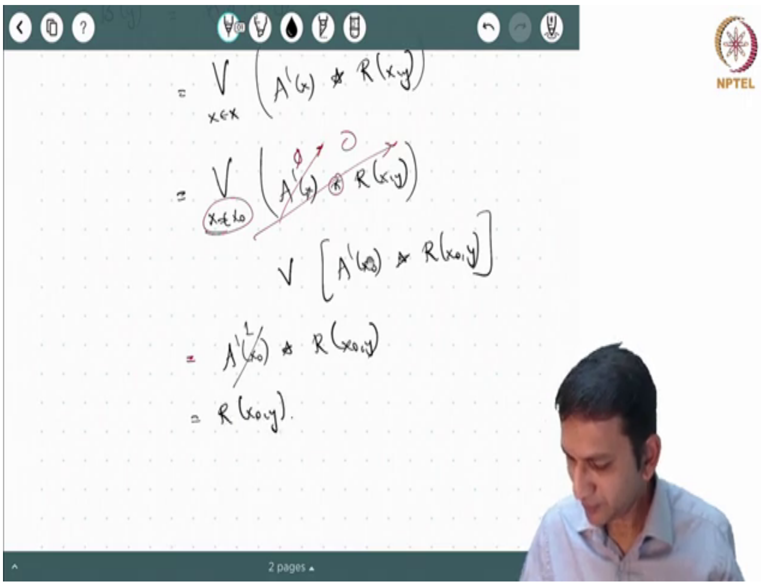

Now, we could take this as all those x not equal to x naught A dash of x star R of x y, max A dash at x naught star R of x naught comma y. We know that A dash of x for x not equal to x naught is 0 and this is a t-norm which means this entire thing is 0. So, what we are left with is essentially A dash of x naught star R of x naught comma y, but we know that this is 1. So, 1 star this is R of x naught comma y.

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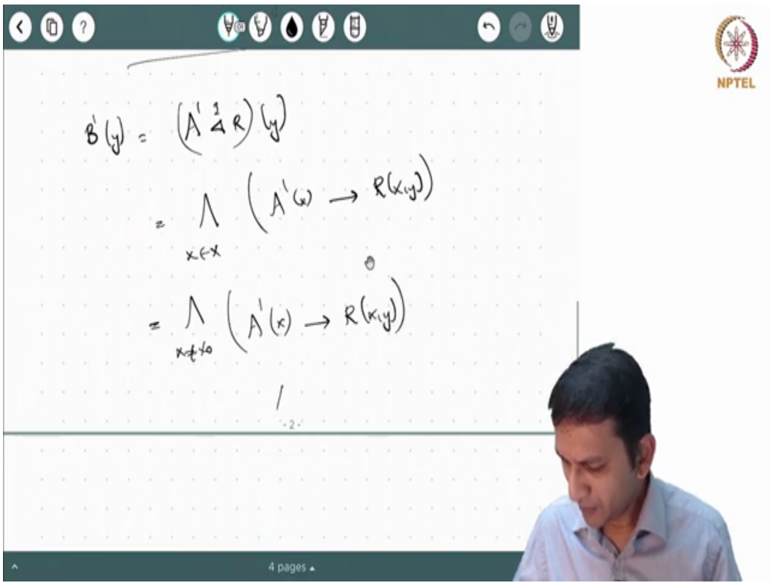

Now, this is the case when we are looking at sup-t composition. Let us look at infi-composition. So, here once again B dash of y is A dash delta I R of y. Now, this is given as infimum of x element of x A dash of x implies R of x y. Now, once again we can split it like this, x not equal to x naught A dash of x implies R of x y infimum with A dash of x naught implies R of x naught comma y.

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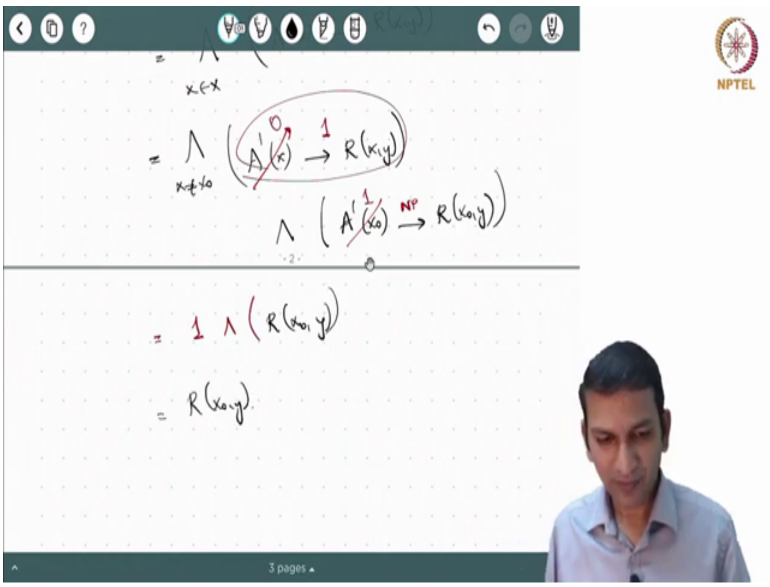

Now, once again if you see A dash of x 0, but 0 implies anything is 1, because it is an implication. So, this entire thing is 1. So, what we get is 1 and A dash of x naught is 1 and we have assumed that implication as NP, which means this essentially becomes R of x naught comma y. Infimum of these two will be R of x naught comma y.

So, in the case we consider sup-t composition or an infi-composition with I satisfying NP, we see that the corresponding B dash is nothing but R of x naught comma y, where x naught is the input. So, essentially if you look at R as the fuzzy relation, we are essentially picking that row that corresponds to x naught in the discrete case.

Well, now it is clear now, that B dash depends only on R and not on the composition. That is when we are given an input and we use a singleton fuzzifier. Now, we need a crisp output for this x naught. So; that means, we apply the defuzzification on B dash and so the corresponding y naught for x naught is nothing but g of B dash of y which is essentially g of R of x naught comma y. It does not matter what the relation R is.

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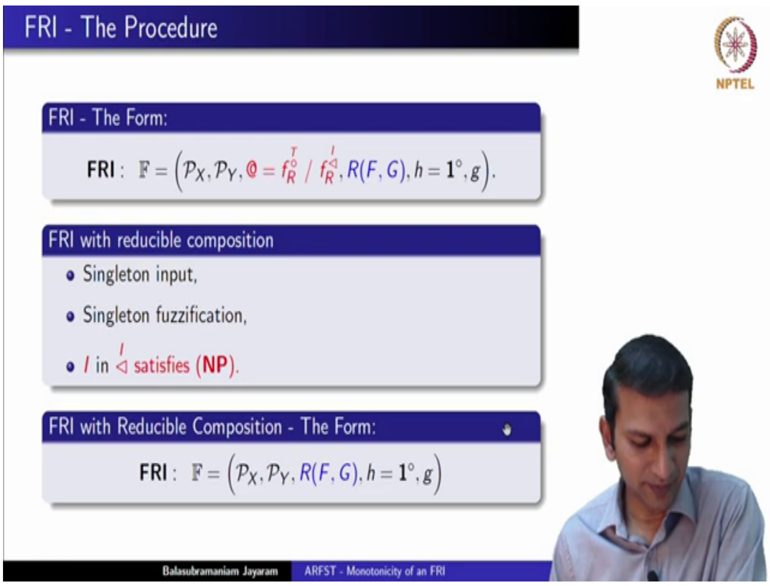

Well, so now, this is the original form of any FRI, but now in the case of singleton input, with singleton fuzzification and whether your sup-t or infi with I satisfying NP, we see that it reduces to this form, where the composition does not play any role. Of course we are fixing the fuzzifier to be singleton fuzzifier.

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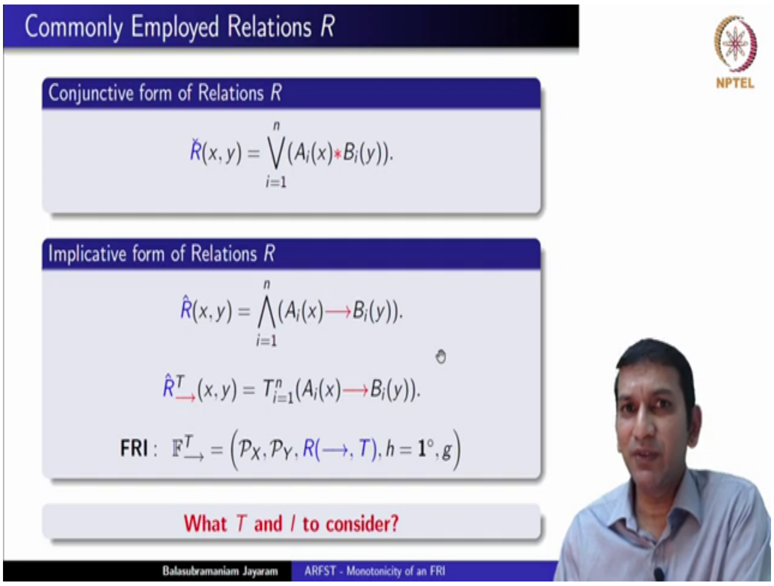

Now, what are the commonly employed relation that we know? If the rules are of conjunctive form we know that we use the R check relation, if they are of the implicative form we will use the R cap relation. But now, we want to generalize a bit; instead of considering min for or in for i is equal to 1 to n is essentially a min because n is finite, we could in fact, generalize and consider any T, t-norm and let this arrow implies some of the implication, the general implication.

They need not be related; T and this implication they need not be related. They need not be related in the form of residuated lattice or something like that. Now, these operations need not come from residuated lattice, presently when we are discussing monotonicity. So, the FRI turns out to be like this. For F we use the implication there is no composition here and for g we are using any t-norm. Of course, h is the singleton fuzzifier.

So, we have fixed at least three components; the F, g and the fuzzifier here. Of course, now the question is what are T and I, which pairs to consider. We want to break free from this residuated lattice structure. We want to see whether we can move out of it and go a little more general.

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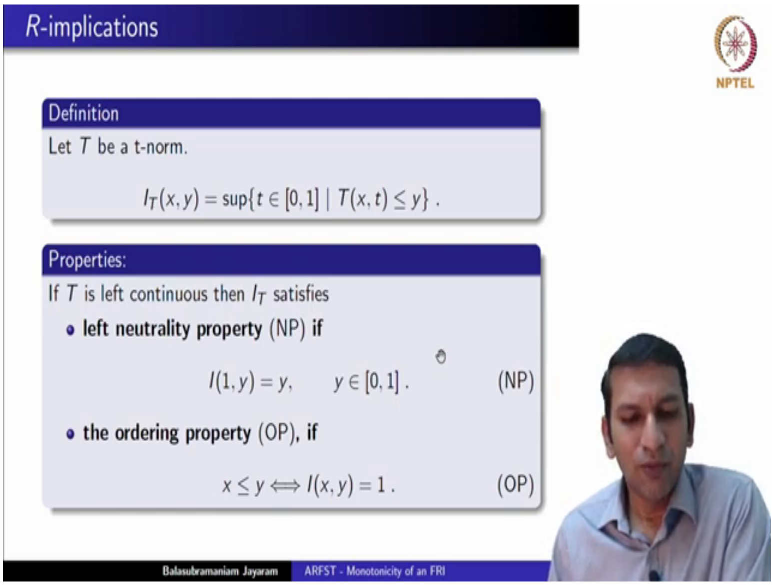

But let us look at R-implication themselves. We have a t-norm T and the corresponding R-implication is given like this. What we do now is that if T is left continuous, then I T satisfies, among the many properties it satisfies, it satisfies at least these two properties.

It satisfies NP, which is important because we want to remove the composition from the picture and we know that it also satisfies ordering property. In fact, to satisfy the ordering property border continuity is enough, but ok, let us keep left continuity for now to see this.

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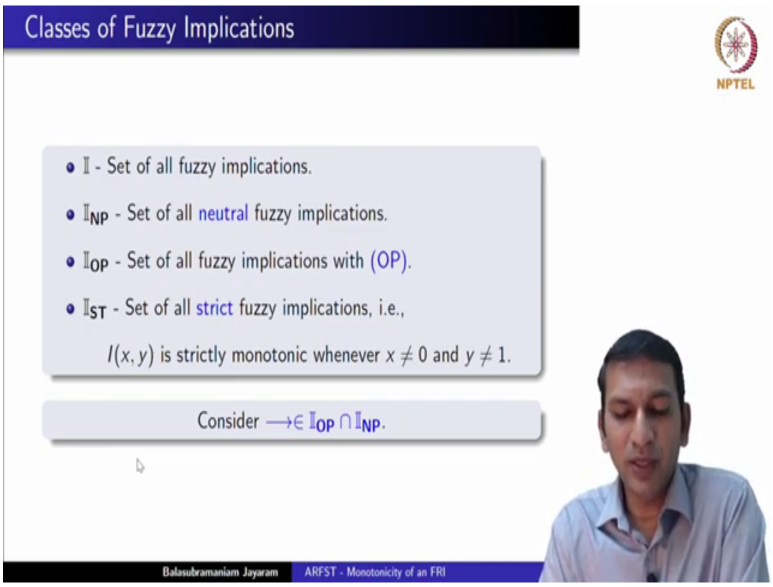

So, now what we would like to do is consider the implications which contain the set of R-implications obtained from left continuous t-norms. Towards this end, let us look at some classes of fuzzy implication. Let the set I denote the set of all fuzzy implications. By I NP we set denote the set of all neutral fuzzy implication, those implications as I said I of 1 comma y is y.

By I OP we denote the set of all fuzzy implication satisfying the ordering property. By I ST let us denote the set of all strict fuzzy implication. That means what? They are strictly monotonic, when x is not equal to 0 or y is equal to 1. So, we know that these are the two boundaries which are fixed, they are 1. So, they are constant.

So, everywhere else, they are non-constant. They are strictly decreasing or strictly increasing depending on which variable we fix. Let us look at these classes in the setting of the set of all fuzzy implications to see how nicely they partition the entire set of fuzzy implications.

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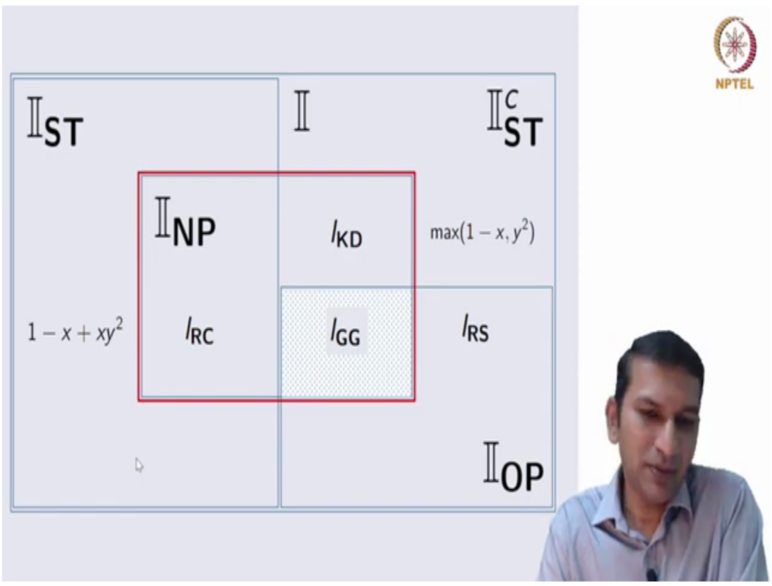

Consider set of all fuzzy implications I. The set of all strict implications of course, form a subset of it. So, they form a partition with respect to its complement, the set of all non-strict implications. Now, if we consider the set of all odd implication satisfying OP, clearly it falls in the complement of set of all strict fuzzy implications. Because when it has OP, whenever x is less than y, then it is 1. When x is equal to y also it is 1.

So, the entire portion above the main diagonal they are all constants. So, it is clearly not a strictly monotonic fuzzy implication. So, this is contained entirely in the complement of strict fuzzy implications. However, if we consider the set of all NP implications, they overlap with all these three classes. How do we say this? Now look at the Goguen implication, it has OP it satisfies OP and it has NP. So, clearly it falls in the intersection of OP and NP.

If you take the Kleene-Dienes implication, it satisfies NP, but it is it does not satisfy the ordering property; maximum of 1 minus x comma y and neither is it strict. So, it is neither satisfying OP nor is it strict; however, it satisfies neutrality property. If you consider the Rescher implication, which is one above the main diagonal and 0 elsewhere, it is it satisfies OP, but it does not satisfy NP. So, clearly it falls here.

If you consider the Reichenbach implication, it satisfies NP, it does not satisfy OP. In fact, it is a strict fuzzy implication which satisfies NP. So, clearly it falls on the intersection between NP and strict fuzzy implication. Now, are there any fuzzy implication which are purely strict, but does not belong to NP? Yes, consider this implication 1 minus x plus x y square, clearly it is strictly monotonic both the variables. However, it does not satisfy NP because I of 1 comma y will be y square.

Similarly, are there implications which are purely non-strict, but does not fall intersect with NP or OP? Yes, look at this max of 1 minus x comma y square. It is not strictly monotonic, it does not satisfy OP and it is not neutral either. So, this entire thing nicely partition set, but what we want is in fact, consider only implications coming from NP. So, this is the class that we need to consider.

Presently we will consider the case where the implications come from OP intersection NP. So, we will consider the case where the implications come from the set of all fuzzy implications satisfying both OP and NP.

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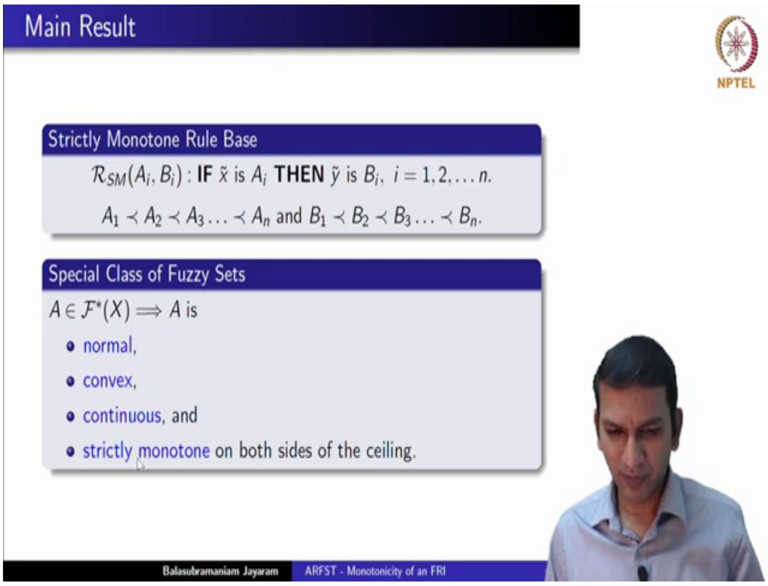

Now, let us look at the main result of this lecture. Let us assume that we are given a strictly monotone rule base. What do we mean by this? So, we have n rules and A_1, A_2, A_3 are orderable backed by their indices and the corresponding consequence are also orderable.

We will consider a special class of fuzzy sets from which we will pick the antecedents and consequence. So, by F star of X, we will denote the set of all fuzzy sets on X, such that if A belongs there then A is normal; that means, it has a point of normality. It is convex, it is continuous, continuous in the sense of the function that we understand as a function from X to 0, 1. Also it is strictly monotone on both sides of the ceiling.

So, here in fact, because it is normal the ceiling is also the kernel, but we want that this the kernel could be an interval, because it is a convex fuzzy set. So, either side of the ceiling the kernel we want this to be strictly monotonic. So, we could think of a triangular membership function or a trapezoidal membership function or even any other membership function of this type.

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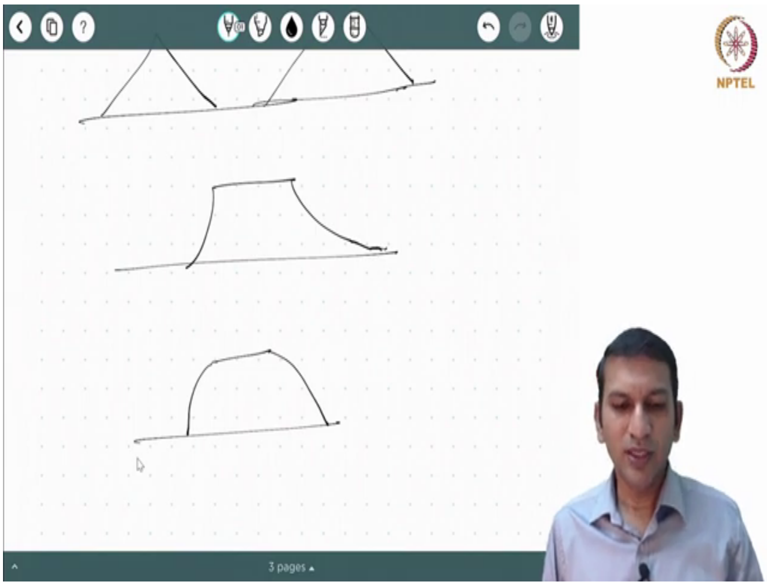

For instance, you could have triangular, trapezoidal or you could also have that the kernel, you could also have something that goes like this. Or you could have that the kernel something that falls like this. So, these are some of the fuzzy sets, which actually fall in this special class of fuzzy sets on X.

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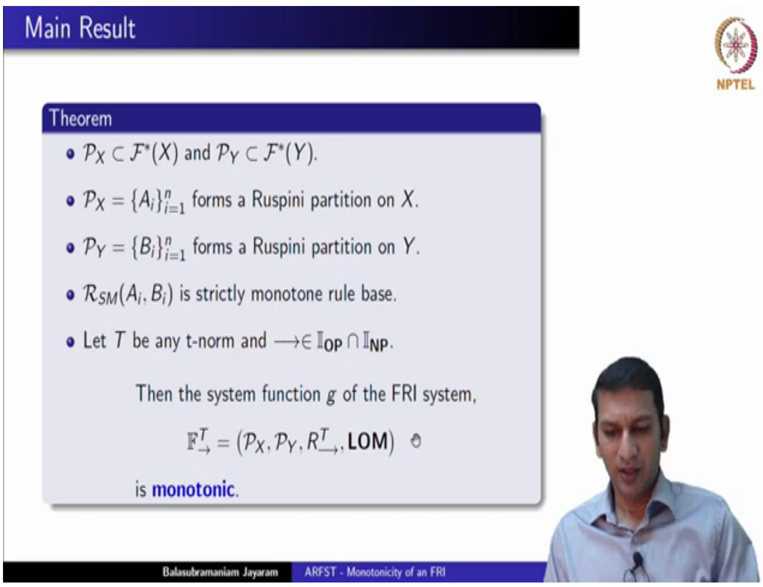

Now, let P X and P Y, which consists of the antecedents and consequence, let them come from the particular class of fuzzy sets that we have discussed now. That means they are normal, continuous, convex and strictly monotone on either side of the kernel. In this P X which consists of these antecedents, they also form a Ruspini partition on X. This collection forms a Ruspini partition on X. Let us insist on the collection of consequence to form a Ruspini partition on Y that happens.

And the rule base that we are considering is a strictly monotonic rule base; that means, the order between the antecedents is carried over to the consequence also the corresponding consequence also. Let T be any t-norm and the implication let it come from the class of fuzzy implications which satisfy both OP and NP.

Then we can prove the system function g of the following FRI system, where we have F X, P Y. And this R T implication is the one where we substitute T for the aggregation and the implication coming from this class for the implication there relating the antecedent to the consequent of each of the rules and let the defuzzifier be the LOM defuzzifier, which is the least of maximum.

If that happens, we see that it is actually monotonic. The system function is monotonic as a function of X to Y, as a function from X to Y. Now, note that the fuzzifier is not specified because we have fixed it to be the singleton fuzzifier. Well, let us go ahead and prove this result.

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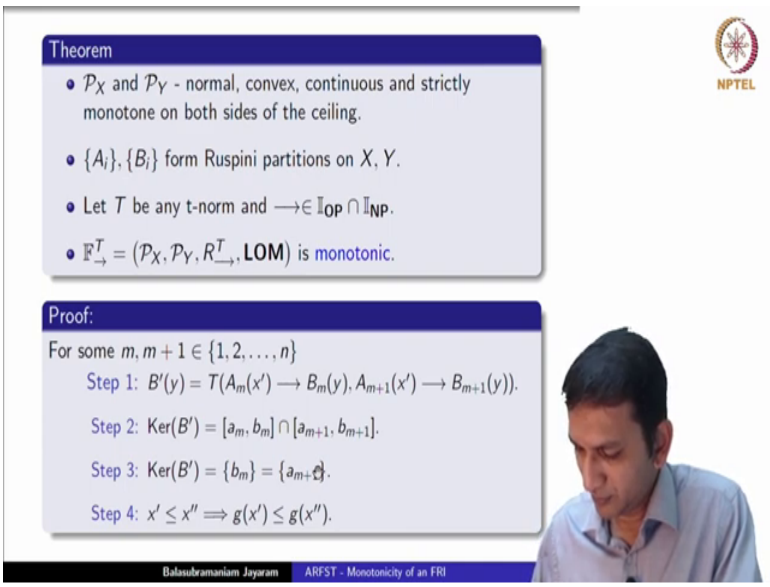

Let us have the required part here and then we will go ahead and prove the result.

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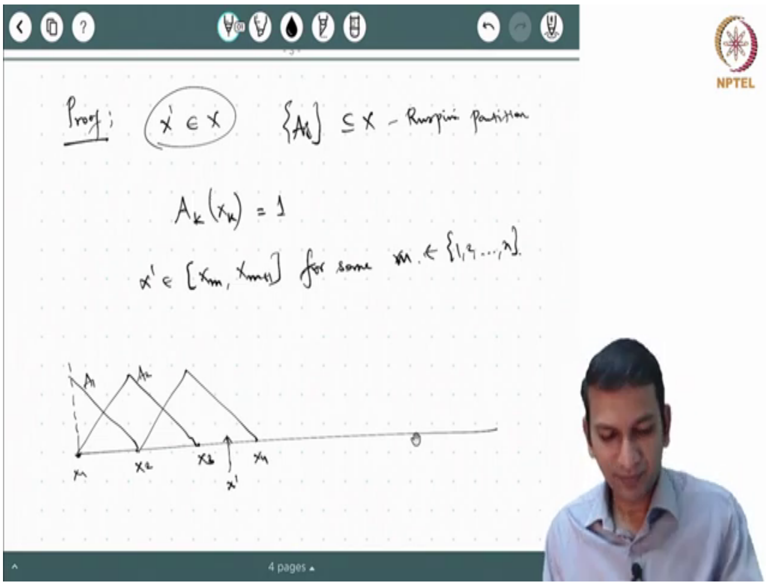

So, to begin with let us assume that we are given an x dash element of x. Now, we know that the antecedents A i coming from x they form a Ruspini partition. Now, we know that these are also normal fuzzy sets, because we are picking it from picking them from the special class of fuzzy sets.

So, we could assume that these A k's, each A k attains normality at some x k. Clearly that means, when you pick x dash, x dash is going to fall in the interval x m, x m plus 1 for some m for instance. So, we could consider this as x 1. So, x this is A_2 is A_1 is A_2 A_3 so on and so forth.

So, if you pick any x, it is going to fall any x dash. It is going to fall between x m and x m plus 1. So, that is the first observation that we can make. For some m, m plus 1 we know that x dash falls there. Let us go ahead and prove in step 1, how B dash of y will look like.

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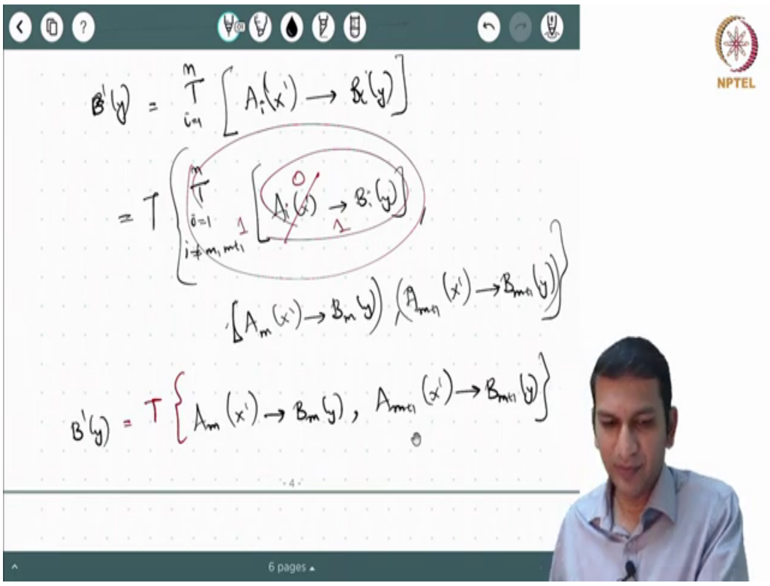

Note that B dash of y is equal to this is the output, we know that this T of i is equal to 1 to n, A i of x dash implies B i of y because we are talking about singleton defuzzification. Now, this can be written as, now x dash belongs to m, m plus 1, it does not belong to any other interval.

Which means, if you write it as T of i is equal to 1 to n, i not equal to m or m plus 1 A i of x dash implies B i of y comma this, we are absolutely write like this; T of thus A m of x dash implies B m of y comma A m plus 1 of x dash implies B m plus 1 of y.

Now, clearly because we are talking about singleton fuzzy implication and x dash does not belong to the interval other than x m, x m plus 1. So, A i of x dash is 0. When this is 0 this entire thing is 1 and t-norm over entire thing is 1. So, this is essentially T of A m of x dash implies B m of y comma A m plus 1 of x dash implies B m plus 1 of y. So, this is our B dash of y.

So, notice this is what we have here. Next, what we need to find is the least of maximum. First, we claim that this B dash is normal, hence we go about finding the kernel. We can show the kernel is in fact the intersection of these two intervals for some values A m, B m and A m plus 1, B m plus 1. Let us look at what they are.

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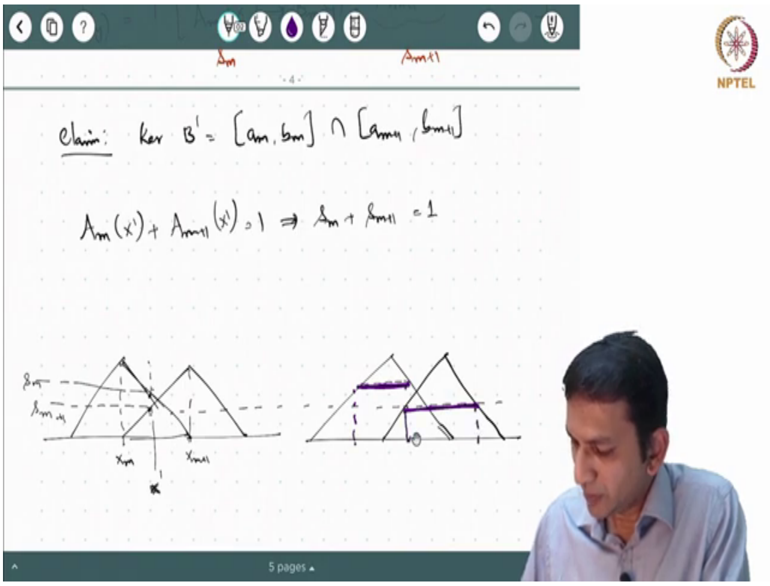

Claim: Kernel of B dash is in fact a m, b m intersection a m plus 1, b m plus 1. First let us look at this. Let us write this as s m and this as s m plus 1, note that these are actually membership values of x dash in a m and a m plus 1. Now, we know that a m and a m plus 1, they are coming from P x which forms a Ruspini partition; that means, we know that A m of x dash plus A m plus 1 of x dash is equal to 1, this implies s m plus s m plus 1 is equal to 1. Now how would it look like?

So essentially, we have these two A m and A m plus 1. So, this is x m, this x m write here, this is x m plus 1. So, we assume that let us assume that the x what we have is here, the x dash. Now so this is our s m plus 1 and this is our s m, plus note that x dash is between x m and x m plus 1; that means, it falls on the decreasing part of A m and increasing part of A m plus 1.

Now, what we are doing is we are using these values essentially to modify B m of y, B m also the B i's also form a Ruspini partition ok. So, when we do that we take this value and this simply. So, what we have is this. Consider this part. What this is? So, it become clear presently. So, now what is kernel of B dash?

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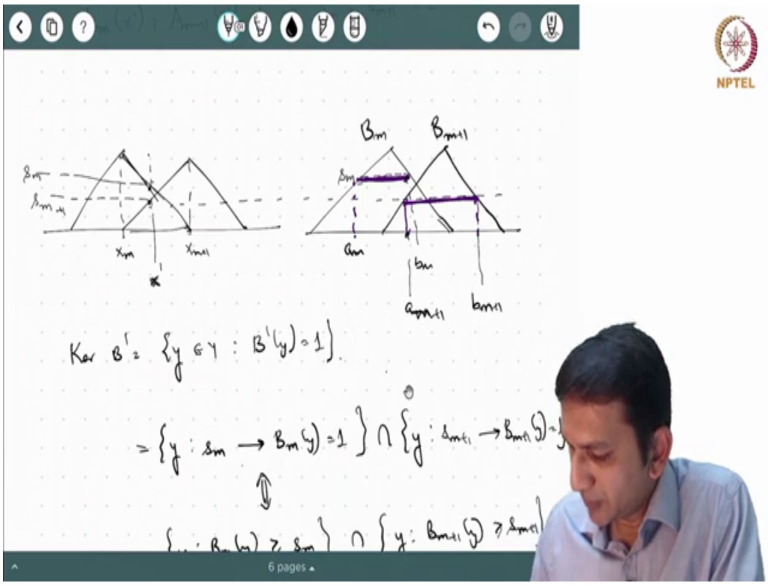

Kernel of B dash is set of all y in y such that B dash of y is equal to 1. But look at it, B dash of y should be 1. Now, this B dash is T of these two quantities; that means, both the quantity should be 1. So, this is essentially set of all y in y such that A m of x dash, now we are writing it as s m. Set of all s m implying B m of y is equal to 1, all those y such that this happens, intersection y, such that s m plus 1 implying B m plus 1 of y is equal to 1.

Now note that implication we have picked it up from this class of OP implications; that means, this is equal to 1, if and only if B m of y sorry is greater than or equal to s m, intersection y such that B m plus 1 of y is greater than or equal to s m plus 1. So, this is essentially B m of s m the alpha cut of this intersection B m plus 1 the alpha cut with respect to s m plus 1.

So, now, this is what you have seen here, essentially the alpha cuts. So, let us take this as a m and this is b m and let us take this as a m plus 1 and this as b m plus 1.

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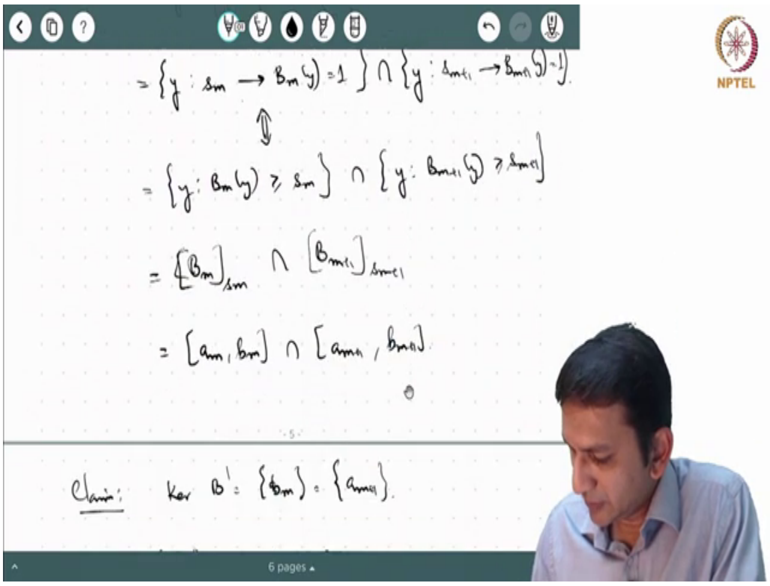

We know that B i's are continuous and convex. So, the alpha cuts will be interval. So, we take this as a m comma b m intersection a m plus 1 comma b m plus 1, this exactly what we wanted here, we have found this. Now, interestingly in fact, what we can show is that this intersection is non-empty not only that this intersection is in fact singleton. How do we show this? Right.

We need to show kernel of B dash is in fact, b m which is also equal to a m plus 1. Now, look at this. What is b m? b m is essentially that element of x element of y, such that the B m of at b m is actually going to take the value s m.

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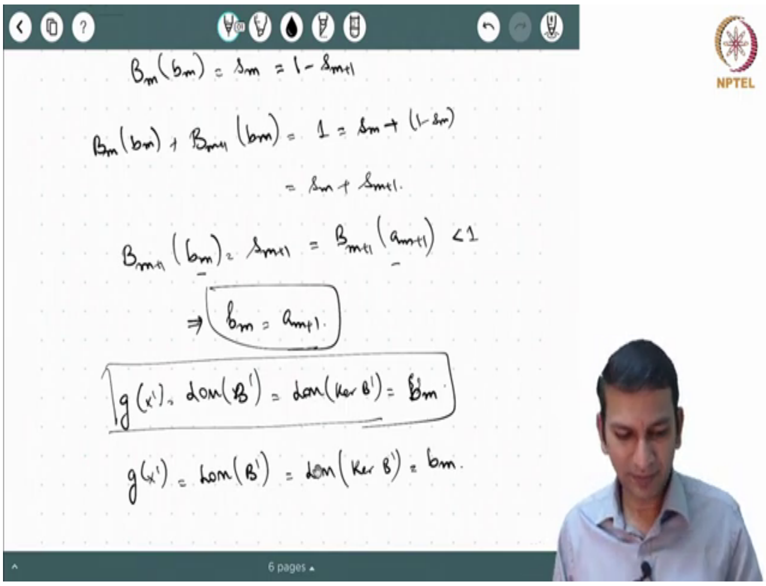

So, B m of b m is equal to s m. Now, what this means is we also know by Ruspini partition that s m plus s m plus 1 is equal to 1. So, b s also forming because they also form a Ruspini partition; that means, this is equal to s m plus 1. Now, we know that at this point B m, if you take this figure as the reference figure, we see that b m belongs to both capital B m and B m plus 1.

See; that means, it is clear that B m of b m plus B m plus 1 of b m is equal to 1. So, this is equal to s m plus some quantity which is essentially 1 minus s m which is nothing but s m plus s m plus 1. So that means, we see that B m plus 1 at b m is in fact equal to s m plus 1.

But we also know B m plus 1 at a m plus 1 is also equal to s m plus 1 and note that this B m plus 1 and this value s m plus 1, because the x dash we have assumed that it may not exactly be the point of, it may not belong to the kernel. When it is not 1, what we know is B m plus 1 on either side of the ceiling is strictly monotonic, either increasing or decreasing.

In this case B m plus 1 is in fact increasing and by the strictness we see that this implies b m is in fact equal to a m plus 1, which means we get the kernel of B dash. So, this b m is actually equal to a m plus 1. So, this is what we get as kernel. So, now, if you are given an x dash, if you want to find the kernel essentially the kernel in the setting that we have taken, where the antecedents and consequence they form Ruspini partitions on the corresponding x and y.

And these sets are in fact, normal, convex, continuous and strictly monotone both sides of the ceiling, both sides of the kernel we see that for a given x dash the kernel of B dash is in fact, a singleton. So, it does not matter whether you apply the least of maxima or the max of maxima or the mean of maxima, all of them will exactly be the same, any such defuzzifier you apply, it will essentially be B m itself.

So, from here we see g of x dash, if it is equal to LOM of x dash of B dash is equal to LOM of kernel of B dash, this is equal to essentially singleton. So, this is b m; the value that we have got right. Well, now, this what we have. Once more what we have written here is we are taking a g of x dash, we know that it is LOM of B dash, it is nothing but LOM of kernel of B dash because we want to find the least of maxima.

So, we have shown that B dash actually is normal. So, we are finding the kernel of B dash, but the kernel of B dash is singleton. So, we have got b m here where the final step now is to show that this mapping in fact, is monotonic ok, we will show this also.

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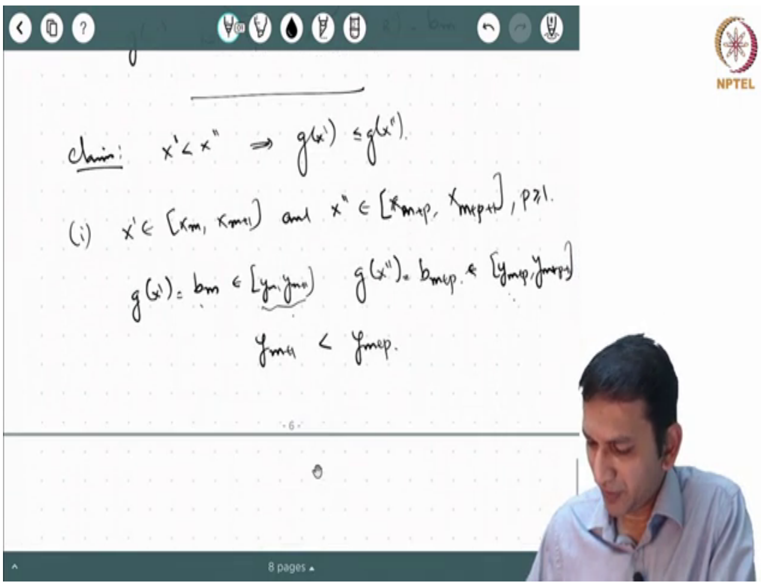

So, we want to show x dash is less than x double dash, this implies g of x dash is less than or equal to g of x double dash. Now, there are two cases here. x dash may belong to x m comma x m plus 1, because it has to fall in the it is coming from x. So, it has to fall in one of these intervals for some m.

And it is possible that x double dash can fall on the interval, which is little farther away. So, let us write it as it might be falling at x m plus p comma x m plus p plus 1 for some p greater than or equal to 1. So, it falls in an interval, a little farther away, because x dash is strictly less than x double dash. Then clearly g of x dash is some b m and g of x double dash is equal to some b m plus p.

Now, note that this b m itself actually belongs to you know in a way we can say y m, y m plus 1 and this will belong to where this is the part where b m and b m plus 1 they intersect. The supports intersect and this is coming from y m plus p comma y m plus p plus 1 and clearly this interval is in terms of ordering. So, we know that y m plus 1 is strictly less than y m plus p. So, clearly b m is smaller than b m plus p. So, in this case there is nothing to prove, it follows quite easily.

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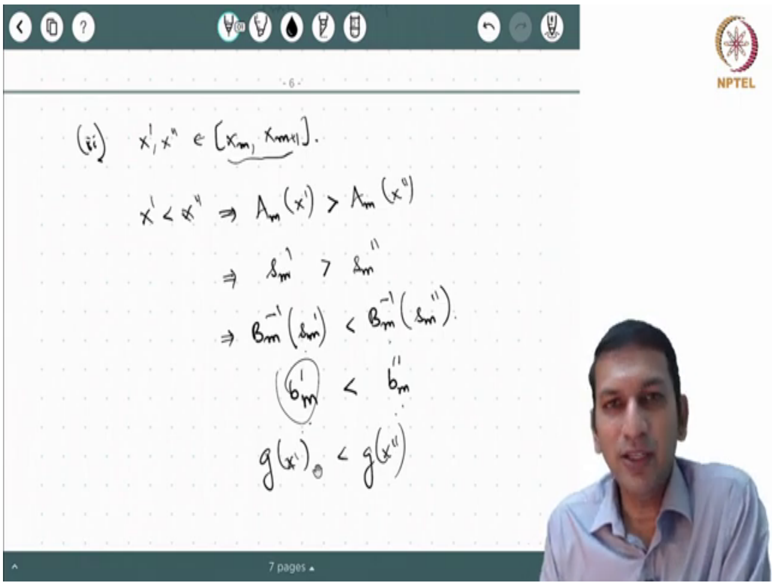

Let us look at the case when both x dash and x double dash they fall in the same interval. In this case again is quite simple because what we have is x dash is less than x double dash. This implies if you look at this region x m x m plus 1, we know that A m is in fact falling down. So, what we have is A m of x dash is greater than A m of x double dash. Now, if we take this as s m dash and this as s m double dash, so, this is what we have.

Now, if we apply B m on this, because that is what we need, we see that this is less than B m of s m double dash. Now, what is B m of s m dash? It essentially the point at which; sorry let us explain B m inverse. Now, we know that B m is strictly decreasing there. So, B m inverse is also strictly decreasing. So, B m inverse of s m dash is essentially the b m that we are looking for, the value b m belonging to y at which b m takes the value s m dash.

This b m dash, this is b m plus 1 dash because this is the value that B m takes s m double dash, which is essentially also falling on b m plus 1, which is what we have seen as a m plus 1 dash. So in fact, we should denote it by b m double dash. So, we see by the decreasing of this of this, this is smaller than this. So, once more; this b m is that point at which B m takes the value s m dash and b m double dash is that point at which B m takes the value s m double dash.

We know that these are what are defining the corresponding alpha cuts and by Ruspini, the kernel is essentially at the b m dash or a m plus 1 dash and in the other case it is either b m double dash or a m plus 1 double dash. And clearly, we see that b m dash is smaller than b m double dash which means this is our g of x dash which is smaller than g of x double dash. Now, you see that in both the cases g is in fact, monotonic.

Now, this shows that if you consider the antecedents and consequence in the monotone rule base to be normal, convex, continuous and strictly monotone on both sides of the ceiling and forming a Ruspini partition. You could use any t-norm and an implication which satisfies both NP and OP, the and if you would like to use any kernel based defuzzifier, then it will be the system function will be monotonic. Why any kernel based defuzzifier? Because the kernel turns out to be singleton kernel.

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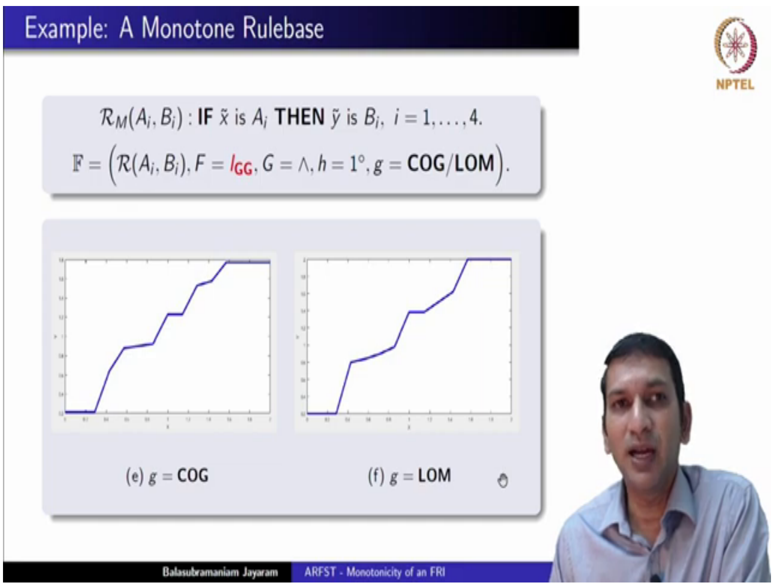

Now, the question comes up about illustrating them. So, let us pick one particular implication, the Goguen implication which we know satisfies both OP and NP and let us take a set of monotone rule bases. So, allow me to present to you the corresponding system function, if you use either the center of gravity defuzzifier or the LOM defuzzifier.

In this case we have four rules. The antecedents and consequence are ordered according to the level set based ordering and they are they do form a strictly monotone rule base. We see that the output for of the system that we obtain, the system function even for COG is quite monotonic, it is monotonic and in the case of LOM it is monotonic.

We see this. This is Goguen implication which is also an R-implication, which is a G-implication also. Let us look at another implication, which we know is not an R-implication, but satisfies both OP and IP and NP.

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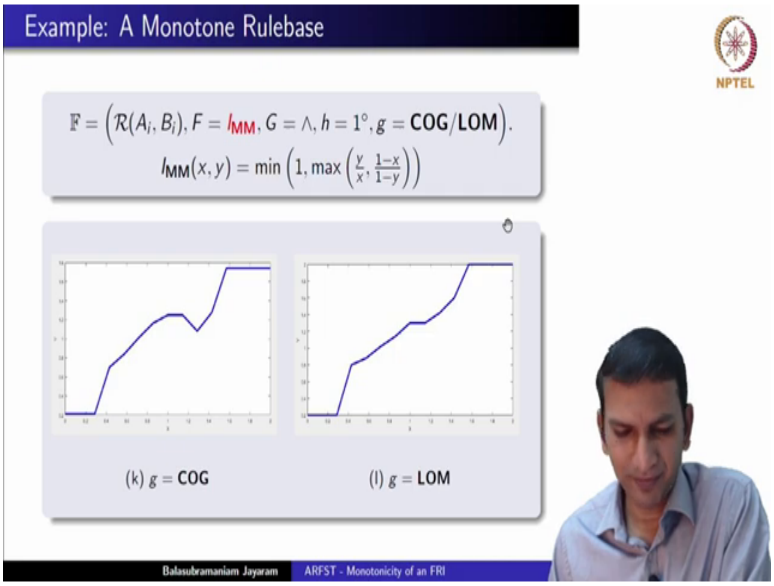

So, we denote it by the MM. So, it is essentially what we call the reciprocal of the Goguen implication, but it can clearly be seen that it cannot be obtained as an R-implication of some t-norm. Now, for this when we use the COG defuzzifier, we see that the system function is not monotonic. However, if you use the least of maxima, the kernel any kernel based defuzzifier, we do see that it validates our result stating that we do get monotonic system function.

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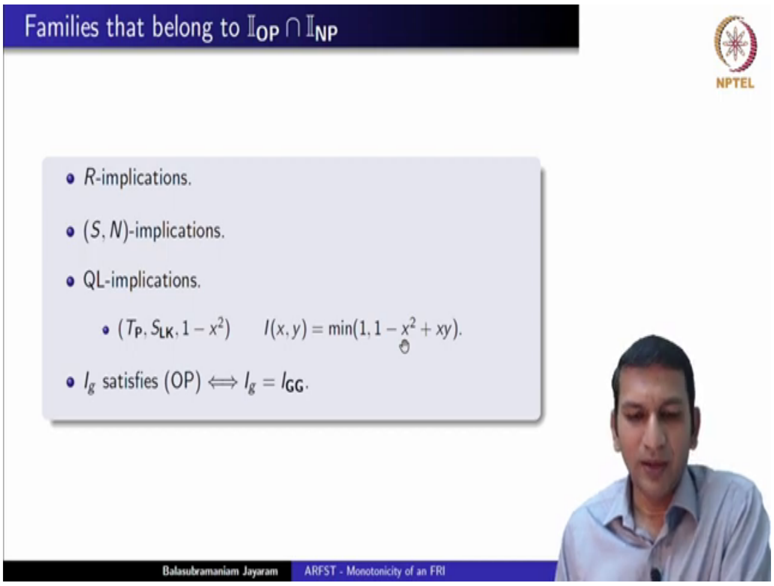

Now, the question that comes up is ok, we know that R-implications do satisfy OP and NP, but are there other families? Yes, R-implications are just 1. We have seen that S, N implications there are many members of the S, N implication which do satisfies satisfy both OP and NP.

There are also QL implications, which do satisfy OP and NP. For instance, if you consider the QL implication obtained from the product t-norm Lukasiewicz t-norm and the negation 1 minus x square we obtain this fuzzy implication. Clearly we see that this is this satisfies OP.

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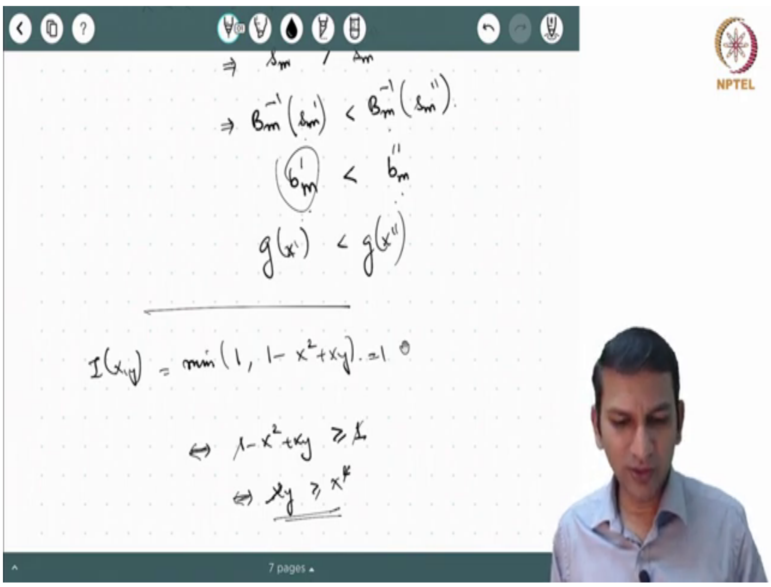

For instance, the implication we have is this, is minimum of 1 commA_1 minus x square plus x y. Now, this is equal to 1 implies 1 minus x square plus xy is greater than or equal to 1, if you cancel this implies x y is greater than or equal to x square, since x not assume to be at a 0 or 1. What we get is x is less than or equal to y, this is if and only if. So, x is less than or equal to y. So, clearly this satisfies OP.

Any QL implication satisfies NP. So, it satisfies both OP and NP. Of course, from the family of F-implications we do not get any member who satisfies OP and in the family of G-implications we know that the only member that satisfies OP is the Goguen implication.

Of course these are from the basic families of fuzzy implications, but they do exist lot more other families of fuzzy implications, R-U implications, S-N type of implications obtained from uni norms and so on and there again you will be able to get some members which do satisfy both OP and A and NP. So, this class of implication satisfying OP and NP they are quite rich they not only contain R-implications, but also other implications outside of this class.

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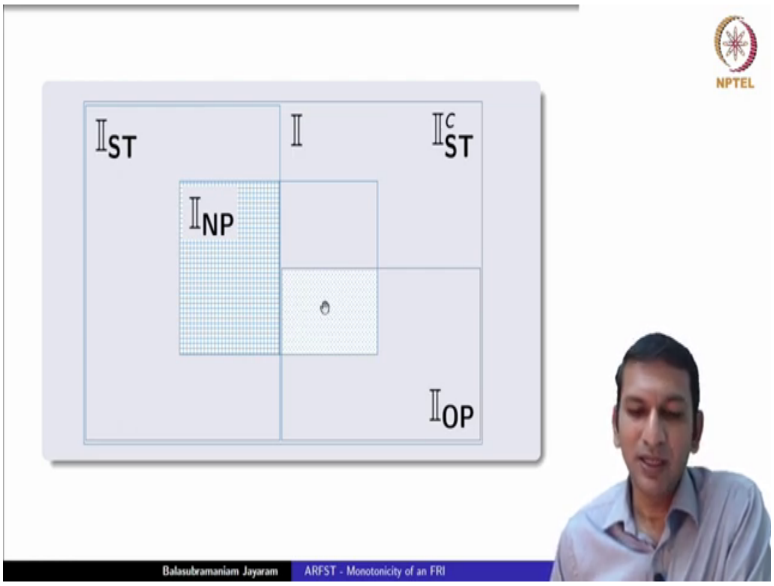

Well, what we have seen is we partition the set of all fuzzy implications into these nice classes. We knew that we had to concentrate only on the set of NP implications and in this case we have considered only that part, where it was intersecting. We consider the intersection between OP and NP. A similar result can be proven for the case, where it is both

NP and strict. Note that we need NP so that we can remove the composition and concentrate only on the R.

What we have seen in this lecture so far is the proof of the such a result when the implication comes from the class of fuzzy implications, which satisfy both OP and NP. That means the intersection of I_OP and I_NP, but similar result also has been proven for the case where the implication comes from the intersection of I_NP and I_ST.

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Well, that is the work related to showing that implications coming from the families of implications which are both strict and satisfy NP, that was a work shown by Sayantan and Jayaram in the paper in 2016. And what we have seen in this lecture about implications coming from a class of implications satisfying both OP and NP, how the system function when you use them is monotonic, that result actually comes from a very recent work of Sayantan Mandal which was published in 2020.

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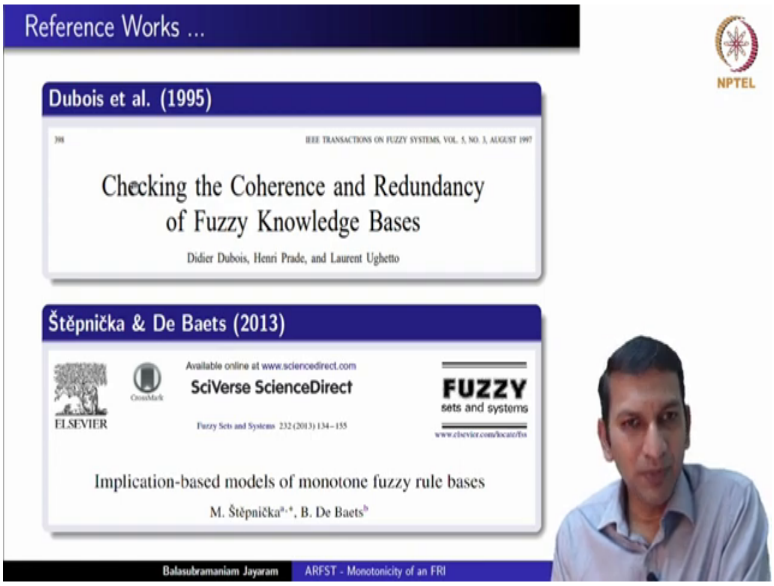

It should be pointed out that, it was very important for us in the proof that we did have B dash to be normal, that mean the kernel was not empty. This relates to a fact of what is called the coherence of fuzzy rules. That means, when you consider a system of fuzzy rules and the corresponding relation we want that for any given x dash there exists some y such that R of x naught y is 1, this is called the coherence. And this coherence and redundancy has been quite well discussed in a paper by Dubois and others.

Also as was mentioned in the last lecture when we consider implicative models if to discuss the monotonicity the rule bases were transformed into what are called at most or at least fuzzy rules. That means, the antecedents and the corresponding consequence themselves were transformed and it was that set of rule base, that was considered to discuss the monotonicity.

However, you would note that in the discussion that we have had during this lecture, we have not had to resort to doing any such transformations on the antecedents or the consequence.

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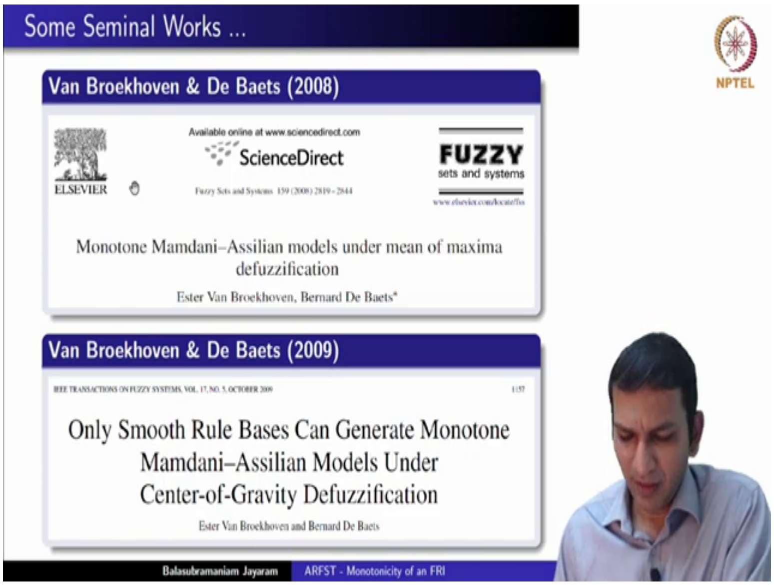

We have predominantly discussed only implication based rules, but of course, there also exists works which deal with the R check relation where we use a t-norm to relate antecedent to consequent. So, this work is contained in the first or the seminal works done by Broekhoven and Bernard De Baets in these two papers in late 2010. With this we will wind up the discussion of monotonicity of an FRI.

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In the next lecture, we will discuss the monotonicity of similarity based reasoning schemes. Glad you could join us for this lecture. Hope to see you soon in the next lecture.

Thank you again.