Approximate Reasoning using Fuzzy Set Theory
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Lecture - 51
Robustness of SBR

Hello and welcome to the last of the lectures in this week 10 of this course titled Approximate Reasoning using Fuzzy Set Theory. A course offered over the NPTEL platform. In this lecture we will discuss the Robustness of Similarity Based Reasoning schemes.

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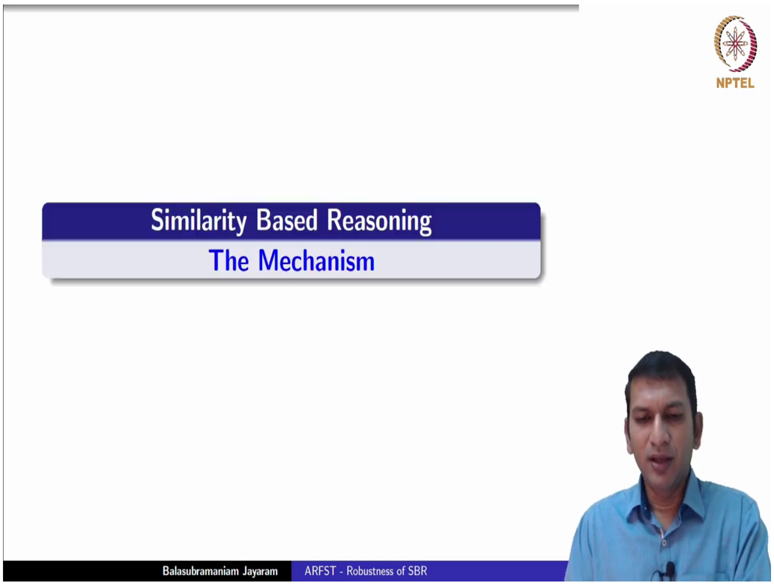

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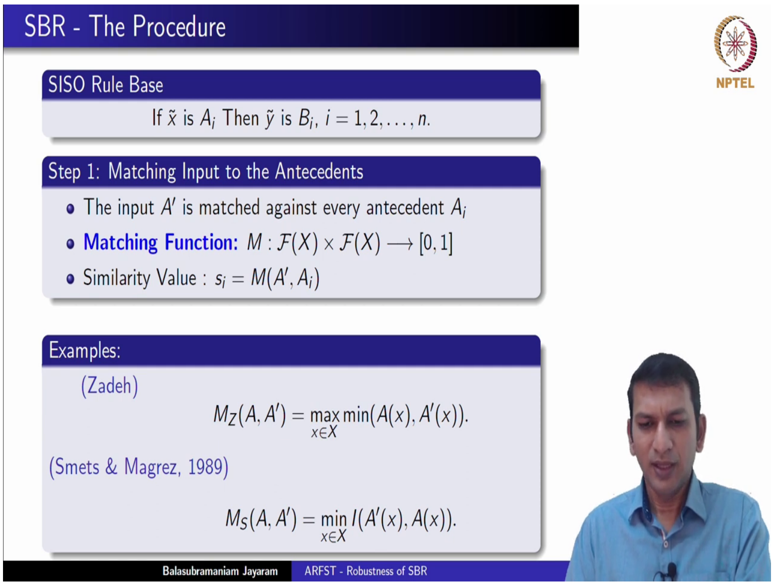

Let us take a quick recap of the mechanism itself. So, we are given a set of SISO rules single input single output rules of the form if x is A i then y is B i n of them. Given an input A dash we use a matching function to compare the similarity between the given input A dash and each of the antecedent A i for this we use a matching function M from F of X cross F X to [0,1].

So, the similarity values are normalized between 0 and 1. Some of the examples that we have seen consistently are the Zadeh's matching function and the one of Smet Magrez.

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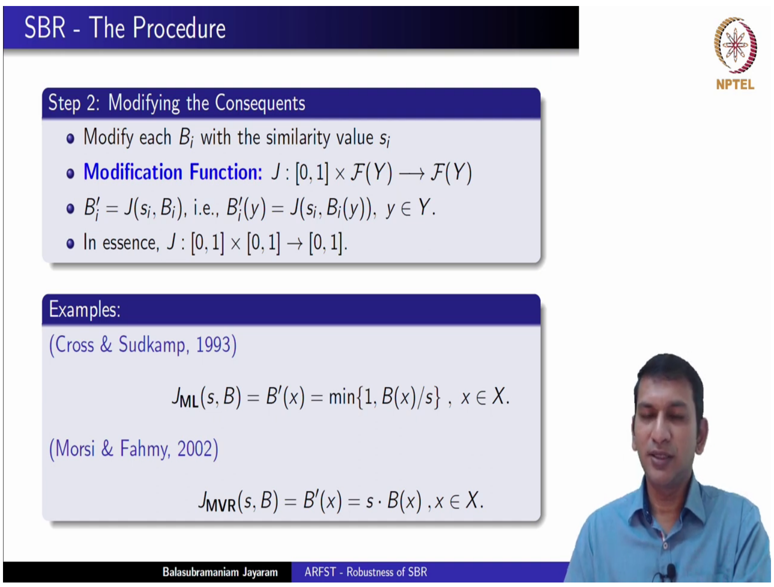

Once an input is given it is matched to each of the antecedent of the rules and the corresponding similarity values s i are held. In the second step we modify the corresponding consequence of the rules using these similarity values. For that we use a modification function.

We know that even though J is essentially the modification function J is essentially a function from [0,1] cross F of Y the set of all fuzzy sets of Y to F of Y. We know that it can be thought of as just a binary function on [0,1]. Because essentially, we are going to act on the membership values of the fuzzy sets defined on Y. We have seen a couple of math modification functions we see that J ML is nothing, but essentially the Goguen implication and the second modification function is nothing, but the product T norm.

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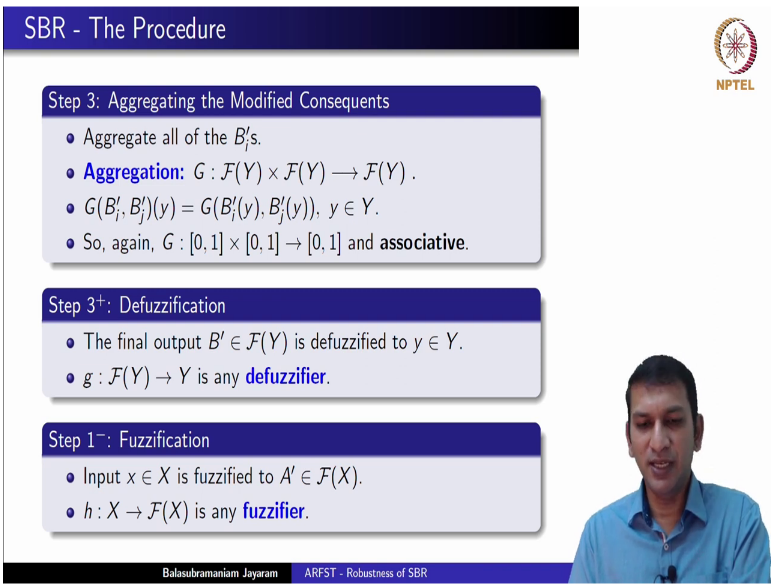

The third step the modified consequents B i dashes that were obtained by using the similarity values s i to modify them using the modification function. All of these B i dashes they are actually aggregative. So, we are looking at a function which takes a few of fuzzy sets over Y and converts them into a single fuzzy set on Y. Once again, we have seen that we could use any binary function on [0,1], but we would like it to be associative. So, that the aggregation is order independent.

We can also have a post processing step because the first three steps they give you only a fuzzy set as an output. You could also defuzzify it to one of the elements on the output domain Y. Similarly, instead of giving an input fuzzy set one may actually have only a value from the domain X. Accordingly we can fuzzify it to an input fuzzy set to be given to this SBR scheme.

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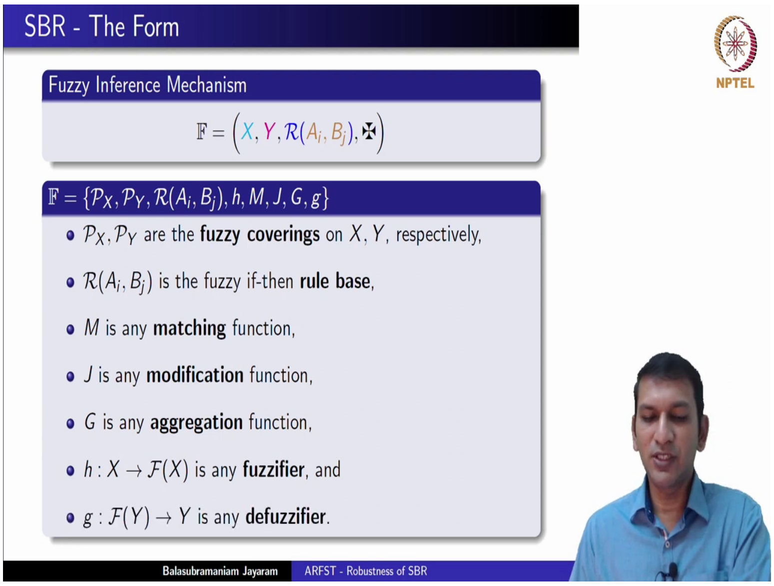

So, in general we said that this is the form of a fuzzy inference mechanism X and Y will denote the input and output spaces R of A i B j denote the set of rules that we have and the other symbol there it captures all the inference operators. In the case of similarity-based reasoning we know that it consists of a tuple of these many parameters or elements where P X and P Y are the fuzzy coverings on X and Y the input and output domains respectively.

R of A i B j gives you the fuzzy if then rule base. The inference operations consist of one the matching function M, the modification function J, aggregation function G and it could also often have the fuzzifier h and the defuzzifier g.

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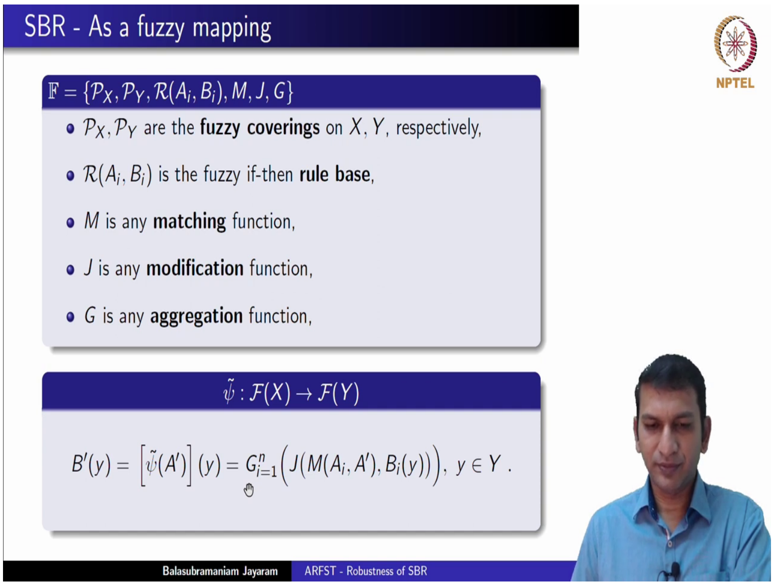

Since we are considering SBR as a fuzzy mapping clearly the fuzzifier h and the defuzzifier g small g are left out the rest of the parameters remain. So, in that sense the corresponding system function psi tilde which can be looked at as a mapping from F of X to F of Y is given as follows if you are giving an A dash to the psi tilde, we get a B dash which is a fuzzy set on y.

So, B dash has to be specified on each of the elements y in the domain output domain Y. It is given like this. So, first we take A dash compare it using a matching function and get the similarity value s i. Next, we use that similarity value to modify the consequent B i using the modification function J. And such modified consequence we aggregate using G.

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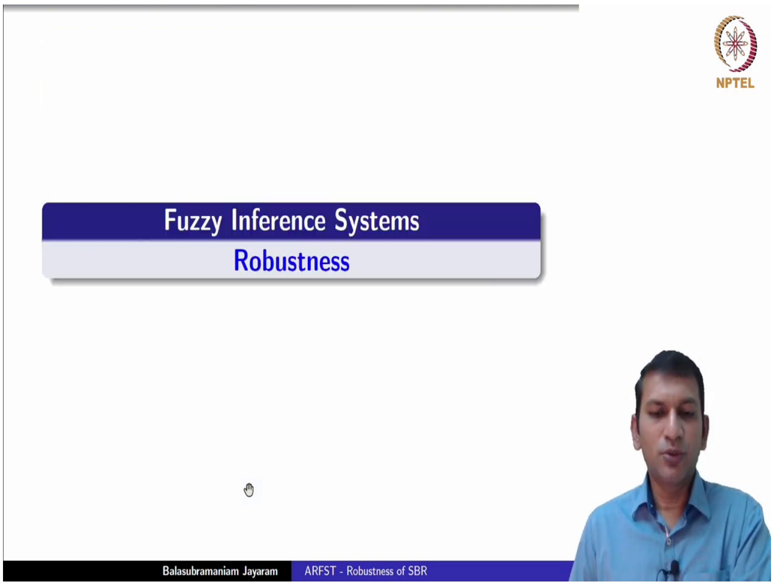

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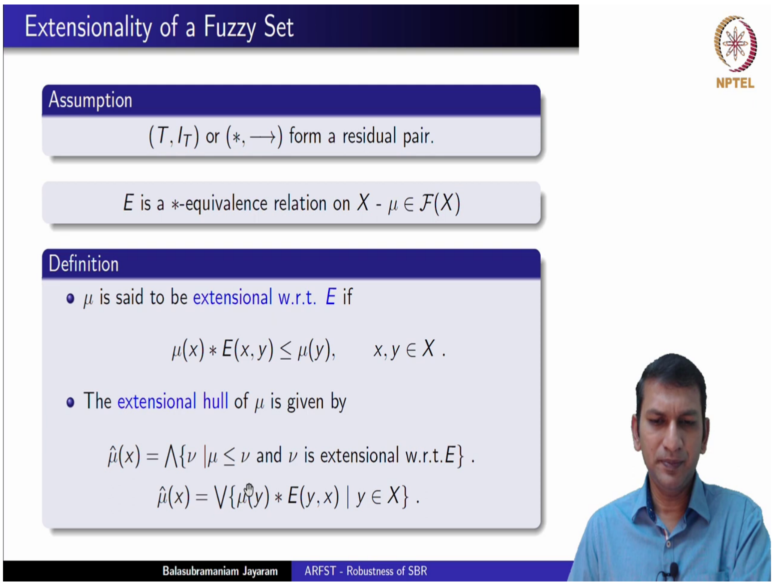

We have defined what is robustness towards defining robustness we need the concept of extensionality. We made an assumption that we will remain within the realms of residuated lattices which means T the operations are coming from residuated lattices and the most important ones are the left continuous T norm T and the corresponding residual implication which we denote by I T. And in the case of in fixed notations we are using the star and the implication of the arrow.

So, we are given a star equivalence relation E and if you are given a fuzzy set mu, we say it is extensional with respect to E if this inequality is held it was, we have seen this being interpreted many times before. So, essentially you can think of it like this if x and y E of x y is 1; that means, we say x and y are equal with respect to this relation or in put in other words x and y are indistinguishable with respect to this concept that their relation E represents.

Then we insist that the membership value of y to this concept should be at least as much as the membership value of x to this concept. Not all fuzzy sets are extensional with respect to a particular relation E. In that case we extend it and we define the extensional hull of mu as follows. It is the smallest extensional fuzzy set with respect to E which also contains mu, we denote it by mu cap and we have a simpler formula how to obtain this mu cap given mu and E.

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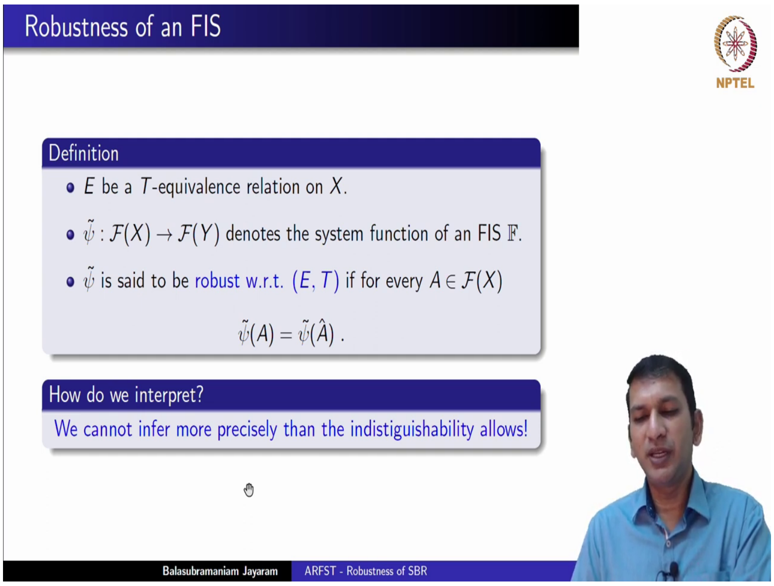

Well, what is robustness? We have a space X on which we have star equivalence or T equivalence relation defined. And we say the system function obtained from an FIS psi tilde is robust with respect to this given T equivalence relation E if for every A and F of X we have that psi tilde of A is equal to psi tilde of A cap where A cap is the extensional hull of A.

We have seen that it means we cannot infer more precisely than the indistinguishability allows.

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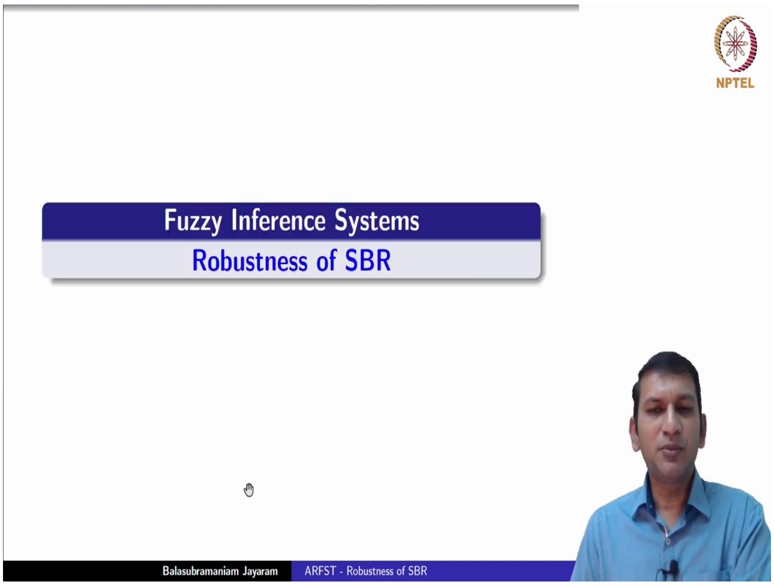

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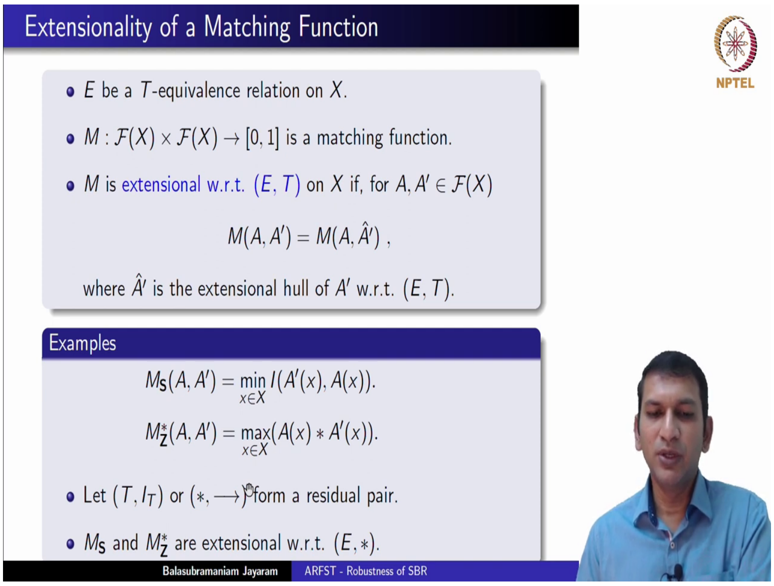

Let us come to the robustness of SBR. For this we need to define the extensionality of a matching function. Let E be a given T equivalence relation on X and M be a matching function; that means, it is a function from F of X cross F of X to [0,1]. We say M is extensional with respect to this T equivalence relation A, if for A A dash any two fuzzy sets A and A dash on x we want that M of A A dash is equal to M of A A dash cap.

Now, what is A dash cap? It is the extensional hull of A dash with respect to the T equivalence relation. So, essentially all we are asking is if you match A dash to A the matching function should give the same value as if you are matching the extensional hull of A dash to A. Are there examples of such matching functions?

Well, the usual suspects the matching function that we have seen before the Zadeh's and Smet Magrez matching functions. In fact, satisfy this condition we will see this immediately. Of course, we are once again going to assume that the operations are coming from residuated lattice structure. In that sense we can say that both these matching functions are in fact extensional with respect to the star equivalence relation.

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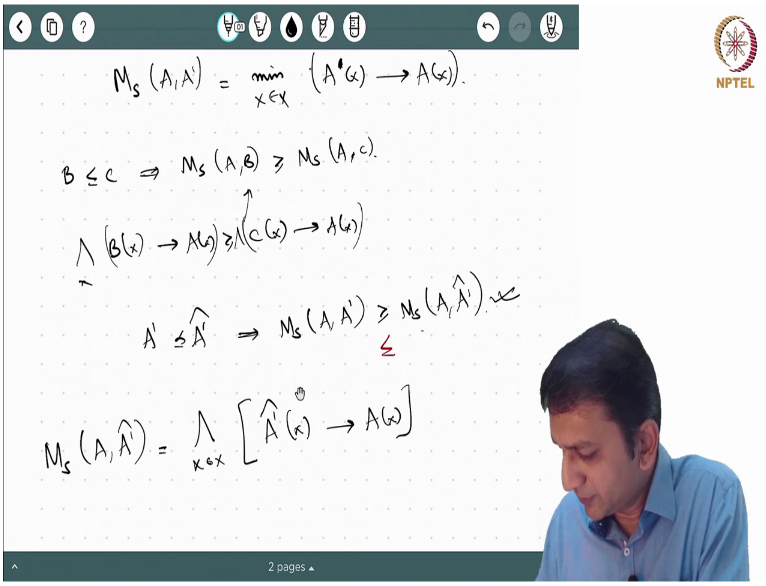

Let us look at this. We will start with the Smet Magrez function is given as min over x element of X, i so maybe it is A dash of x implies A of x is what we have. So, first of all notice that if let us say if you have a fuzzy set B which is smaller than C in terms of point wise ordering, then this implies M S of A, B is in fact greater than or equal to M S of A, C. Now this is clear because if B is smaller than C then B of x is less than or equal to C of x for every C.

Now, input A of x here then we know that this inequality in fact reverses. So, if you take (Refer Time: 09:38) x on either side we get this. So, now we know that given an A dash is always smaller than its extensional hull which means M S of A A dash is greater than or equal to M S of A A dash cap. So, this is an inequality that is always available for us with respect to A dash and A dash cap. So, what we need to prove is in fact the other inequality.

We need to prove that this fellow is actually the right-hand side is bigger than the left-hand side. So, let us start with that M S of A A dash cap is equal to infimum over x a dash cap of x. Note that by definition we are switching places even though A dash appears in the second form. We are actually putting in the first position A dash cap of x implies A of x you know.

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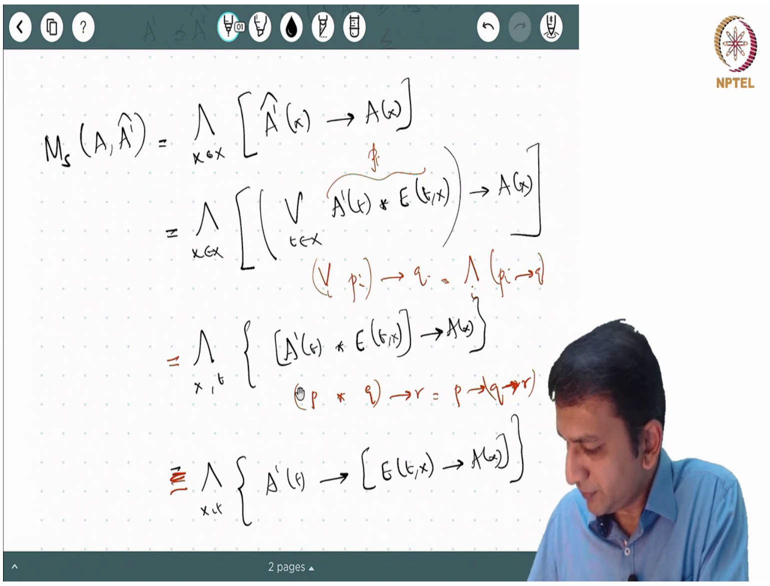

Once again sup over t element of X A dash of t star E of t comma x implies A of x. Now we know that in the residuated lattice setting you can look at this as over i p i implies q which is actually equivalent to we have seen this before and over i p i implies q. So, taking this as p i you can write this as and over x and have infimum over t A dash of t star E of t, x implies A of x. Now, we are in the residuated lattice structure.

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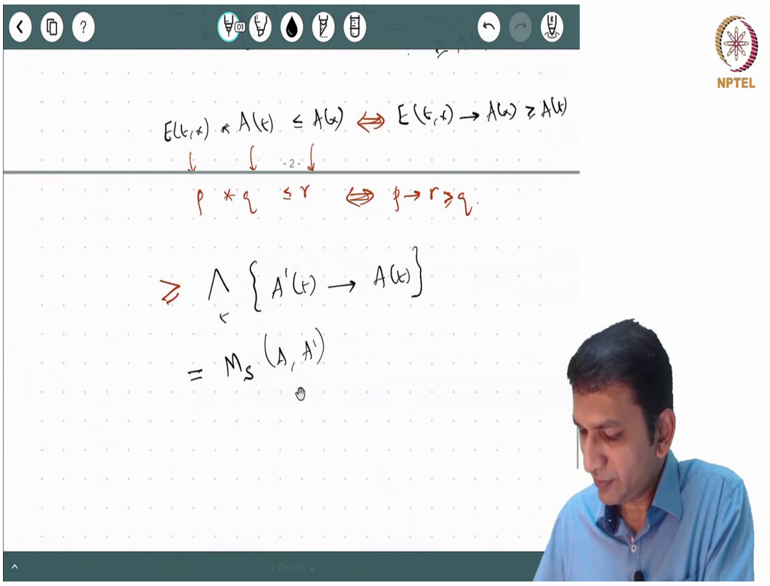

So; that means, we have also this property taking this as p and q p star q implies r is equal to p implies q implies r. So, let us use that property here. This is p for x A dash t implies E of t, x implies A of x. Now, we have seen this earlier too let us work it out. We know that E t of x star A of t is less than or equal to A of x. Now, we are coming from a residuated lattice structure.

So, now, we know this also. If p star q is less than or equal to r then this is equivalent to p implies r greater than or equal to q. So, now, this is p this is q this is r. So, now, we see that this if and only if E of t, x implies A of x is greater than or equal to A of t. Now, so; that means, this quantity is greater than or equal to A of t. Now it appears in the second variable or second position for implication where it is increasing.

So, the same inequality will hold greater than or equal to infimum over x t A dash of t implies A of t. Now this x is supremum of course, and what we see this is essentially M S of A comma A dash. Now what we wanted to show was that this quantity which we have taken here is greater than this quantity and that is essentially what we have shown.

So, from here we see that M S is extensional with respect to the E star equivalence if the operations are coming from the residuated lattice structure.

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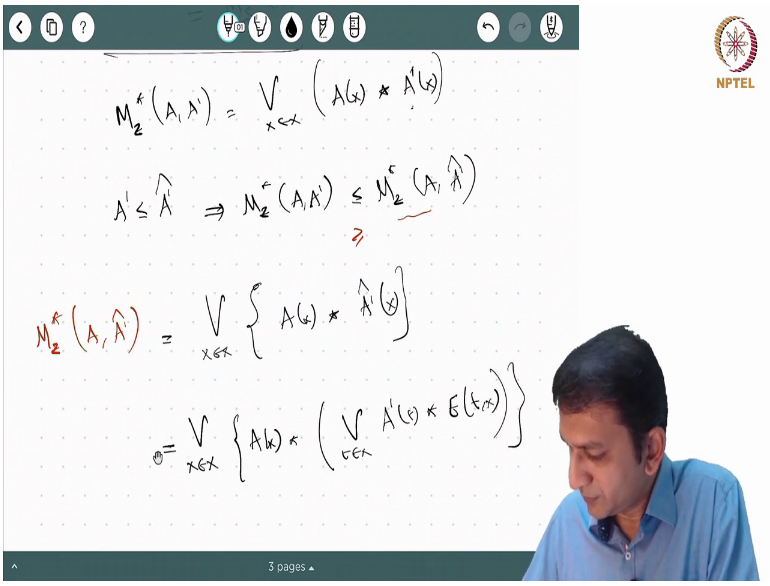

Let us also look at M Z star. Sup over x element of X A of x star A dash of x. It is immediately clear that A dash is less than since A dash is less than its extensional hull this implies M Z of A A dash (Refer Time 14:28) more or than M Z star of A A dash cap this follows from the monotonicity of star. Because if you put instead A dash you could put A dash cap and this will be smaller than that.

So, one way inequality is true. What we need to show is the other way. Now we need to show this is smaller than. Let us start with that A A dash cap and this is equal to supremum over x A of x star A dash cap of x substituting for the extensional hull of A dash supremum t element of x A dash t star E of t, x.

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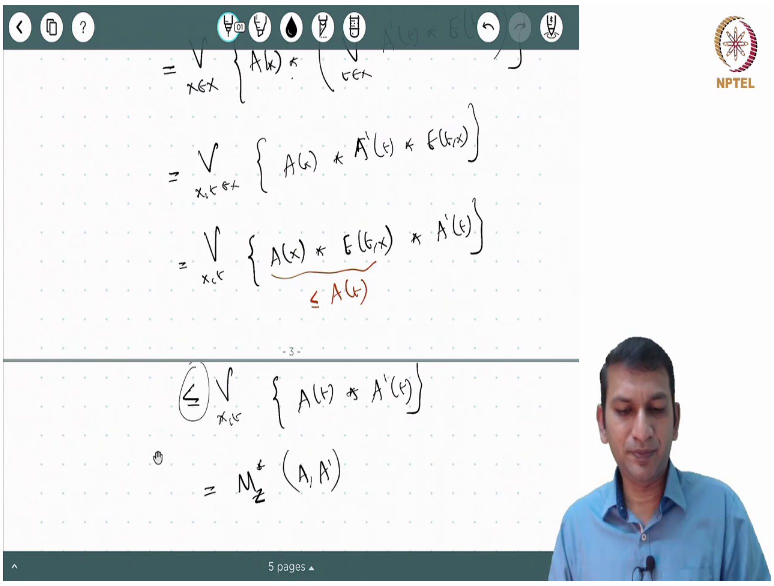

Now, we know that the star is actually a T norm coming from the residuated lattice structure. Which means we have it is left continuous which means it is sup reserving. So, we can remove this outside supremum over x and t element of x A of x star A dash of t star E of t, x. Now, by commutativity and associativity we can also write this as A of x star E of t comma x star A dash of t. Now we know that A is extensional.

So, now, this is less than or equal to A of t which means less then or equal to supremum over x comma t A of t star A dash of t which is essentially equal to M Z star A comma A dash. Now, we wanted to show this right-hand side is smaller than this left-hand side that is essentially what we have shown here because of the extensionality of A.

So, what we have shown is if A is extensional then M Z star is also is extensional preserve on those points A where it is extensional with respect to E.

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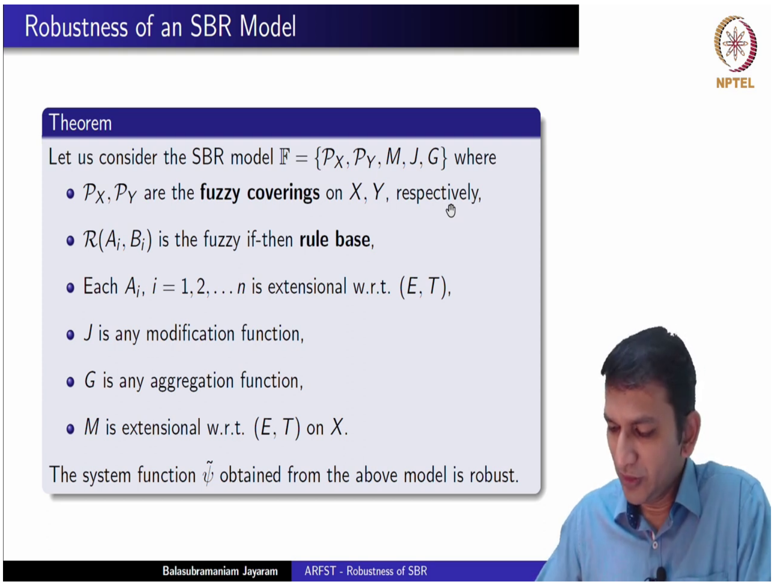

So, these are some examples of matching functions which are extensional. Now, let us come to the robustness of a SBR model. Let us consider this model where we have P X and P Y which are essentially the fuzzy coverings on X comma Y, we have the given fuzzy if then rule base.

And each of the A i's is in fact extensional with respect to a given t equivalence relation E. Let J be any modification function G be any aggregation function, but what we want is m is extensional with respect to E T on X. So, if A i is extensional then we want that M is M of A i comma A dash should be equal to M of A i comma A dash cap. Under that condition we can show the system function psi tilde obtained from the above model is in fact robust.

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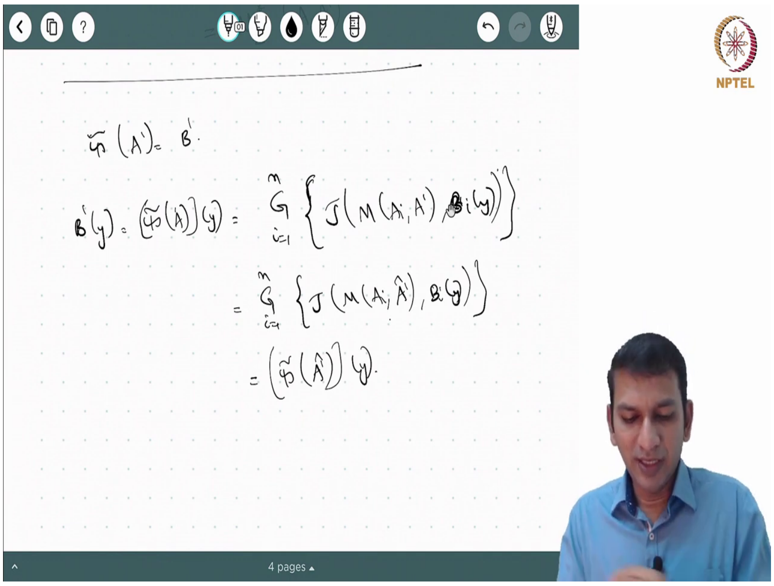

Now, how do we prove this? Let us look at this. So, we know that psi tilde of A dash is equal to B dash. Now, B dash of y is equal to psi tilde of A dash at y. Now, this is nothing, but G of i is equal to 1 to n J of M of A comma A dash comma B i, ok. Now, clearly, we know that M is extensional.

So, this can be written as M of A i comma A dash cap comma B i of y, ok. This is the value that we need to do. Now what is this? This is nothing, but psi tilde of A dash cap at y. So, clearly it follows since M is extensional with respect to the given equivalence relation T equivalence relation E, we see that this system in fact becomes robust.

So, for the robustness of an SBR all we require is that the M is extensional with respect to the antecedence A i which are themselves extensional with respect to the T equivalence relation E. So, we need the antecedents to be extensional like before in the case of FRI's in the case of CRI and BKS and we want that matching function also to be extensional with respect to the T equivalence relation E that we have.

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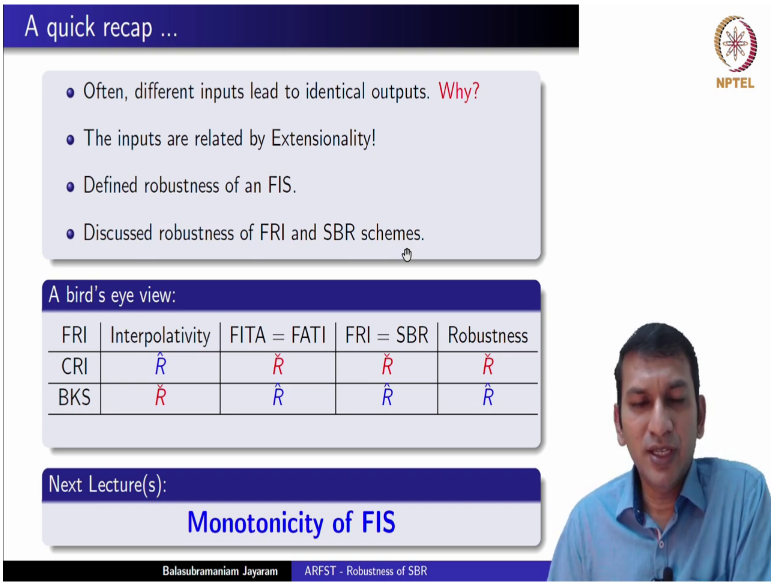

A quick recap of what we have done throughout this week. We noticed in the very first lecture during this week that different inputs can lead to identical outputs, but not always. So, we asked the question why does it happen? Then we came up with the answer that two fuzzy sets will give us identical outputs if they are related to each other in terms of extensionality.

We defined robustness in terms of this, we discussed robustness of FRI schemes that of compositional rule of inference and the Bandler Kohout subproduct and also that of similarity-based reasoning schemes. Once again let us look at the CRI's and the FRI's and SBR in terms of the corresponding relation that we use. In the last two three weeks we have been discussing about interpolativity, continuity and robustness.

If it is CRI, we see that for interpolativity R cap plays a role if it is a solution to that fuzzy relational equation. Then it has a solution and there can also be other solutions; however, when it comes to discussing robustness or whether CRI can be expressed as an SBR or the question of when FITA is equal to FATI it is R check which actually comes into play.

Dually if we consider BKS R check plays a role when it comes to interpolativity, but it is R cap which is important to discuss questions of equivalence between FITA and FATI when a BKS can be looked at as an SBR and finally, to discuss the robustness of the underlying inference scheme.

With this we will wind up our discussions on robustness of fuzzy inference systems. In the week of lectures, we will start to discuss on the monotonicity of a fuzzy inference game. Glad you could join us for this lecture. Hope to meet you soon again in the next one.

Thank you again.