Approximate Reasoning using Fuzzy Set Theory
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Lecture - 52
Monotonicity of an FIS

Hello and welcome to the first of the lectures in this week 11 of the course titled Approximate Reasoning using Fuzzy Set Theory; the course offered over the NPTEL platform.

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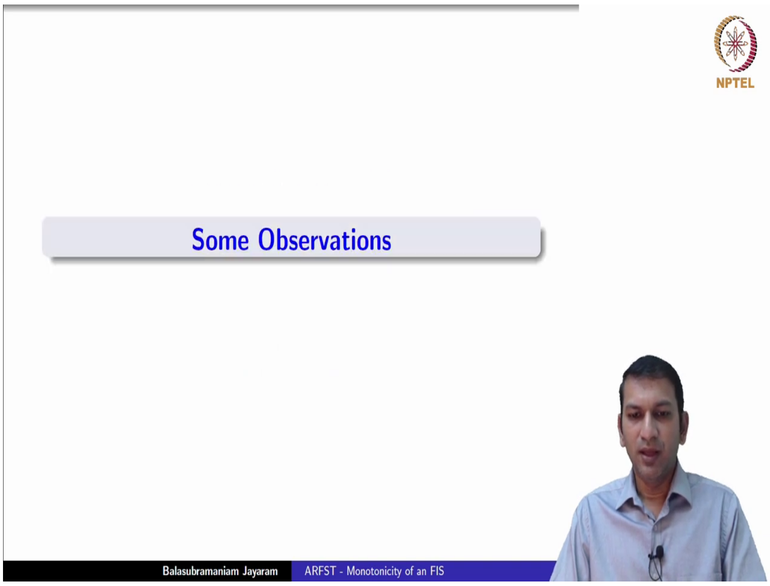

Let us begin this lecture by making a few observations on some of the fuzzy inference systems that we ourselves created using the MATLAB fuzzy logic toolbox a few weeks ago.

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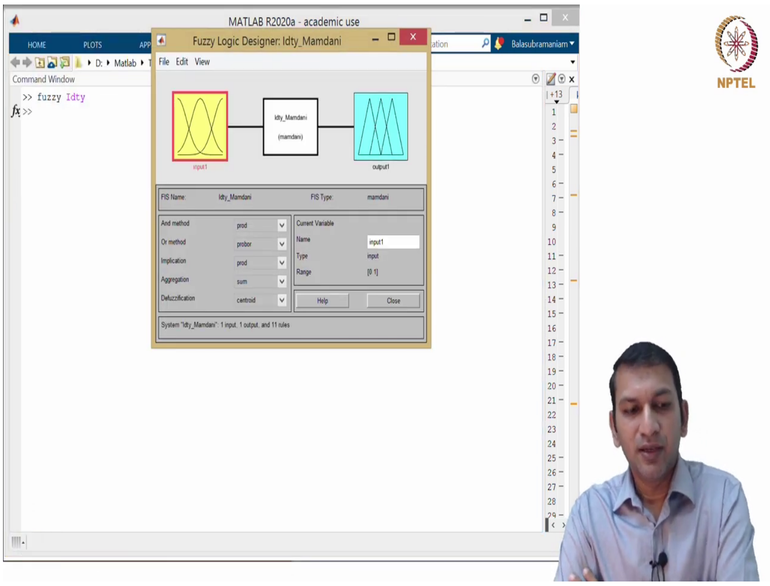

Let us recall we implemented a Mamdani fuzzy system to approximate the identity function from the unit interval to the unit interval. So, this is the fuzzy system that we see.

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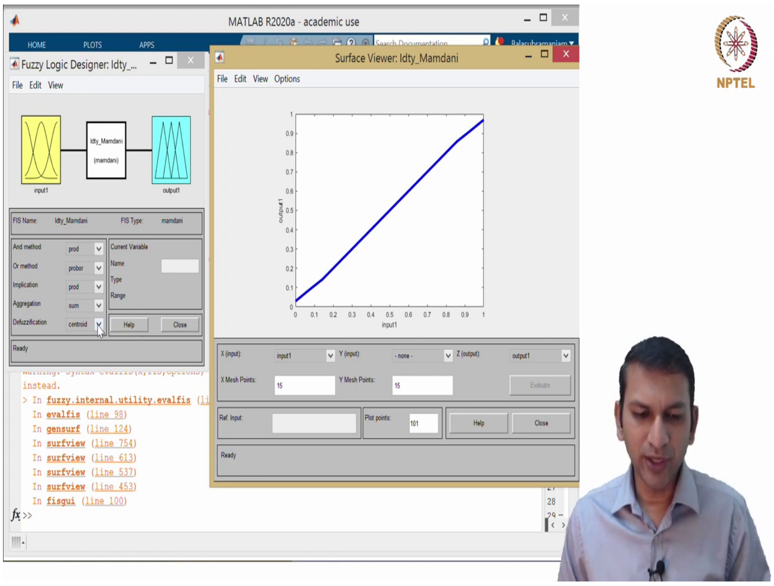

If you look at the surface, this is how it would look like. It is a function from 0, 1 to 0, 1. Now, when we look at the output of this fuzzy system which is one minute; a function from 0, 1 to 0, 1, what we require or what we wanted to approximate was the identity function and as you can see from the graph of this function, it is quite close to being the identity function.

But there are also a few other properties that we can actually see here. Now, this is done with taking the Jades matching function with product for the t norm and the Lukasiewicz t co norm is the instead of the max and we have taken product for the implication and some for the aggregation and centroid for the defuzzification.

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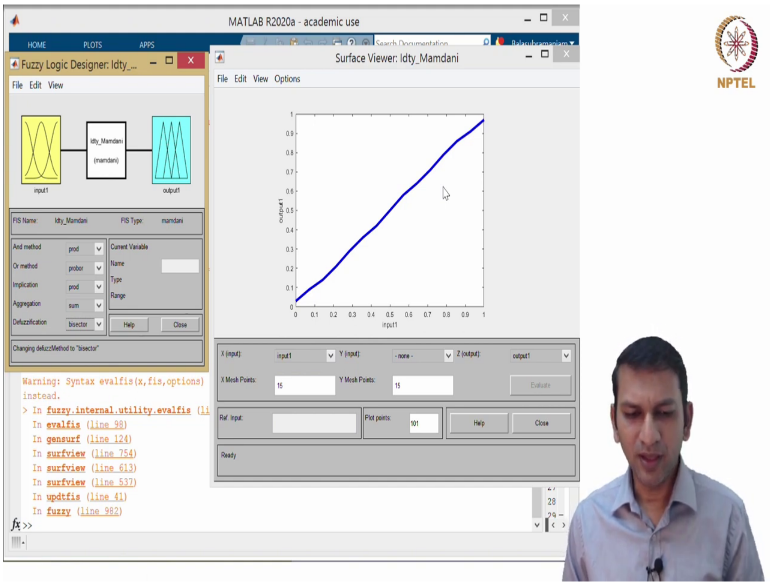

Let us try to change the defuzzification method from centroid to the bisector method. You will see immediately, the graph of the function has changed slightly. However, more or less it still a good approximation to the identity function.

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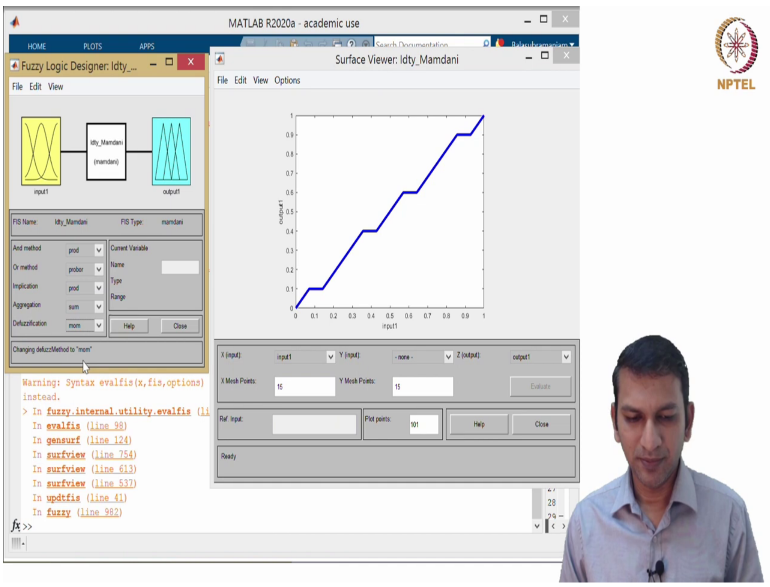

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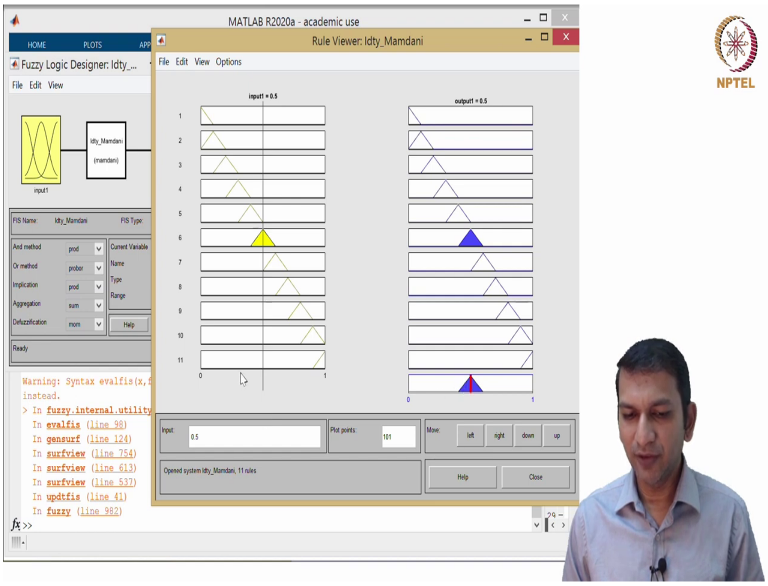

Let us try to change this defuzzification function to the mean of maximum function. That means, essentially when you look at it, when you look at the rules, you see here for the given input here, this is the kind of output you have got. Look at in this case; rules.

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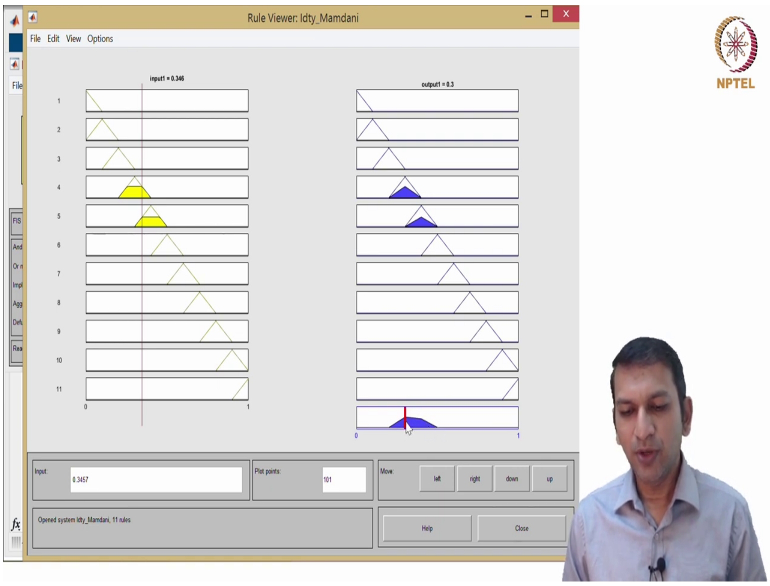

So, you see here for this input, it has excited these two rules. We have aggregated them and what we have here is this combined output fuzzy set; this is the aggregated, modified consequent fuzzy sets and you see here because we are using now mean of maxima function here, it is this is the ceiling of this function. This is what we get. We see that with this function, this is the output of the corresponding fuzzy inference system.

Now, it is clear that when we had the centroid fuzzy centroid defuzzification, it was more or less a smooth curve. So, it appears to be not only continuous; but also differentiable. However, when we changed it to the MOM defuzzifier, then what we see is it is still continuous; but not differentiable. Not only that, it has also lost more or less its approximation properties.

So, on these parts, where it is a constant function, clearly we know that it is not approximating it as well as we obtained with either the bisector method or the center of gravity defuzzifier. Now, however, it is still continuous among the three defuzzifiers that we have used, while the continuity and differentiability may have varied; the approximation capabilities may have varied; one property has remained constant and that is the monotonicity of the function.

It has always been increasing if in the sense of non-decreasingness in the not in the strict sense. So, you see here all of these functions are increasing and if they are not increasing, at least they are not falling behind. Now, let us look at a different defuzzification. We know that we have many options here to choose from the parameters.

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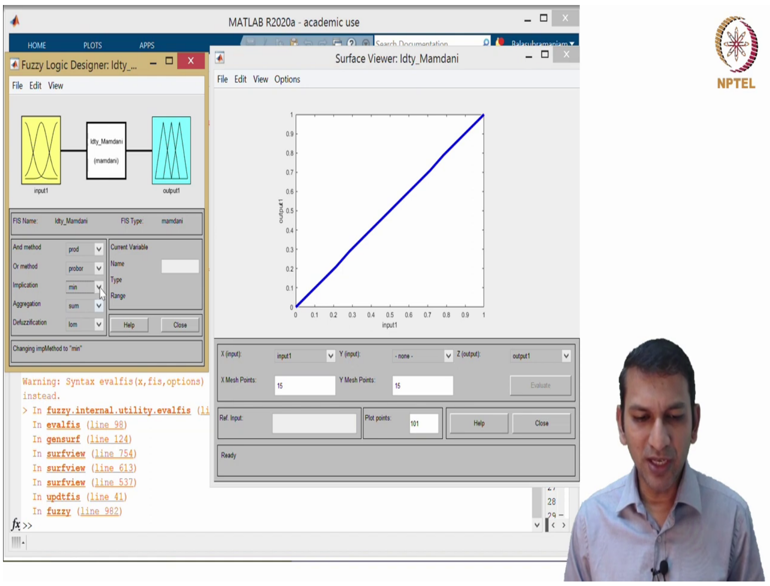

Instead of taking the product for implication, the modification function, let us take the min. Now, what we see is for the same defuzzifier LOM, least of maxima, what we find is we in fact get an exact match. This is exactly the identity function. So, this is differentiable continuous monotonic. It is in fact exact. So, there is no approximation error crept.

So, just by changing one particular function, we are able to move from having only a monotonic, continuous and perhaps not so good approximation to almost a perfect approximation of the function that we have in mind. Having said that, all of these are still keeping the monotonicity property.

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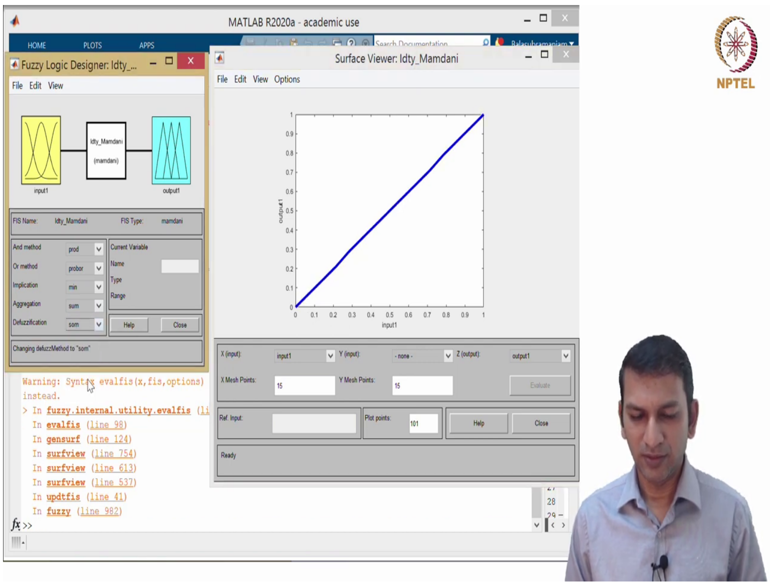

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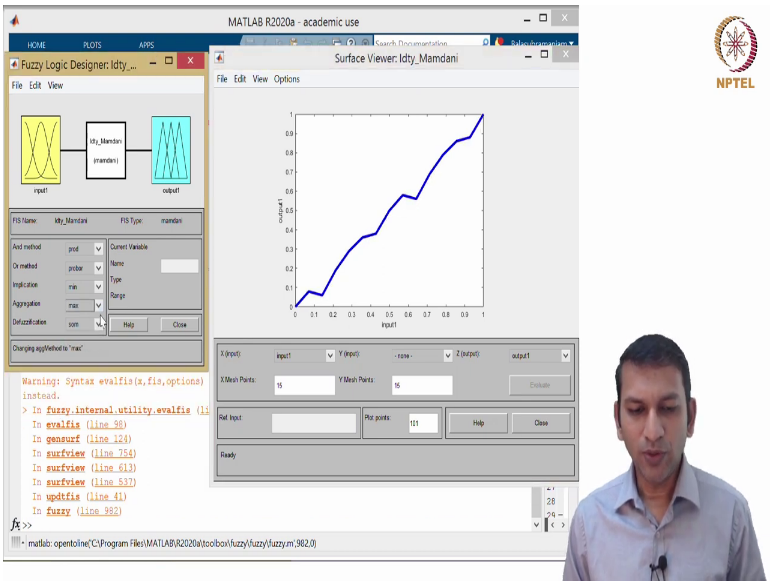

Let us change this from LOM to say bisector. Let us change the aggregation from Lukasiewicz t norm to the max t norm. Now, what do you see here is what was almost a perfect fit earlier in the case of LOM.

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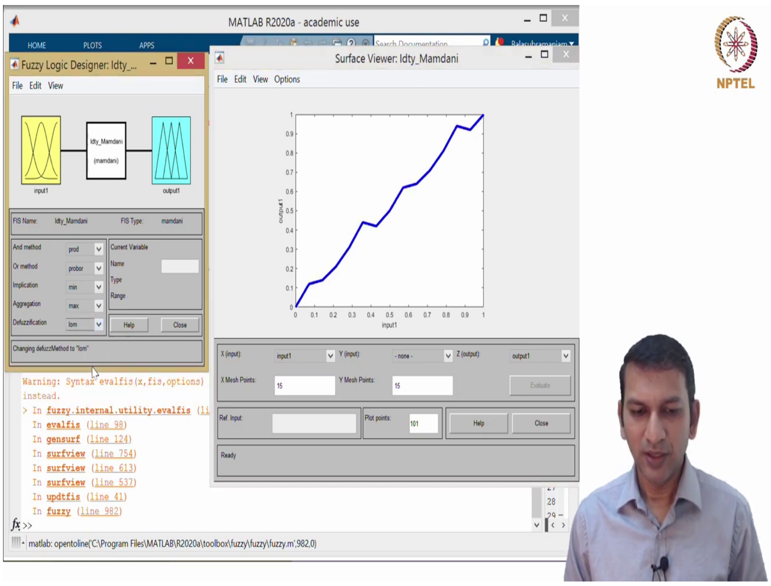

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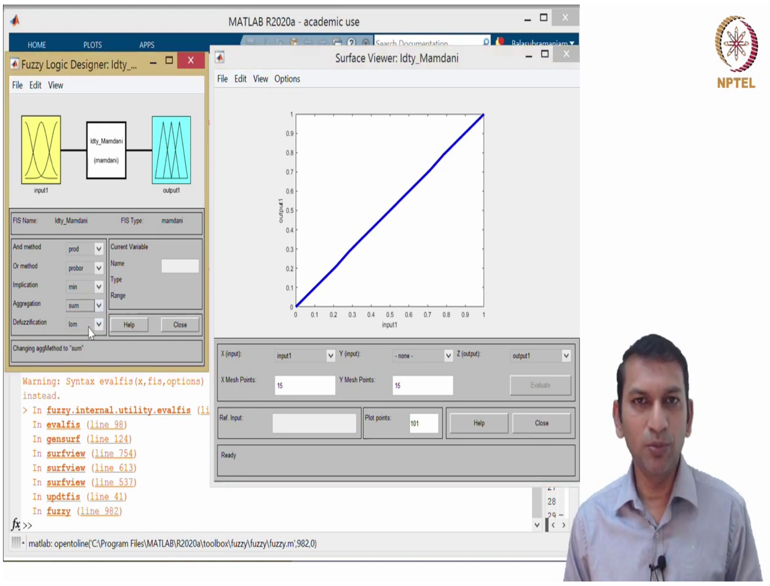

So, note that if we have the sum, it is almost a perfect fit. However, by changing this to max, what we see is in fact not only has the approximation error crept in. But we have lost continuity to you can see these sharp edges. So, in that sense, it is also not differentiable; not only that we are losing monotonously also. For instance, you look at this point, at 0.35, it is higher and as it goes towards 0.4, it is in fact dropping.

So, we seem to have lost all kinds of properties that we had earlier by just choosing one function or one operation in the inference scheme differently. Now, the question now is how do we preserve monotonicity? So, we discussed about interpolativity, continuity, robustness. In this week of lectures, we will discuss how to preserve monotonicity of the system function because often, we will have to approximate functions, systems which have some kind of a monotonic behavior.

For instance, if we look at the rules here, you see that as we go to the right on the input space, we are also going to the right on the output space. That means the antecedents and consequents, they are in some kind of a relationship. They seem to be honoring some kind of an ordering. So, this gives us a motivation to study, when a fuzzy inference system will be monotony and is this monotonicity property really important.

As we have seen let us assume that this is not an identity function, but we are actually capturing the monotonic behavior of a system function. For the example, there is an often quoted example in the literature imagine, we are using a fuzzy inference system to control the opening and closing of the gates of a dam. So, clearly, all it says is after a certain level, as the water rushes in maybe during floods, during rain, what we want to do is to preserve the structural stability of the dam.

We want to start letting out water which means we have to start opening the gates. And imagine the output is how much of water to let out. So, that means, accordingly, the valve should open; the gate should open. Now, we know that there is a monotonic behavior, after a certain level if the water starts to gush in, more water in, more water out. So, there is a clear monotonic relationship.

However, imagine if we did not if the fuzzy system that we have implemented is not monotonic, may not capture the exact behavior that we want. But if it is not monotonic; after some time, it starts to in fact close the gates when it is supposed to open it even more. Let us assume that on the x axis, we have the rate at which the water is coming in.

So, this is the rate at which the water is coming in, you suddenly see that when the water comes in the and the inflow is much higher, suddenly the gate might start to close, if this is the kind of a system that we have implemented to control the opening and closing of the gates. So, in practical applications, monotonicity does make a very play a very important role and thus, make the system usable and trustworthy.

Now, having said this, we have seen in this case at least an example, wherein we try to implement or simulate or approximate a known function and we knew that this function is monotonic. Now, however, when we are actually implementing a system or fuzzy inference system to approximate a function, we might only have an idea that it is monotonic. But how do we say our fuzzy influence system is monotonic, what captures this monotonicity?

Clearly, in this case we have seen, it is the rule base which has captured this monotonicity. You see from here there seems to be some kind of an ordering on the input fuzzy sets and output fuzzy sets in terms on the antecedents and consequents that we have considered in the rule base and this is somehow capturing the monotonicity of the entire system function itself.

This is what we want to study and from this example, we will see how to capture this concept of monotonicity in a more abstract sense so that we could discuss which fuzzy inference systems are monotone towards this end.

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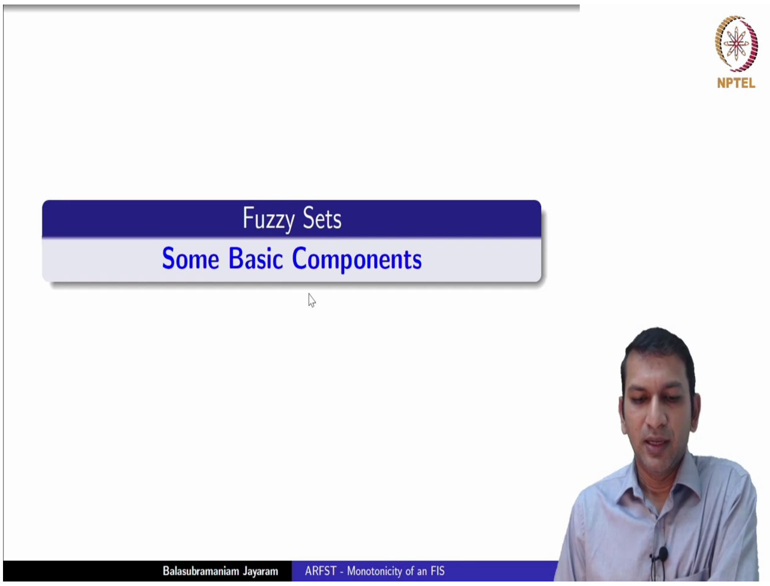

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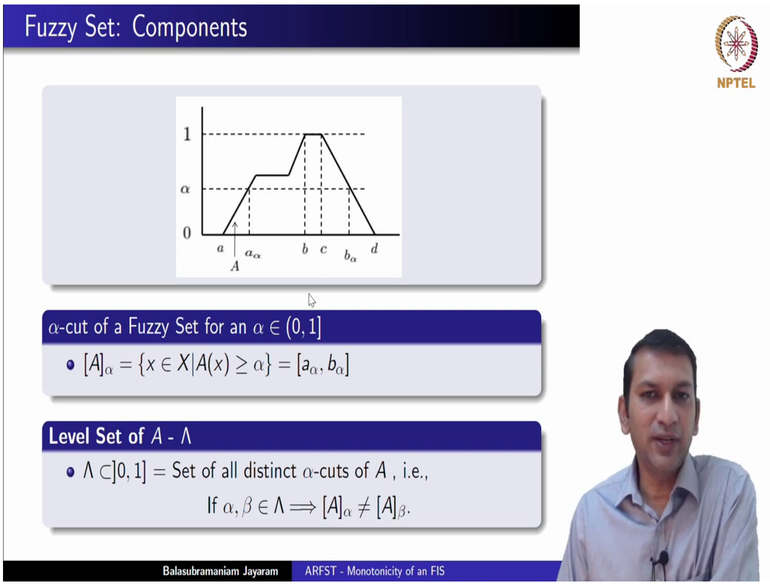

Let us revisit some of the concepts that we have seen earlier in the course. Given a fuzzy set we know what an alpha cut is. For any alpha in 0, 1, the alpha cut is all those elements on the domain whose membership degree is greater than or equal to alpha.

So, in this case for the figure that you see on the screen, the alpha cut for particular alpha is essentially a alpha b alpha, the interval and we know that the levels it consists of all distinct alpha cuts and this is also characterized by the fact that if alpha and beta come from lambda, then the corresponding alpha cuts are in fact different.

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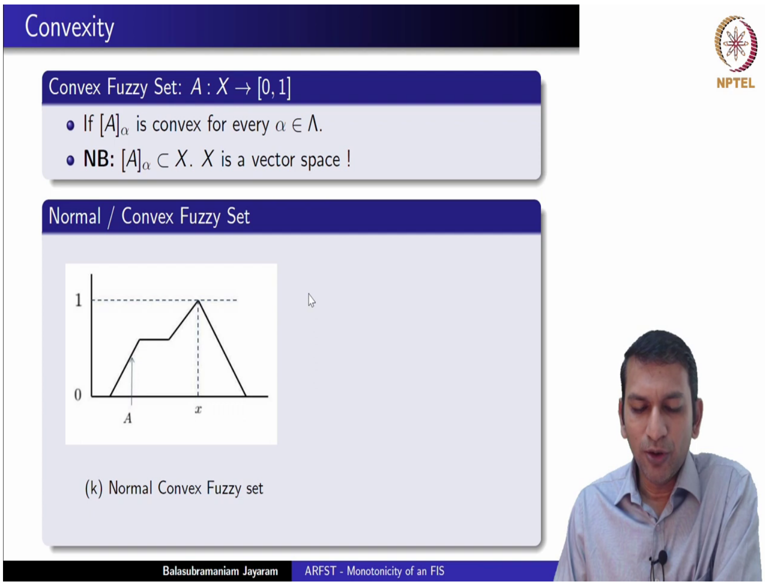

We have seen what a convex fuzzy set is; a fuzzy set is said to be convex if every alpha cut is convex. And note that an alpha cut is a subset of X which means we need some kind of convexity on X and typically X is taken to be a linear space or the vector space.

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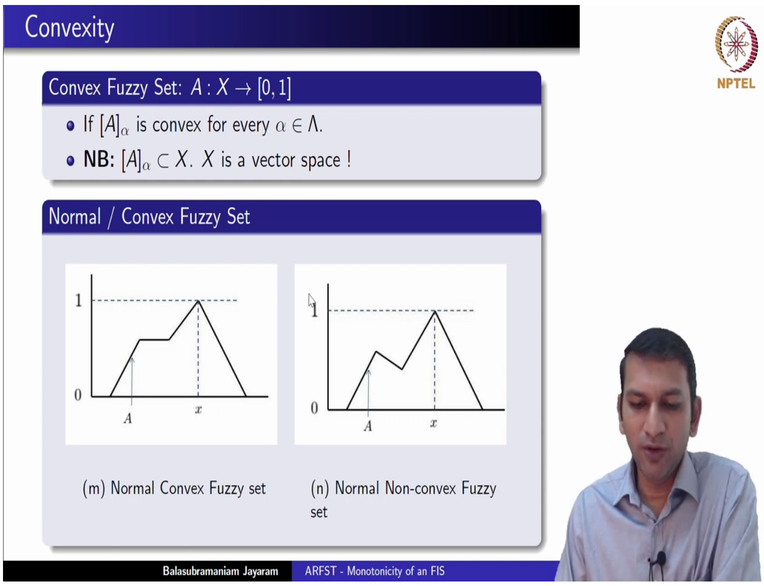

So, this is an example of a convex fuzzy set which is also normal; but convex fuzzy sets need not be normal, normal fuzzy sets always be convex. This is an example of a normal fuzzy set which is a non-convex fuzzy set.

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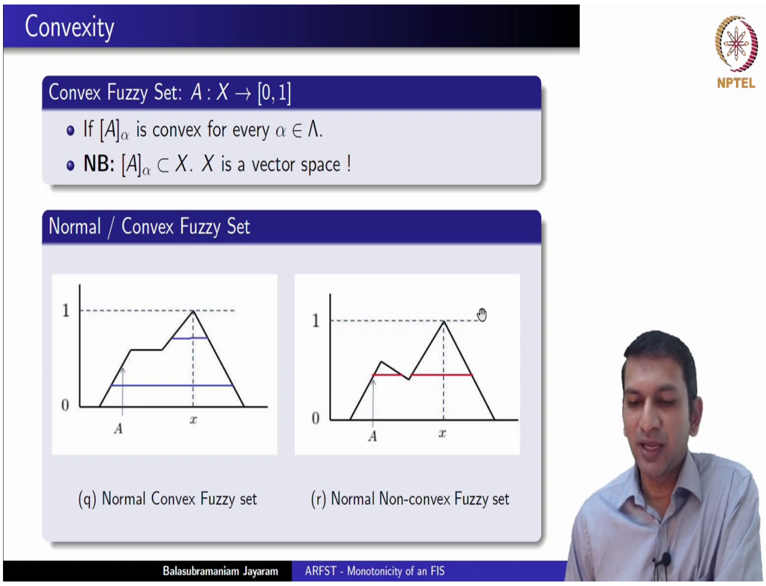

And why do we say this is convex? Because if we look at it each alpha cut is an interval. So, on if you are considering the domain to be r, x to be subset of r, then we know that the alpha cuts for them to be convex, they have to be intervals. You see here every alpha cut in this case is interval; whereas, in this case we see that at least there exist one alpha whose alpha cut as a subset of x, subset of r is not convex.

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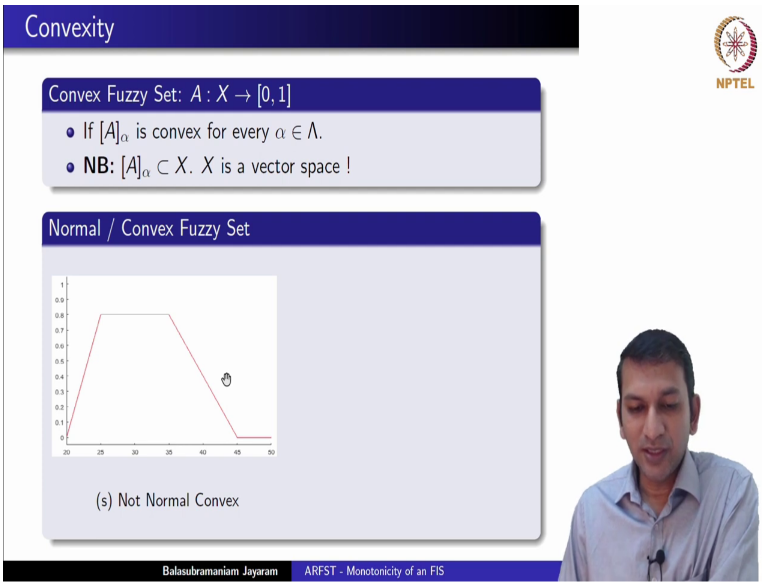

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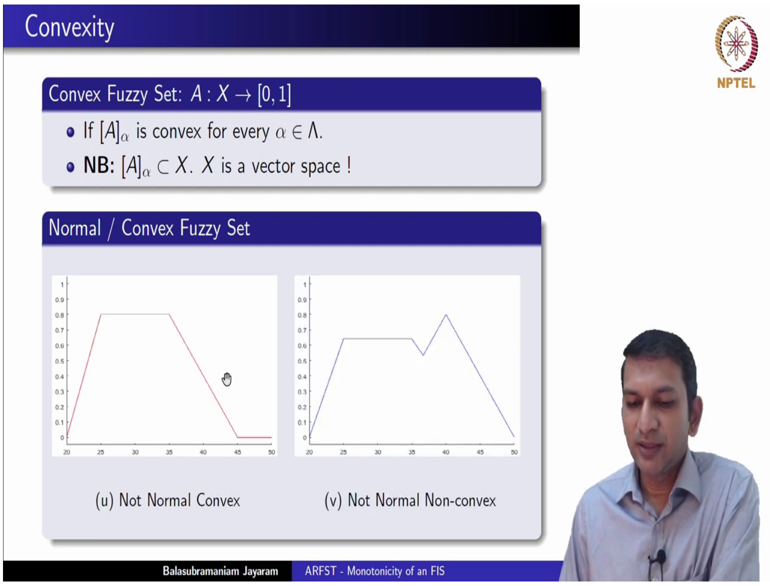

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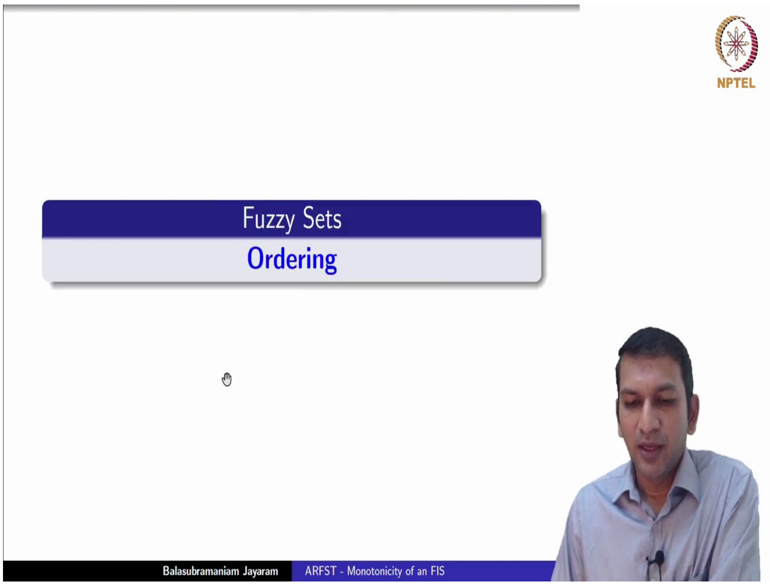

This is not normal; but convex and this is neither normal, nor convex. Now, looked at we looked at the rule base that we just had for the identity function and we were wondering what kind of relationship exists between the antecedents and the consequents. We have already defined a couple of types of ordering. Let us revisit them.

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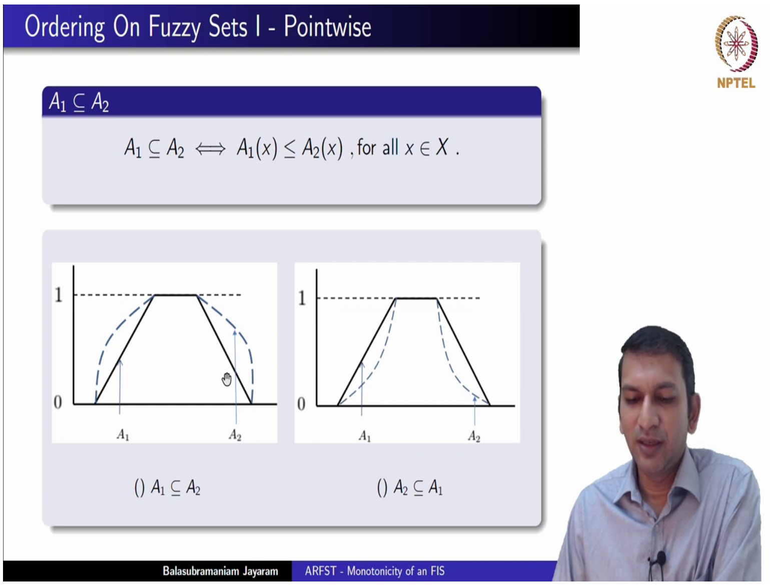

The first type of ordering that we saw was the point wise ordering. Looking at fuzzy sets as functions from x to 0, 1, this is the first or the natural ordering that one would think of. So, we say that A 1 is contained in A 2 if and only if for every X A 1 of x is less than or equal to A 2 of x. It is essentially the point wise ordering. So, we have seen the examples if you take these two fuzzy sets, clearly A 1 is contained in A 2; here the other way around A 2 is contained in A 1.

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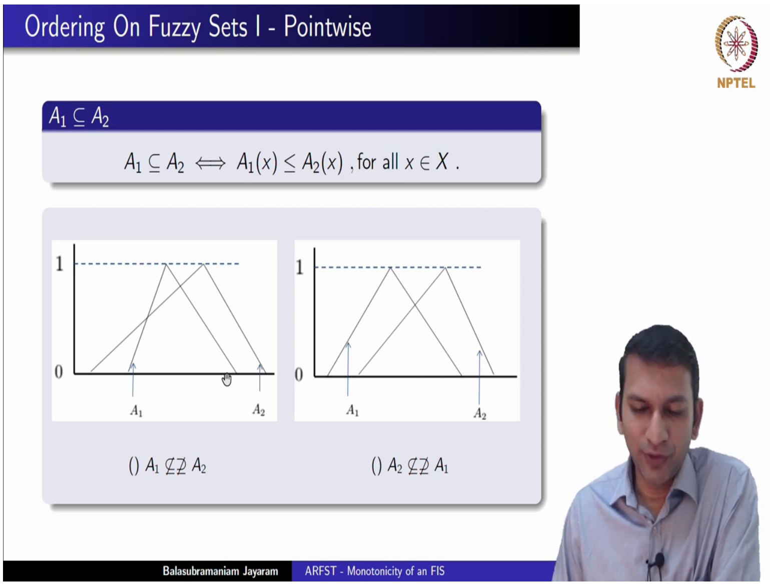

However, if you recall in the fuzzy system that we have constructed this is not the kind of relationship that exists among the antecedents. For instance, it appears more like this; they are like shifted copies, shifted probably overlapping shifted copies of and of the of each of them. The antecedents are they look like shifted and overlapping copies of each other. Now, we see that if we want to consider such fuzzy sets, this ordering is perhaps not really useful for us.

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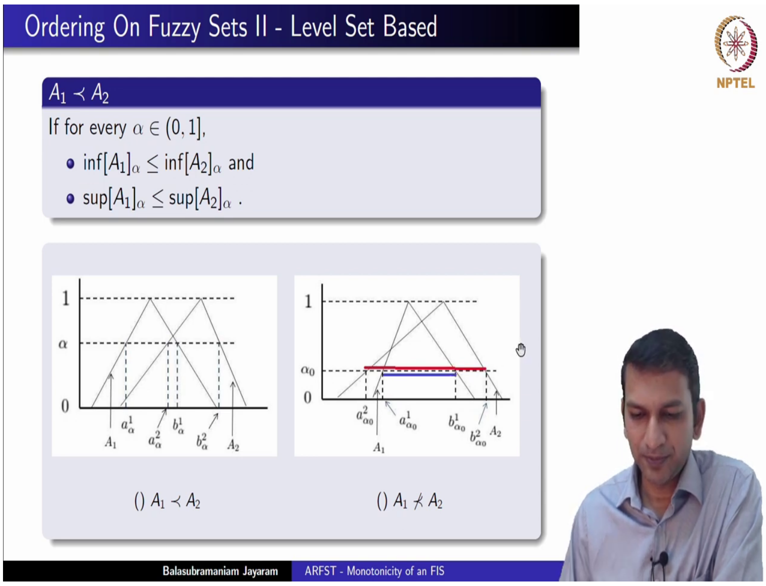

And this led us to introduce another type of ordering based on the alpha cuts which we call the level set based ordering. Here we say A 1 is contained in A 2 or smaller than A 2, less than A 2; we use this symbol. If this these two inequalities are valid. So, what does it say? For each alpha cut the infimum should be smaller than the corresponding infimum of the alpha cut of A 2. The infimums of the alpha cuts and the supremums of the alpha alpha cut they should be comparable and one should be smaller than the other.

This is essentially the ordering put in words. In that sense, we see here these two fuzzy sets are in fact orderable with respect to levels and based ordering; while these two fuzzy sets are not orderable and we get an idea that perhaps this is the ordering that is required for us to talk about the relationship between antecedents and consequents, when you are looking into the monotonicity of a fuzzy inference system.

Now, why is this orderable with respect to this ordering? For any alpha, if you take for in the case of this particular alpha, we see the interval is essentially the red line and the other for A

2 are the same alpha, it is the blue line. So, essentially, the interval consisting of all those red points on x and blue points on x.

And clearly, the infimum of these two intervals is comparable; the supremum of these two intervals is comparable and we see that one is smaller than the other. However, for this the figure on the right side, we see that at this particular alpha naught, this is the alpha cut of A 1, this is the alpha cut of A 2 and we see that while infimum of A 1 alpha naught is smaller than A 2 alpha naught the infimum of it; the supremum of A 1 alpha naught is not less than or equal to supremum of A 2 alpha naught.

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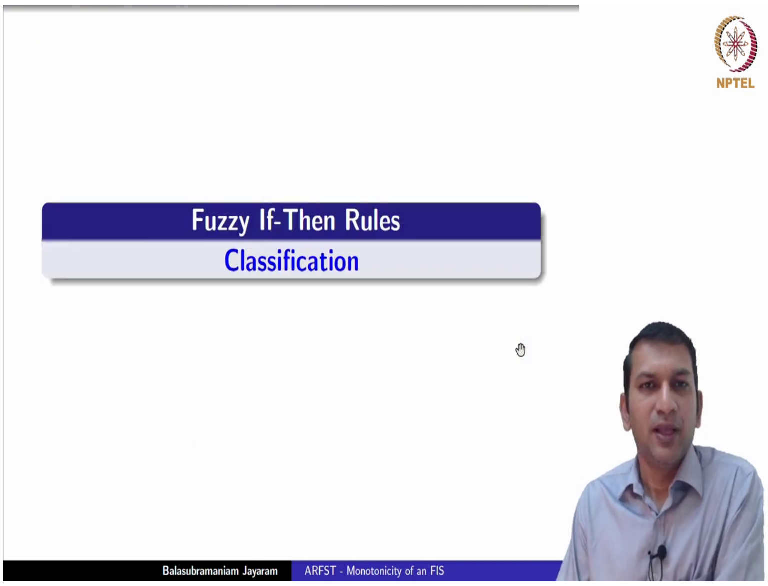

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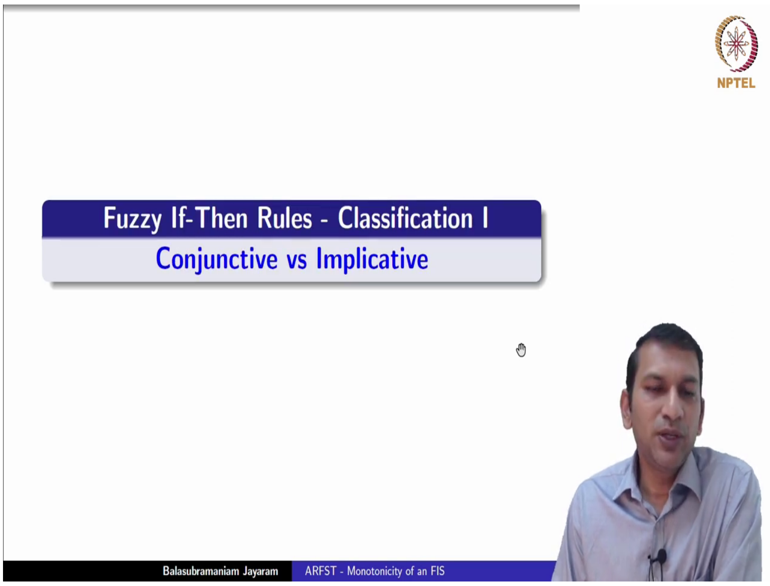

Well, we have seen many types of classification of fuzzy If-Then rules. The first one was that of conjunctive or implicative, where we looked at it has positive examples or as constraints.

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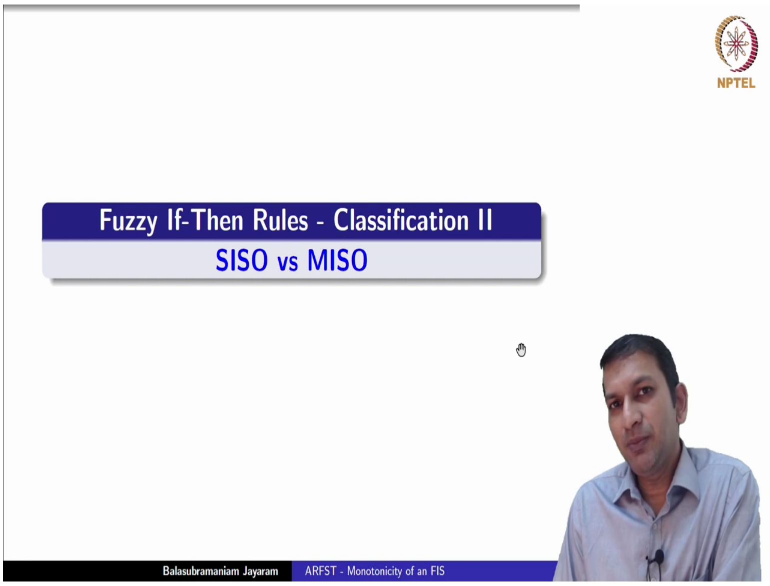

The second type of classification was that of whether they are single input or multiple input based on the you know that based on the arity of the inputs.

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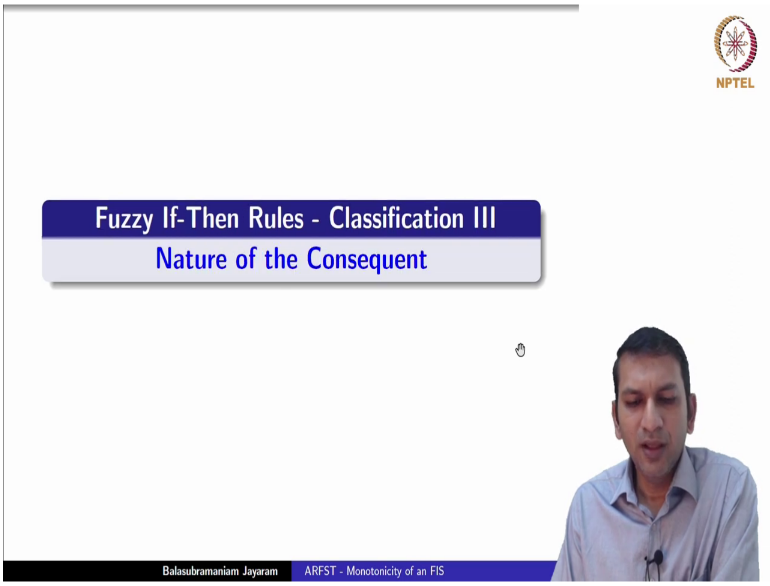

The third type, we saw was based on the nature of the consequent; whether the consequents were fuzzy sets or functions of x as in the case of tsk fuzzy systems.

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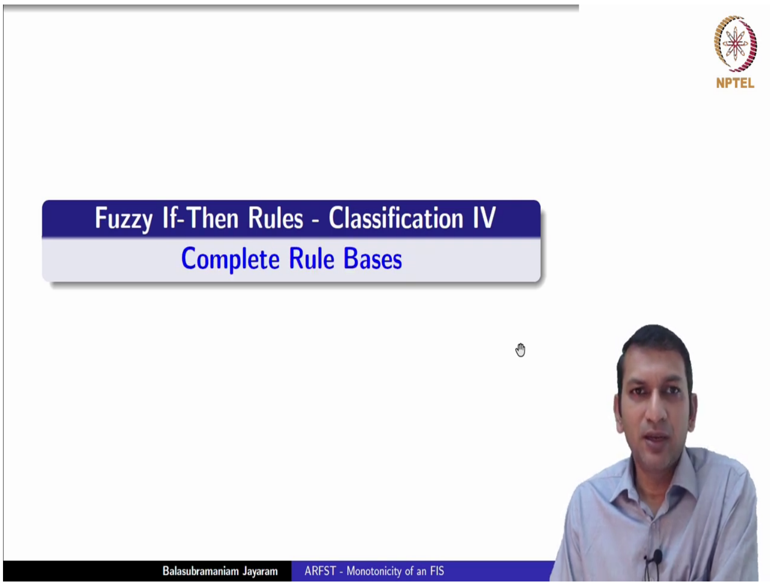

The fourth type that we have seen are those rule bases which are either complete or incomplete; complete means if you collect all the antecedents, they form a fuzzy covering of the input space. Let us look at yet another classification of fuzzy if than rule bases.

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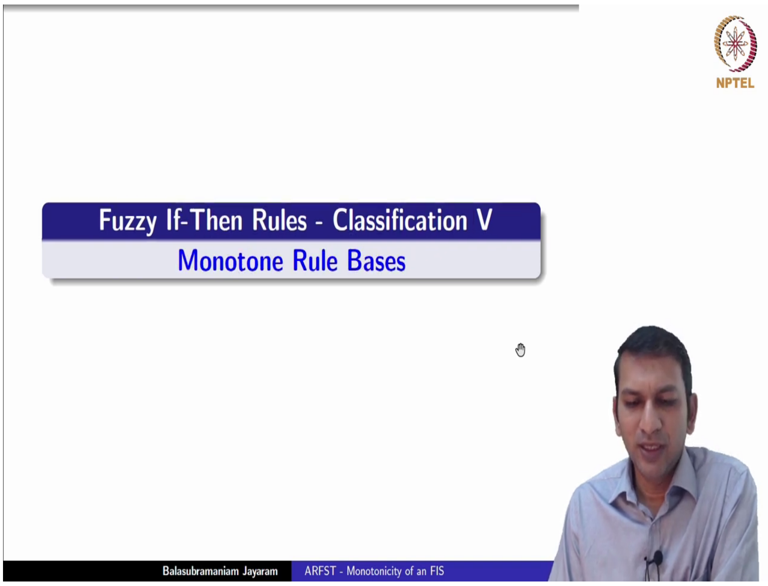

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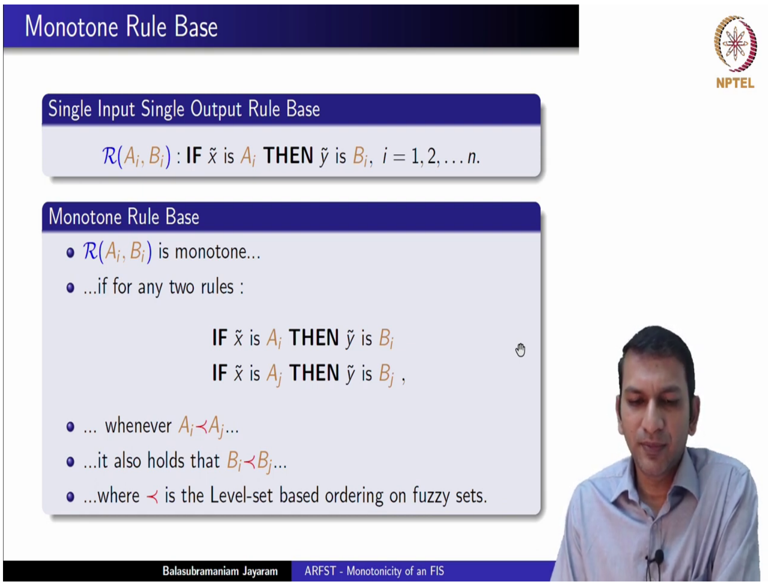

These are called monotone rule bases. Let us be given a set of SISO If-Then rules; Single Input Single Output rules we say; this rule base is monotone. If we pick any two rules let us say we pick these two rules, IF x is A i, THEN y is B i; IF x is A j, THEN y is B j; whenever A i is smaller than A j, then it should hold that B i is also smaller than B j and for this, we are using the level set based ordering on fuzzy sets.

So, essentially, there exists some kind of a monotone relationship between the antecedents and the consequents. This is when we say a given rule base is a monotone rule base.

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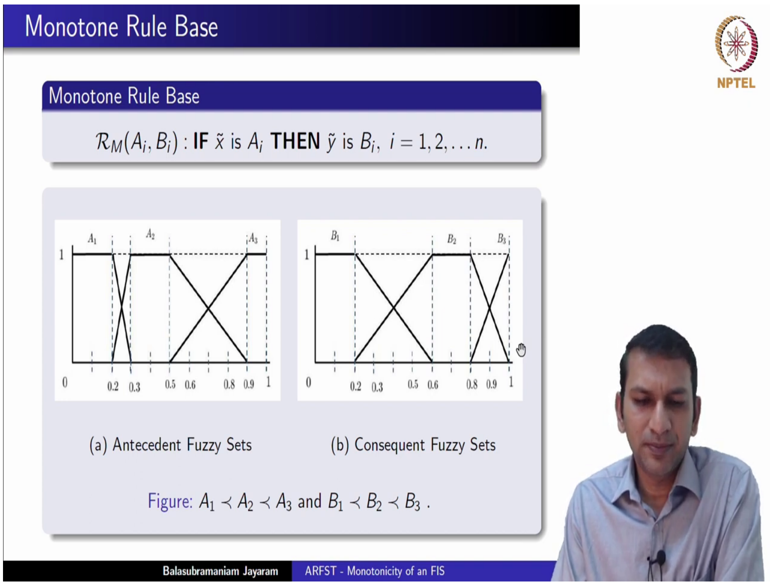

Let us look at a couple of examples. So, let us take these three fuzzy sets; A 1, A 2, A 3. Let us take them as the antecedents fuzzy sets and let us take this B 1, B 2, B 3 as the consequent fuzzy sets. Clearly, A 1 is smaller than A 2 smaller than A 3 with respect to the level set based ordering and similarly, B 1 is smaller than B 2 smaller than B 3 with respect to the level set based ordering here. So, they can very well serve as the antecedents and consequents of rule base, if we want to construct a monotone rule base.

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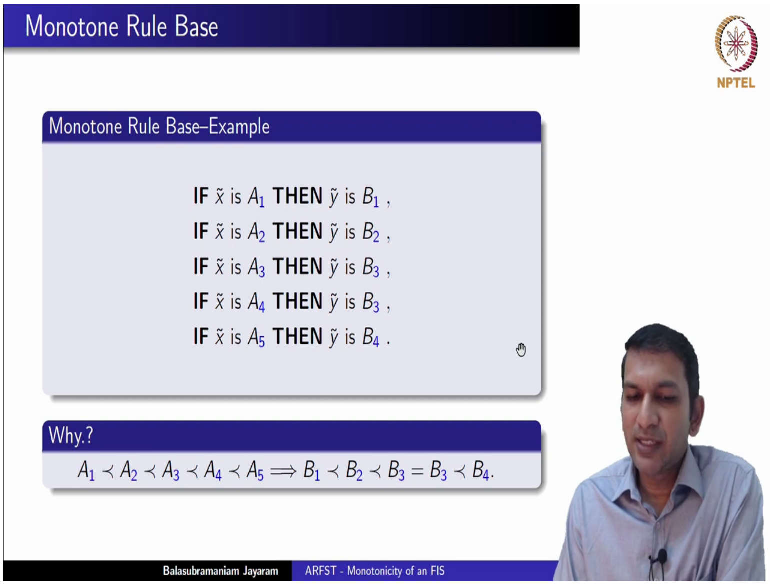

Now, consider a rule base given like this. It will be a monotone rule base, if as we have seen earlier, this is the ordering that holds. So, in this case, we have assumed that IF i is smaller than j, THEN A i is smaller than j, B i is smaller than j, B i is smaller than B j. Now, this will be an example of monotone rule base.

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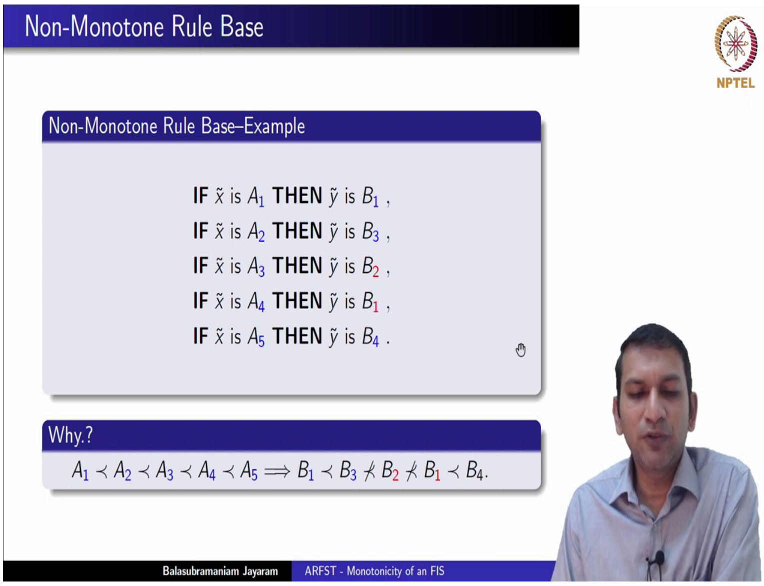

However, given that ordering on the input and output fuzzy sets, the partition the coverings p x and p y, if we have such a rule base, we see that while B 1 is smaller than B 3, B 3 is not smaller than B 2. So, clearly A 2 is smaller than A 3, but the corresponding consequence are not of the same order. They are not relatable with the with respect to the same ordering or in that order, we see that it actually the order reverses. So, clearly, this is an example of a non-monotone rule base.

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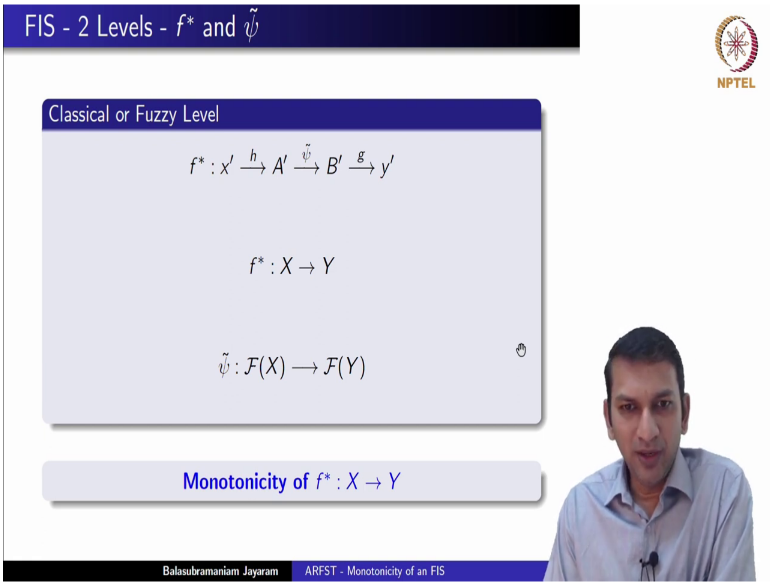

Now, when we want to talk about monotonicity, at what level of the system function should we talk about monotonicity? Note that we could talk about a fuzzy inference system at two levels; either we are given an input from the domain which x dash which we use a fuzzifier to fuzzify give it to the fuzzy inference system which we looked at as a mapping from f of x to f of y obtain an output B dash and further, we apply a de fuzzifier and obtain a y dash.

So, we could look at fuzzy system as a mapping from x to y or we could also just look at fuzzy system as a mapping from f of x to f of y. We have seen in the case of monotonicity, the de fuzzifier also plays a role and hence, typically monotonicity of fuzzy inference system is discussed at the level of looking at the fuzzy inference system the corresponding system function as a mapping from x to y.

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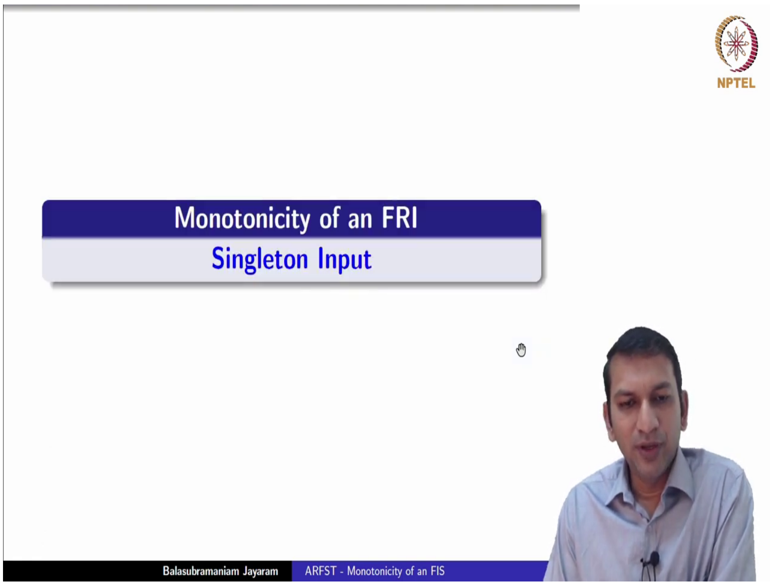

Now, we have seen one example of a mapping which was an example of a similarity based reasoning because we have used the fuzzy logic toolbox in MATLAB to come up with approximating the identity function. Now, we have seen that there by using different defuzzification methods or functions, we could break the monotonicity of the output function, but what happens in an FRI? Is it still true, when we apply a de fuzzifier? Let us look at a particular example.

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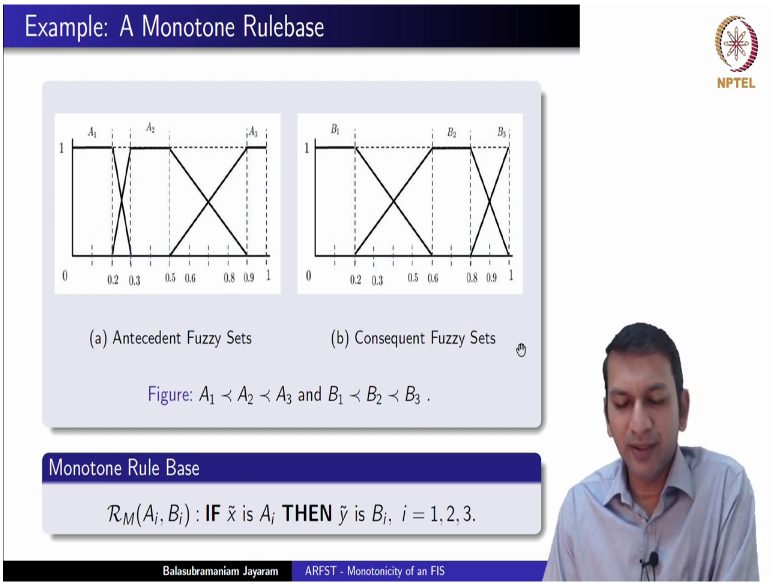

Now, let us consider for the monotone rule base exactly the antecedents and consequents that we saw a few slides back; clearly these three fuzzy sets are orderable with respect to the level set based ordering and let us construct a rule base consisting of three rules such that A i is related to B i.

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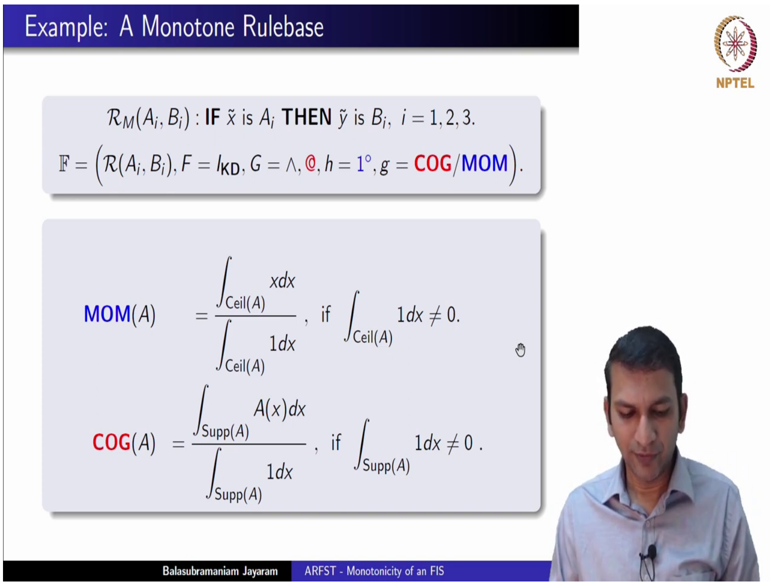

Now, we know to choose an FRI means we need to choose the components in the FRI. We have already chosen the rule base as we have seen in the previous slide. For relating the antecedents to the consequents of each of the rules and obtaining the relation, let us for the moment use the kleene dienes implication and of course that means, we are looking at implicative type of rules which means we need to use a conjunctive operation to aggregate it.

So, let us use the minimum for aggregation for the moment because we are going to consider singleton input and singleton fuzzifier, we will see that in the next lecture that the composition does not really come into play. So, for the moment, it does not matter what composition we take.

So, that is why this composition operator is indicated in red and for the fuzzifier, let us take the singleton fuzzifier; that means, at a particular given an x naught. So, we are looking at getting an input from x because monotonicity as it was mentioned is discussed when we look at a fuzzy inference system as a function from x to y.

So, that means, what we are given as an x. We need to suitably fuzzify it to be given to an FRI and we are going to use a singleton fuzzifier for that; that means, whatever x naught is given, the input fuzzy set a dash takes 1 at x naught and 0 everywhere else. And for g because we need to defuzzify it to get a value y which belongs to the output domain y.

For the de fuzzifier, let us consider two different defuzzification mechanisms that of the center of gravity method and the mean of maximum. So, how are they given? This is how the corresponding formulae are given. Let us not worry about this formulae now. Because we have a visual idea of what happens as we have seen in the previous case.

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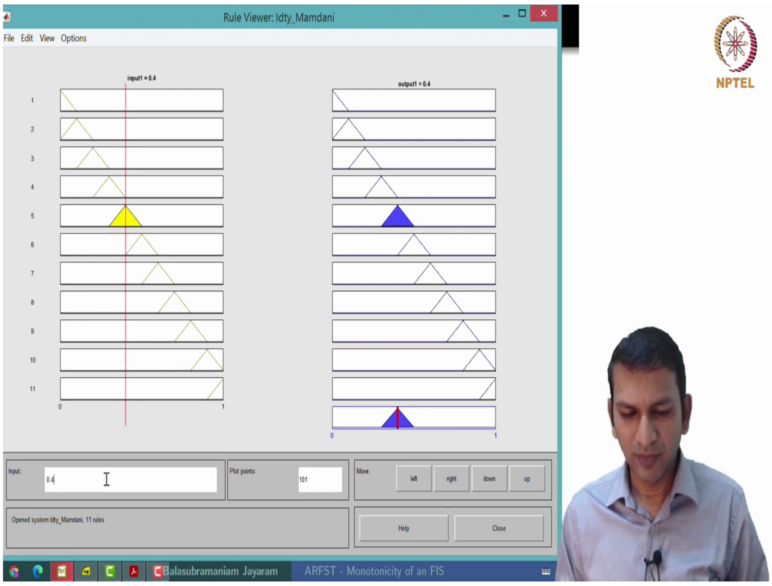

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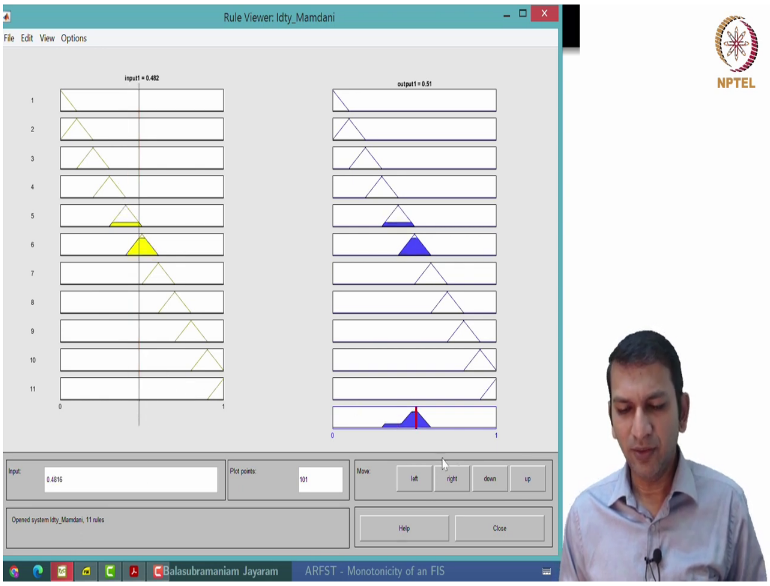

So, if you look at it; so, now, in this case when we see the output is yeah. So, based on that, we see here now that the maxima are here. So, the mean of maxima is somewhere in the middle. So, now, this is essentially the mean of maximum method that we are using here. So, it considers only the maxima and then, puts it at the average of them; so, in the middle of them. So, center of gravity we understand, it takes the entire area and finds the center of gravity and then, maps it down to the element at which that value is taken.

Well, if we consider this FRI system, where the rules are obtained from the kleene dienes implication, there are three rules. So, we use an aggregation which is the min here; singleton fuzzifier and either of these defuzzification mechanisms and the functions.

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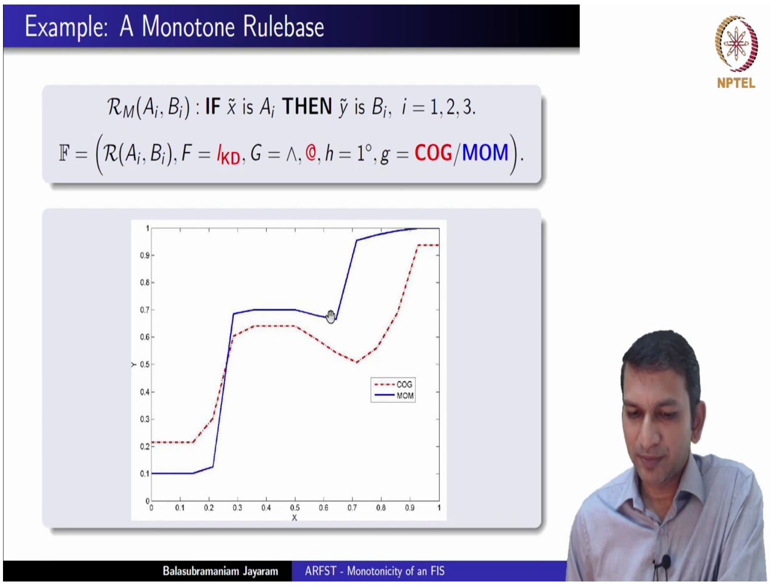

This is the output that we get between 0, 1 to 0, 1. So, as a mapping, we see that this function of course, we are not trying here to approximate the identity function because we do not know what is the function that we are trying to approximate, that is captured by the rules that we have come up with. So, the ground truth is given by the rules. So, we have this a A 1, A 2, A3 being related to B 1, B 2, B 3. We have considered them to be in ordering with respect to the levels at based ordering.

And we have related them to the corresponding consequents which are also following the level set based ordering. So, we while we actually do not know what is the function that we may be approximating, but we do know the whatever function that we approximate, the system function the output function should be monotone because there is a monotone relationship between the antecedents and the consequents in the given rule base.

So, when the ground truth says there is some kind of monotonicity in the system function that we that we are trying to capture. Then, we expect that our fuzzy inference system. In this case we are using an FRI should also capture that and the defuzzified output the corresponding mapping looked at as a mapping from x to y should also be monotone.

However, if you see here, if you use the center of gravity defuzzification method, we see that this function is not monotone. At 0.5, it is at 0.6 something; it is above 0.6 and at 0.7, it is almost well below 0.5, it is quite close to 0.5. So, you see that this function, output function is not monotonic. What happens if you use the mean of maxima? We see that when you use the mean of maxima again, you see that the curve dips here which means it is not monotonic.

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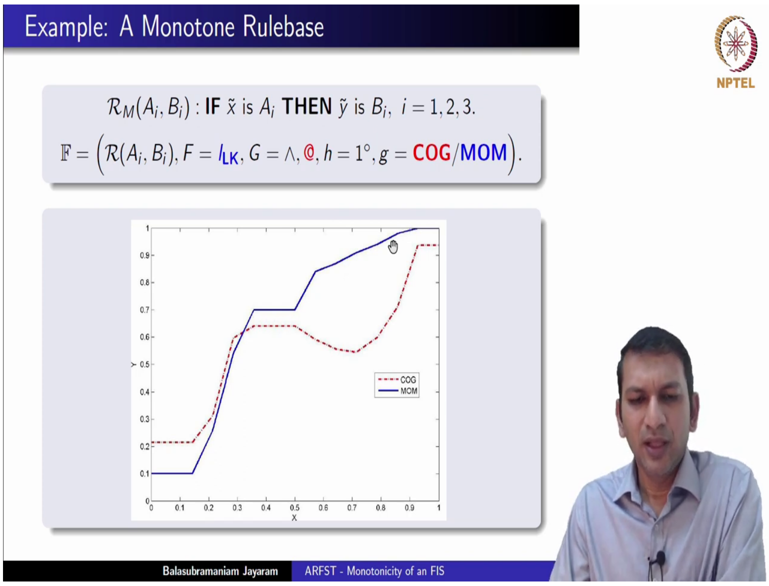

However, if we change the implication to the Lukasiewicz implication and retain everything else as the same, this is what we find. Once again, the center of gravity method, if we use that as a defuzzifier, we see that it actually dips. So, the output function is not monotonic. However, if you use the mean of maximum method, the monotonicity of the system is retained is preserved.

So, clearly, if you consider the first four components fixed and if you are changing only the defuzzifier, clearly there is a role played by the defuzzifier in ensuring the monotonicity of the system. This is something that we will investigate going forward.

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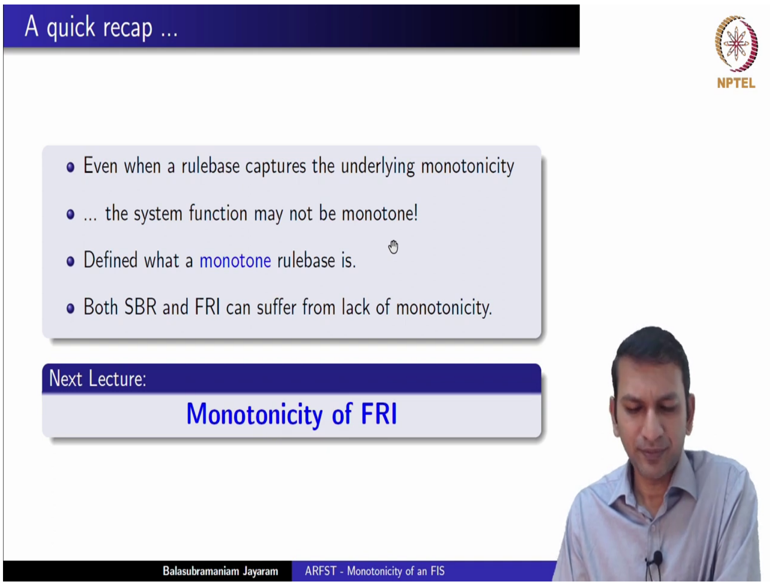

A quick recap of what we have seen so far in this lecture. We have seen that even when a rule base captures the underlying monotonicity, the system function may not be monotone and remember rule base is the ground truth that is given to us and if the antecedents and consequents are related in some way, if they are related through some kind of a monotone relationship; then, we expect the output function also to be monotone.

We defined what a monotone rule base is and we have seen that both SBR and FRI can suffer from lack of monotonicity even when the underlying rule base is monotone. In the next lecture, we will discuss monotonicity of an FRI as was mentioned. Since we are looking at singleton inputs and perhaps, usually the singleton fuzzifier, in the case of an FRI, the composition operator does not play much of a role.

We will see this in the next lecture. So, we will whether we talk about CRI or BKS, it does not matter. What matters is essentially the rule base; the relation that you capture from the rule base and also the de fuzzifier.

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This was the first main paper to appear in a mainstream journal which discussed the Monotonicity of Mamdani systems. This was a paper from University of gin researchers Van Broekhoven Se Baets. They also followed it up with another paper which clearly said under what conditions we could use the center of gravity defuzzification to generate Monotone mamdani models.

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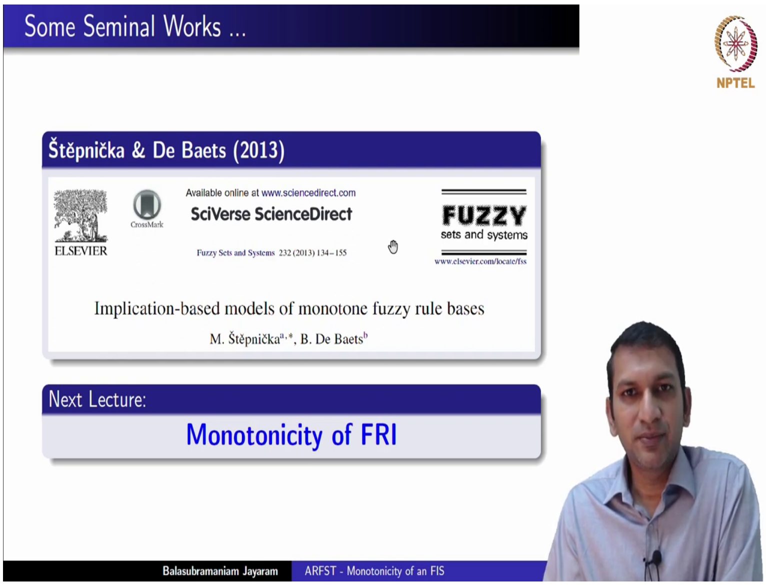

Quite a few years after that the same kind of a study was done by Stepnicka and De Baets; but in this paper, they looked at FRIs with implication based models. That means, they used implication to relate the antecedents and consequents. Perhaps, a word about the different approaches towards discussing monotonicity should be mentioned here. So, many of the works either they fix the operations and then, try to tweak the de fuzzifier so that they can get a monotone output or they convert the rule base itself to a different form.

For instance, we are given a rule base as A i implies B i. But now, what many works have considered is to extend them, convert them, transform them into what are called at least and at most models of the rule base, where they modify the antecedents and consequents of the rule base itself and consider this transformed rule base and study the monotonicity properties.

In our work in this lecture series, what we would see is without transforming the rule base and with as general a class of operation that we could consider when do we ensure monotonicity; what are the conditions required to ensure monotonicity, this will be the focus in this lecture series. We will not touch upon those works.

So, in the next lecture, we will discuss monotonicity of FRI. Glad you could join us for this lecture. Hope to see you soon in the next lecture.

Thank you again.