Approximate Reasoning using Fuzzy Set Theory
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Lecture - 50
Robustness of BKS

Hello and welcome to the third of the lectures, in this week 10, of this course titled Approximate Reasoning using Fuzzy Set Theory. A course offered over the NPTEL platform.

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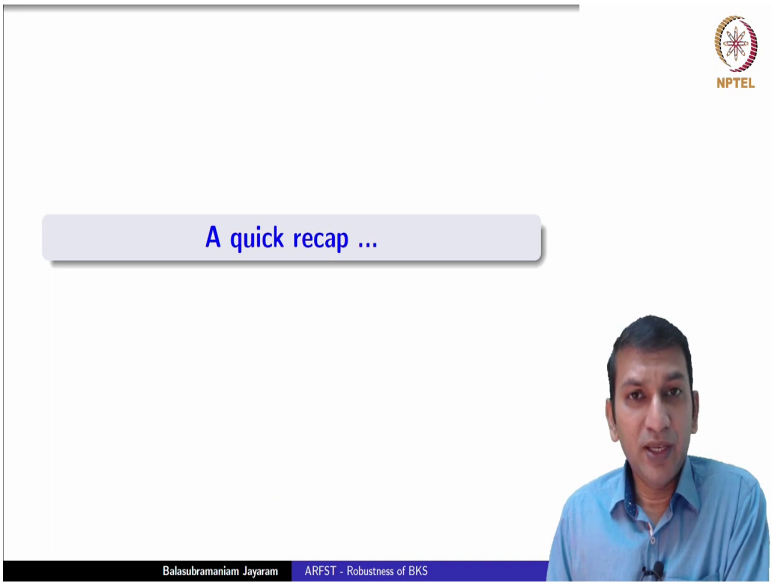

A quick recap of what we have seen in the last two lectures.

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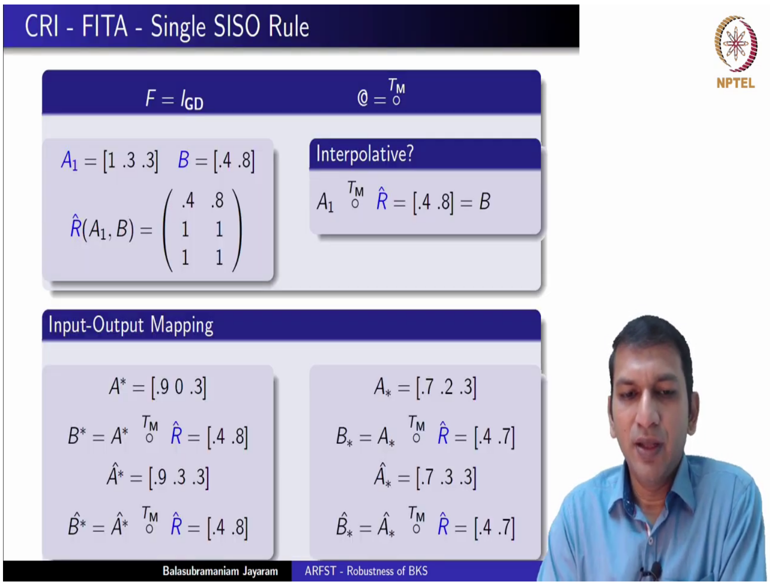

We began by looking at the inference obtained from a single SISO rule, single input single output rule, in the case of CRI inference, where we found for certain pairs of inputs, we obtained identical outputs. While, for certain pairs of inputs we did not obtain identical outputs.

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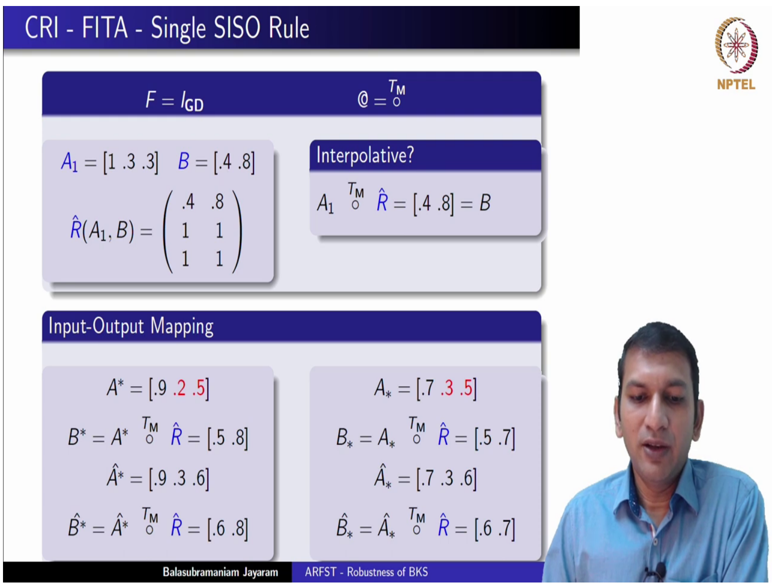

Now, this was the case whether the system was interpolative or not; so, it egged us on to understand why this was happening?

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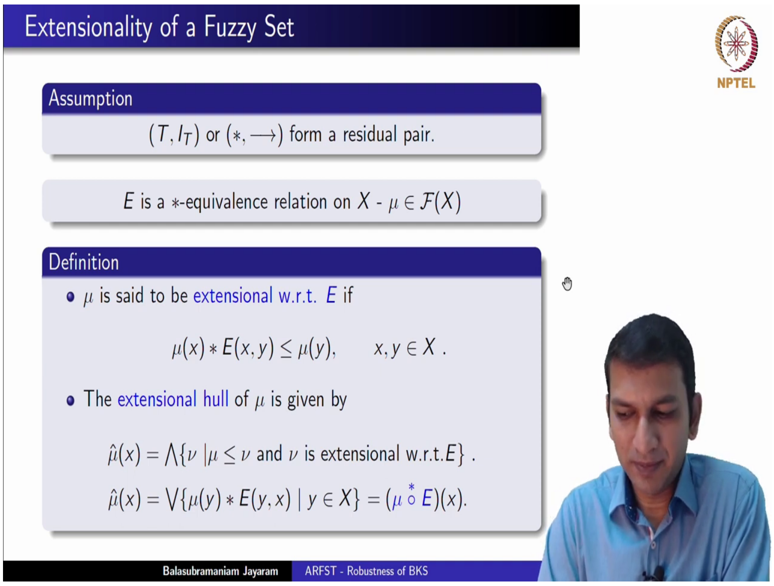

Towards decoding this we introduced the concept of extensionality of fuzzy set. Of course, the underlying space we are in the realms of residuated lattices. So, we will consider T to be left continuous T norm and I_T implication to be the corresponding residual implication.

We have an star equivalence relation on X. And if you are given a mu which is a fuzzy set over X, we said mu is extensional with respect to E if this inequality is held. It is clear, essentially it says that the membership value of y to the concept mu should be related to how much it is related to an X and the membership value of x itself to the concept mu.

Once again note that if E of x, y is 1; that means, with respect to this relation x and y are indistinguishable, then this inequality enforces the fact that the membership value or the degree of belongingness y to the concept mu should be at least as much as the membership value of the degree of belonging of belongingness of x to the concept mu. We have seen that not every fuzzy set can be extensional with respect to E and a particular T norm star.

So, to make it extensional, the extensional hull was defined as follows. It inverts, it is essentially the smallest extensional fuzzy set containing mu. So, nu is extension with respect to E and star, it contains mu, and it is also the smallest such extensional fuzzy set of mu which is called the extensional hull which we denote by mu cap. We also have an easier way of obtaining mu cap, given a mu; this essentially taking the sup star composition of mu with E.

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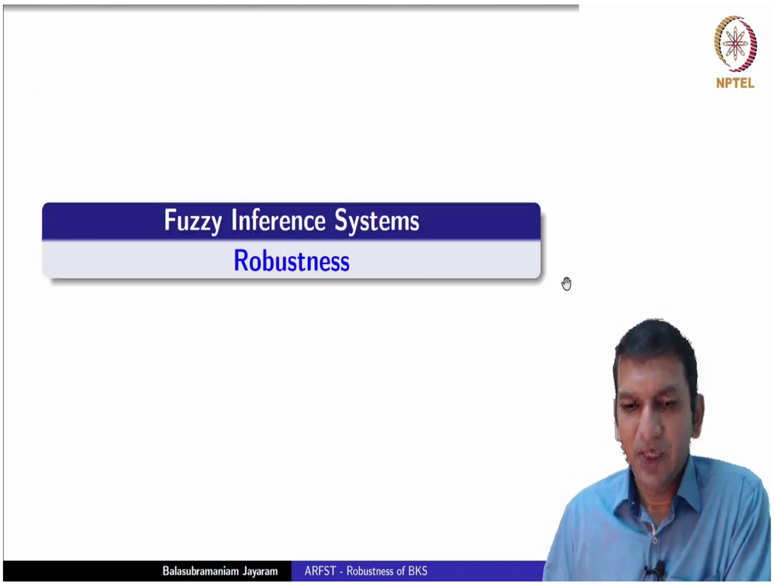

Well, then next we introduce what we mean by robustness of a fuzzy inference system.

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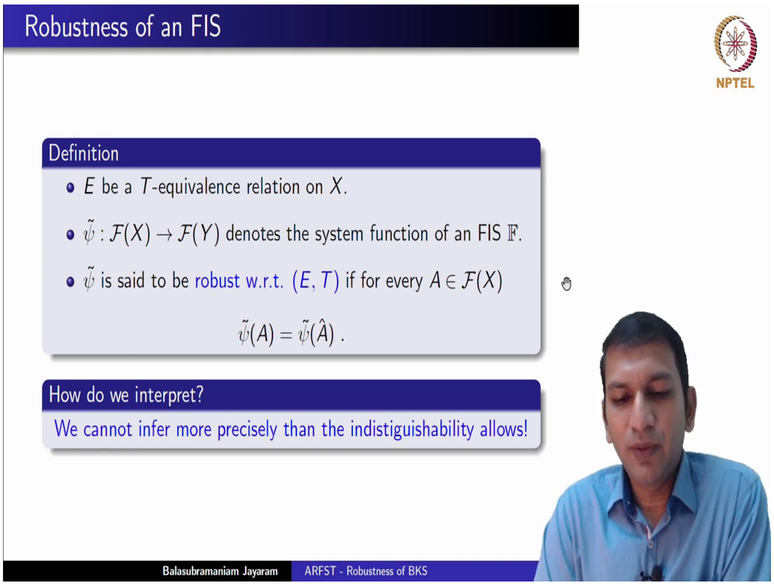

On the space X, we have a T equivalence relation E. And if you are given a psi tilde which is the system function of a FIS of a fuzzy inference system F, psi tilde as a mapping from F X to F Y, we say this is robust with respect to a T equivalence relation E if for every A in F of X we see that psi tilde of A is in fact, equal to psi tilde of A cap.

We saw that one way to interpret it is that we cannot infer more precisely than the indistinguishability allows.

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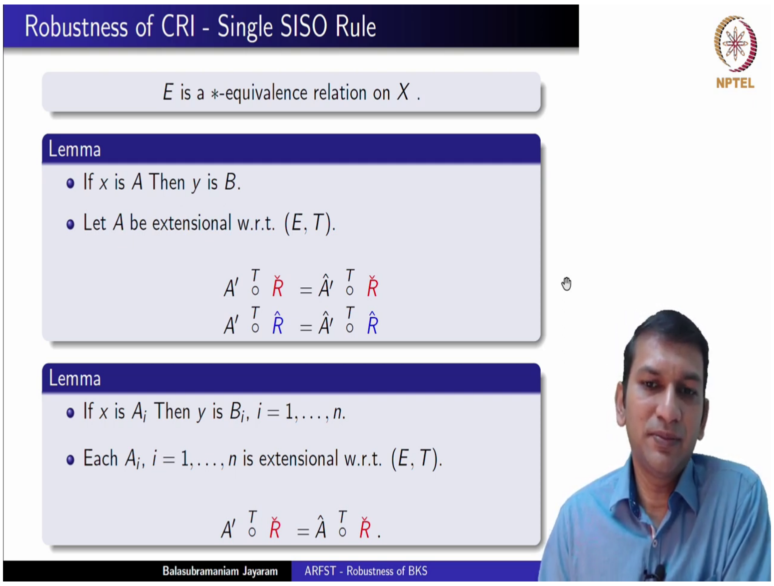

We have seen in the last lecture that CRI also enjoys robustness both in the single rule case and also in the multiple rule case. So, we have E which is a star equivalence relation. If you are given a single rule, If x is A Then y is B, and if the antecedent A is extensional with respect to this T equivalence equals relation E, then we found that it does not matter whether you use R check or R cap, it is robust.

That means, where the input A dash; the output obtained from input A dash or its extension hull, both are identical. Similarly, if you have a set of fuzzy inference rules, single input single output. And if each of the antecedents is extensional, then we see that CRI is robust. Of course, we need to use the R check relation.

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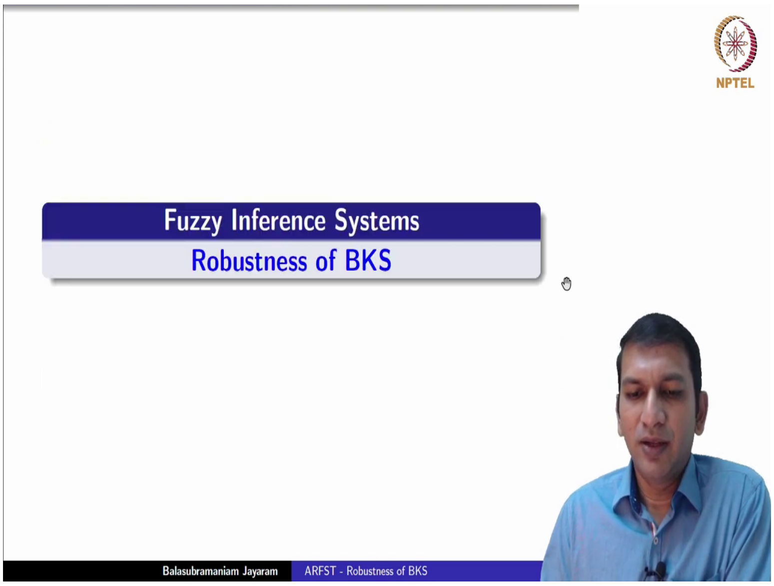

Let us study the robustness of Bandler Kohout Subproduct inference scheme.

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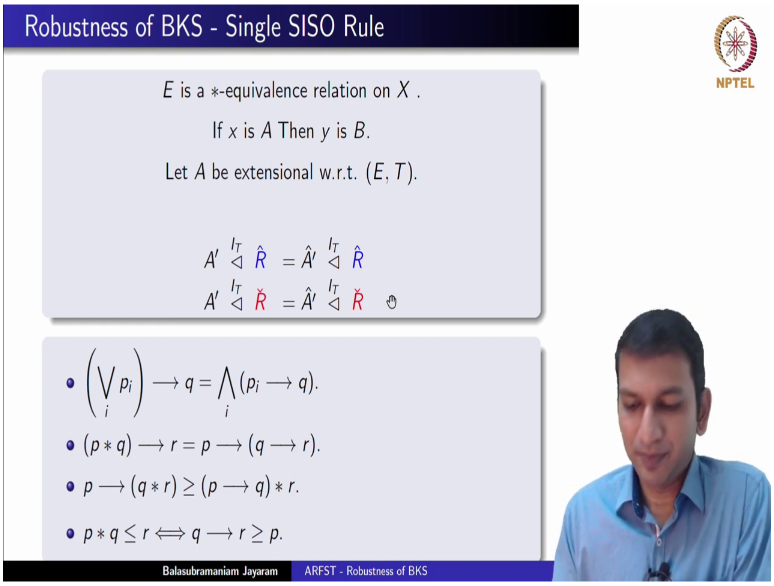

Once again, we have E which is a star equal installation on X. Let us begin by studying a single SISO rule. If x is A Then y is B. Let us also assume that A is extensional with respect to this T equivalence relation E. Then what we can show is the Bandler Kohout Subproduct where, the implication is given with respect to the implication is taken as the R implication, then with both R cap and R check the BKS inference scheme enjoys robustness.

That means, it does not matter whether you will A dash or its extensional hull, the outputs obtain will be identical. Let us prove this.

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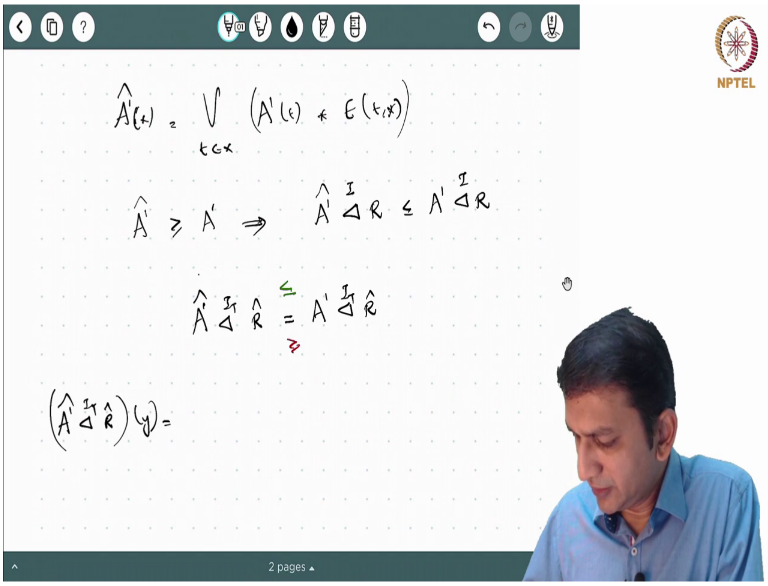

So, now, recall that if you are having a A dash of cap of x this is, sup x element of x. Sup t element of x A dash t star E of t comma x. It is clear that A dash cap is greater than or equal to A dash. We have seen this. But this now implies A dash cap, Bandler Kohout Subproduct with any implication I because of its antitonicity in the first variable is in fact, less than or equal to A dash for any R.

So, now, what we need to prove is that with I_T and R cap is in fact, equal to A dash Bandler Kohout Subproduct of I_T, R cap. This is what we need to prove. We know this is always true for the Bandler Kohout Subproduct. So, all we need to prove is that this A dash cap Bandler Kohout Subproduct R cap is greater than the right hand side.

Let us go ahead and prove this. So, A dash cap delta I_T of R cap at a particular y is given like this.

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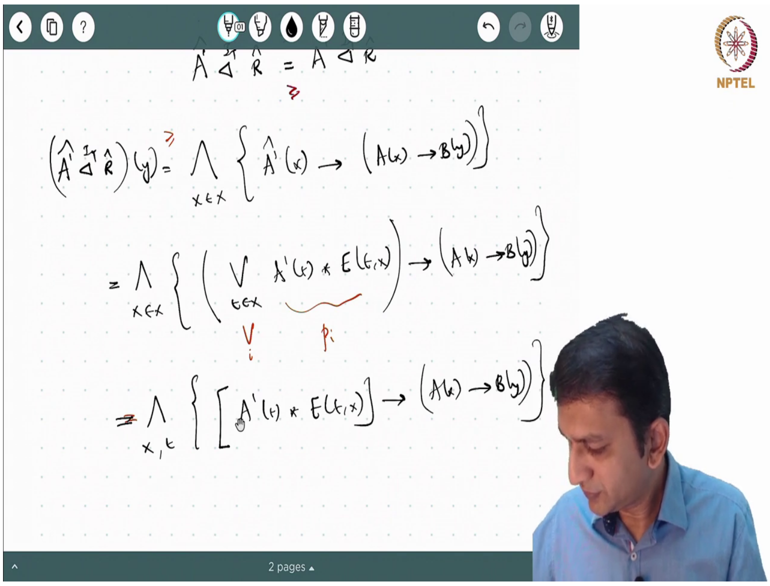

Infimum over x element of x A dash cap of x implies, note that R cap now is A of x implies B of y. Once again, let us expand A dash cap supremum power t element of x, A dash of t star E of t, x implies A of x implies B of y. Now, note that we are in the realm of a residuated lattice. We have a lot of properties available for us. Let us make use of a few of them towards proving this result.

Also note that, what we need to prove is that this quantity is greater. That is the inequality that we would like to prove. In a residuated lattice, this property is true. This essentially distributivity in the first variable where, the joint becomes the meet, the local joint becomes global meet.

So, now, if you look at this quantity here so, now, you can look at this as p i the supremum of joint. So, now using this equality, this property that we have here, the distributivity property, what we can do is write it like this is equal to supremum over x and when t comes out it is again over meet infimum A dash t star E of t, x implies A of x implies B of y.

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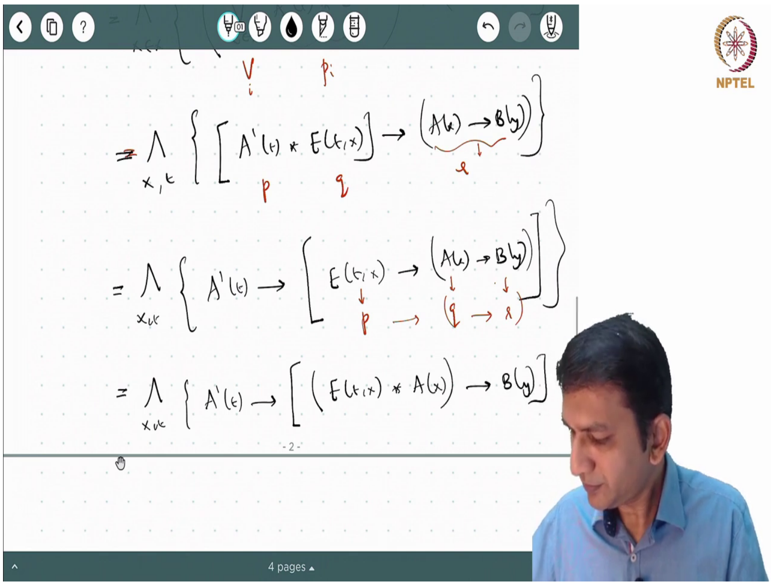

Now, we know the law of importation is valid, so taking this as p, this as q, and this as r, let us write it like this, A dash t implies E of t, x implies A of the x implies B of y. Note that you are considering this entire thing as r, so that is what we have written here. Now, not only is p star q implies r equal to p implies q implies r, because of equality, we know that p implies q implies r can also be written as p star q implies r.

So, now once again we play the same trick here take this as p, q, this as q, this as r, this as p, which means this is p implies q implies r, which can be written as p star q implies r, which what we will write it as now. A of x, t A dash of t implies p star q E of t, x star A of x implies B of y. Now, look at this quantity.

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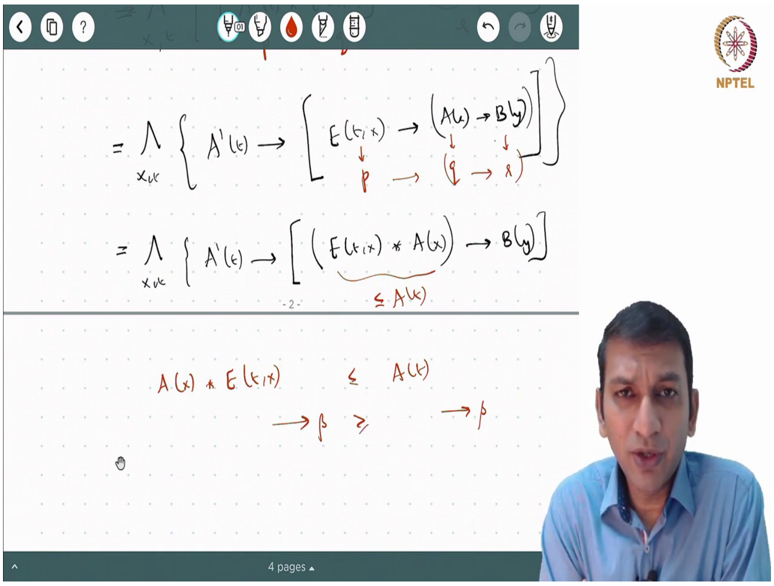

We know that this quantity is less than or equal to A of t because A is extension. However, note that A of x star E of t, x is less than or equal to A of t. Now, however, if you are take this in the first variable implies beta, then what we get is the inequality reverses. We know this because any implication is anti-tone in the first variable; that means, it is non-increasing in the first variable.

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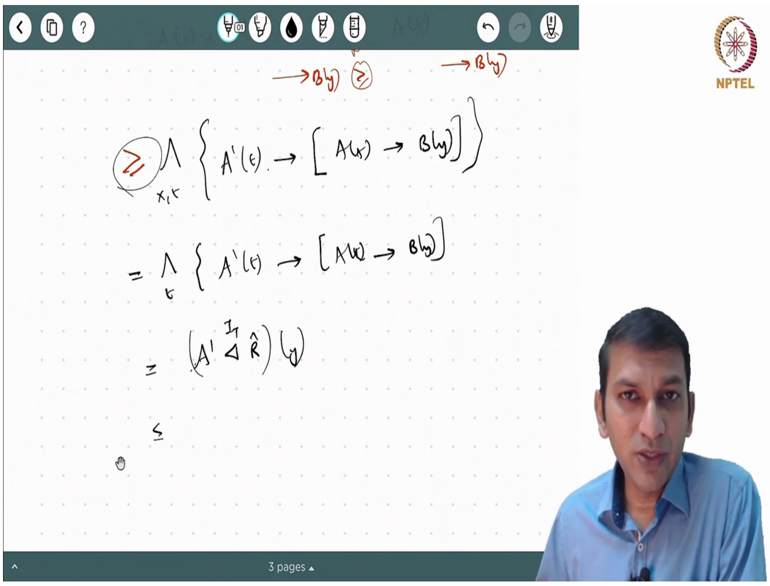

So, now if you put instead of beta, if we put B of y here, then you see that this inequality comes into this inequality greater than or equal. So, now, applying that here infimum over x, t, A dash of t implies A of t implies B of y. So, we have substituted this entire quantity by A of t. Now, what is this? x vanishes is equal to inf of t A dash of t implies A of t implies B of y. This essentially equal to A dash I_T R cap (Refer Time: 12:38).

Now, because of this inequality here, what we started with was that A dash cap delta I_T of R cap we have shown is greater than or equal to A dash, this. The other inequality we know this is always valid. So, what we have ended up doing is showing that this first inequality, first equality is in fact, true. Well, now let us go ahead and try to prove the second equality.

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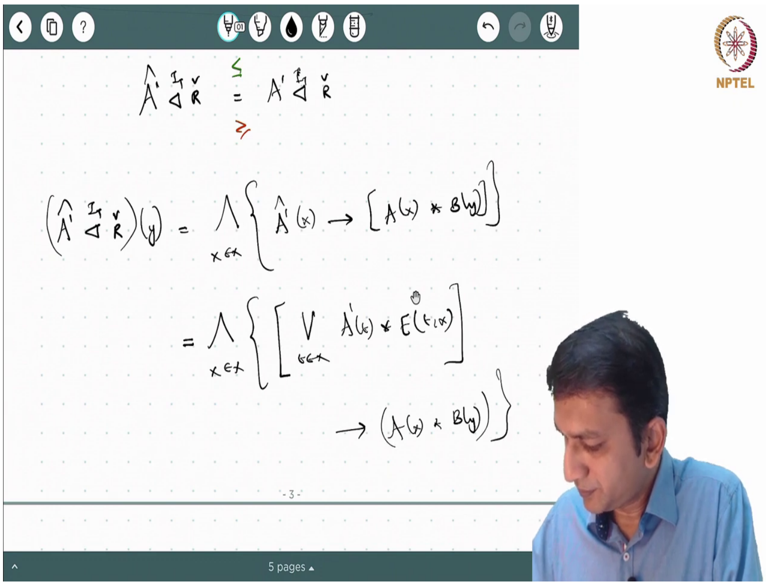

You need to prove A dash cap delta I_T R check is equal to A dash delta I_T R check. Because of the monotonicity there, we know that this is always true. So, what we need to prove is the other way inequality. This is what we want to prove. So, let us start once again A dash cap delta I_T (Refer Time: 13:44) into R check, at any arbitrary y is nothing, but inf of x element of x, A dash cap of x implies know that it is R check, so that means, it is A x star B y.

This is equal to; let us once again open this bracket. Yes, sup t element of x A dash t star E of t, x implies A x star B y.

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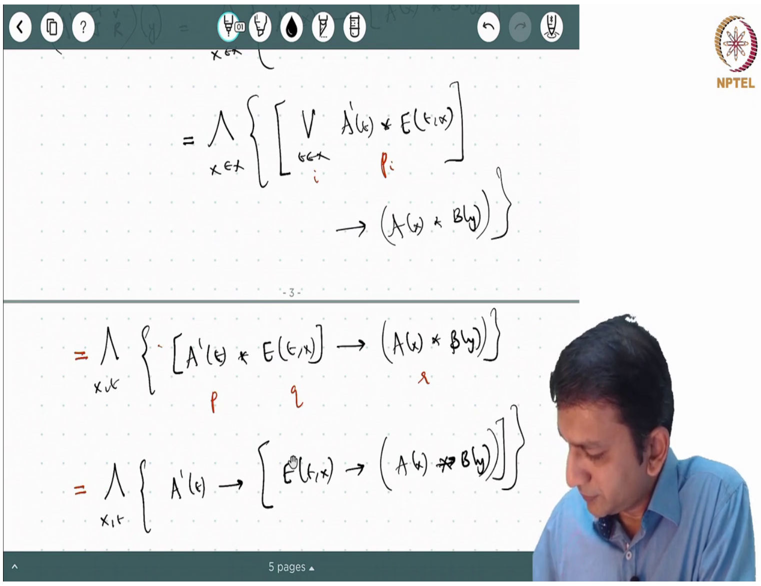

Now, once again, we use the fact that sup p i implies q is nothing but join over i of p i implies to q. So, once again we consider this as p i over i, right. So, now, this is equal to here write it here in x comma t, A dash of t star E of t, x implies A of x star B of y. Now, once again we use the law of importation, p star q implies r is p implies q implies r. So, we are taking this to be p, this to be q, and this to be r is equal to infimum over x, t, A dash of t implies, E t of x implies A of x star B y.

Well, now, we have another property which says that p implies q star r is greater than or equal to p implies q star r.

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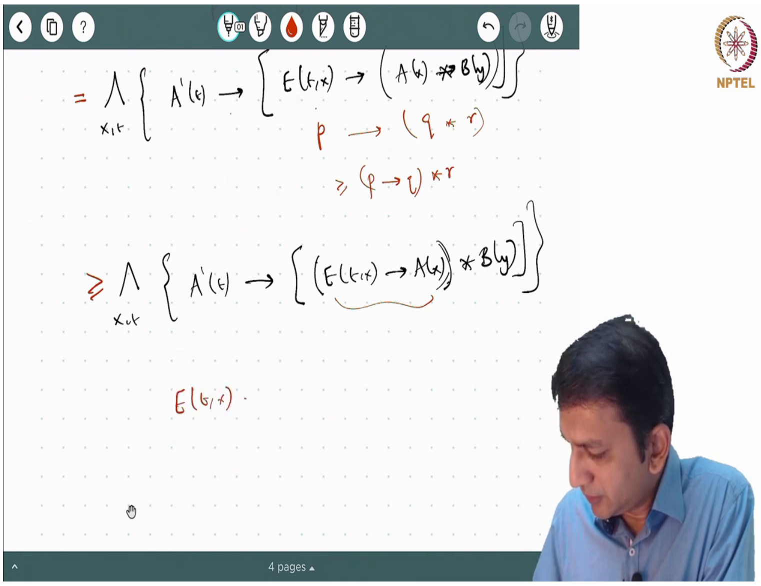

So, now let us take this as p this as q star r. So, we know that p implies q star r is greater than or equal to p implies q star r. Note that it is appearing on the second position with respect to this implication in which implication is increasing non-decreasing. So, that means, what we get here is greater than or equal to inf of x comma t A dash of t implying, E of t, x implies A of x this close, implying star B of y. So, we have managed to bring the correct inequality. But look at this quantity, yeah.

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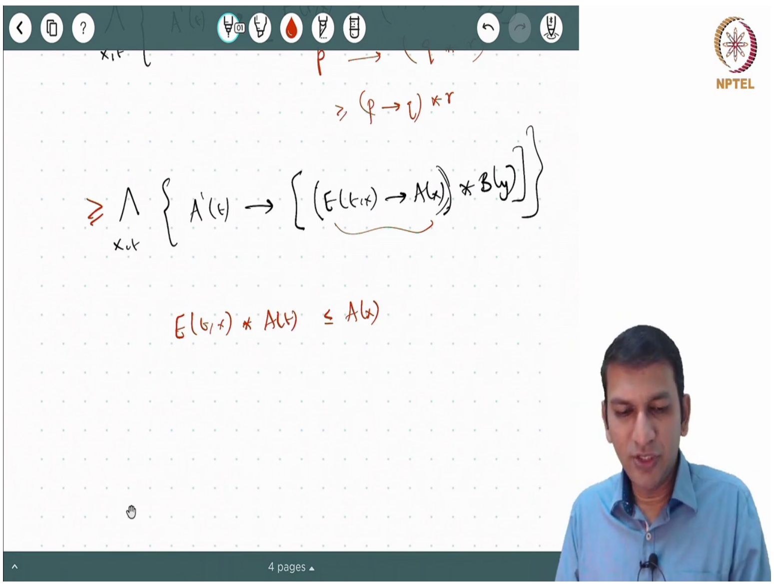

Now, we know that E of t, x star A t this less than or equal to A of x. Now, the star is essentially residuation, and star and implication are related to each other by residuation principle so, now, if you consider this as p, q, r. So, p star q is not equal to r in infimum is equal into p implies r greater than or equal to q and r. So, now, what we have here is E of E x implies A of x is greater than or equal to B of y. So, this entire quantity is greater than or equal to A of t.

Now, remember, this is essentially in the second variable, so the same inequality holds. So, this entire thing can be written as greater than or equal to inf over x, t, A dash of t implies E of t, x. This is what we are replacing it with A of t star B of y. Now, this is clearly equal to and over t, x vanishes A dash of t implies A of t star. B of y, which is essentially A dash delta I_T R check (Refer Time: 19:16). So, what we needed to prove was this inequality in the left hand side we have this.

So, we have shown that it is greater than or equal to right hand side, ok, which shows that the second equality is also valid. That means, as the case was with CRI inference, if you are considering a single SISO rule, and if the antecedent is extensional with respect to the equivalence relation that we are considering, the star equivalence relation we are considering.

If you take the implication to be the residual implication in the lat[tice] underlying residuated lattice, then both R cap and R check are valid relations to be used if you want robustness. So, BKS enjoys robustness when you relate the antecedents, of the antecedent of the rule with the consequent either using the corresponding T norm or the residual implication. BKS enjoys robustness. And note that this was clearly possible, because we had this repertoire of properties that residuated lattice offers us.

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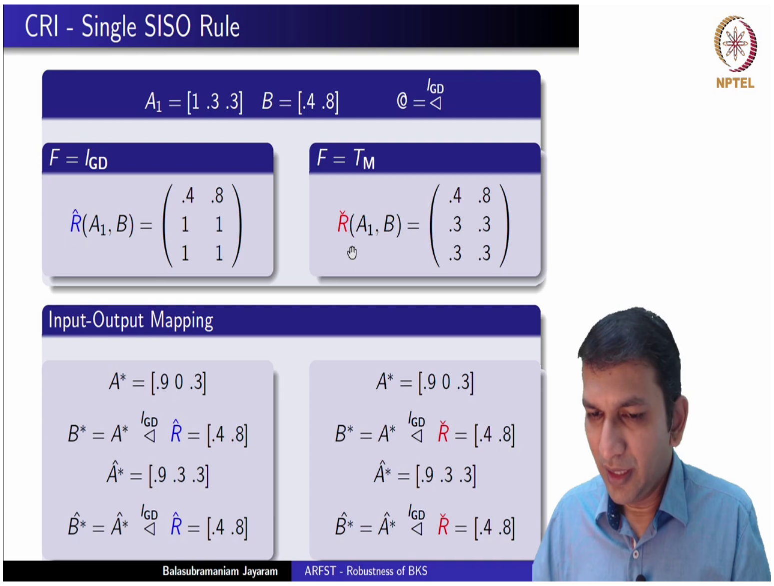

A quick example; let us look at the same single SISO rule that we saw in the CRI case. We will take it and then, but will apply the BKS here. So, once again instead of taking sup min composition, we are considering the Bandler Kohout sub composition with respect to the corresponding residual implication which is the Godel implication.

So, if you take the Godel implication, we know R cap is given as this. We have seen this, before too. And with respect to min this is the R cap. Let us try and calculate what the outputs are for the same pairs, the same pair of inputs that we have considered.

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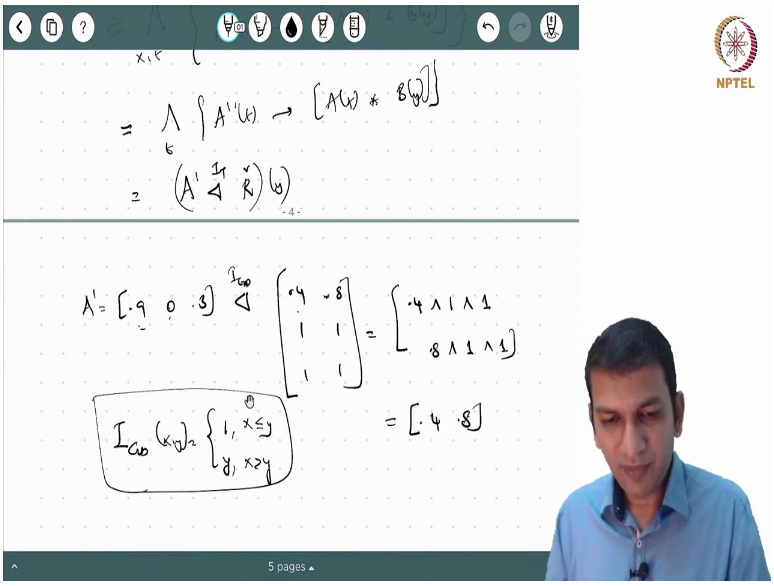

So, note that, we have A dash to be 0.9, 0, 0.3. We need to take the Godel implications with respect to R cap. R cap is 0.4, 1, 1, 0.8, 1, 1 and apply this. We are applying the Godel implication. Note that the Godel implication is given like this. 1, if x is less than or equal to y and y if x is greater than y. So, when you consider 0.9 and 0.4, it is 0.9 is not smaller than 0.4 and it is 0.4. 0 and 1 is 1, and 0.3 and 1 is 1.

The second component is 0.9 and 0.8, so it (Refer Time: 21:57) with 0.8. (Refer Time: 21:59) 1, 1, so the output is in fact, 0.4, 0.8. That is what we have recorded here. We know that its extensional hull is in fact, 0.9, 0.3, 0.3. So, let us try and work out what the values would be for this fuzzy set 0.9, 0.3, 0.3.

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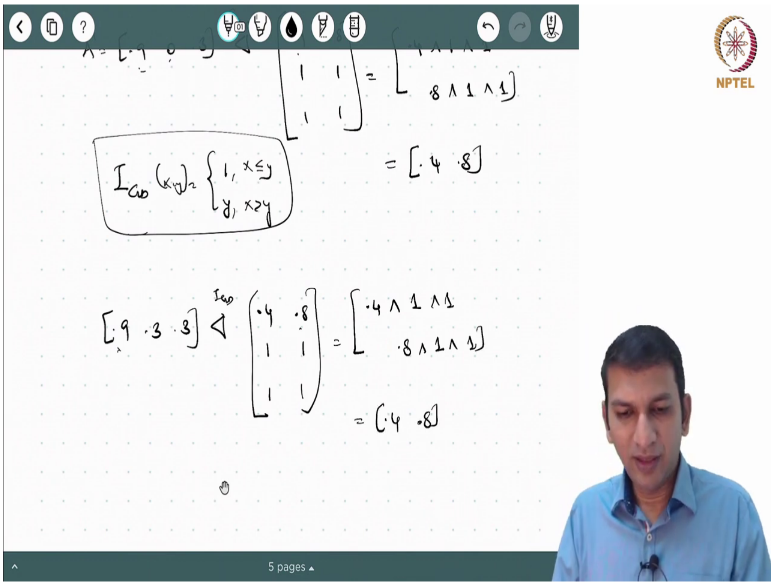

So, 0.9, 0.3, 0.3 (Refer Time: 22:24) 0.4, 1, 1, 0.3, 1, 1. Once again 0.9, 0.4 is 0.4, min with 0.3, 1 is 1; 0.3, 1 again is 1. 0.9 and 0.8, it give us 0.8 with respect to Godel implications getting the inf norm. So, 0.3 and 1 is 1. This is 1. What we see is this 0.4. Once again, if we consider the same pair of inputs with respect to R check, let us calculate this too.

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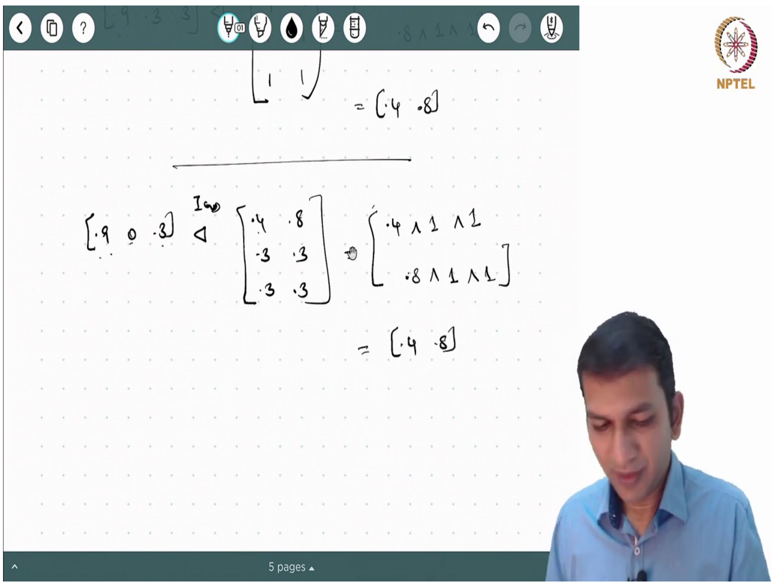

So, we have 0.9, 0, 0.3. Note that we are using the Bandler Kohout Subproduct inference. So, inf over the implication which is the Gradle implication here. And the relation is given as 0.4, 0.8, 0.3, 0.3, 0.3. Note that 0.9, 0.4 its Gradle is this 0.4. 0.3 is 1; 0.3, 0.3 is 1. Similarly, 0.9, 0.8 is 0.8, min with 0, 0.3 is 1; 0.3 0.3 is 1. Overall output is 0.4, 0.8. It can once again easily be seen that 0.9, 0.3, 0.3 will also give us 0.4, 0.8.

So, this is just to illustrate that if what happens in the CRI case, exactly the same thing happens in BKS case even when we change from the sup T composition to the inf I composition. Of course, the T and I, they are coming from the residuated lattice structure. Now, the question, is this is true for a single SISO rule case?

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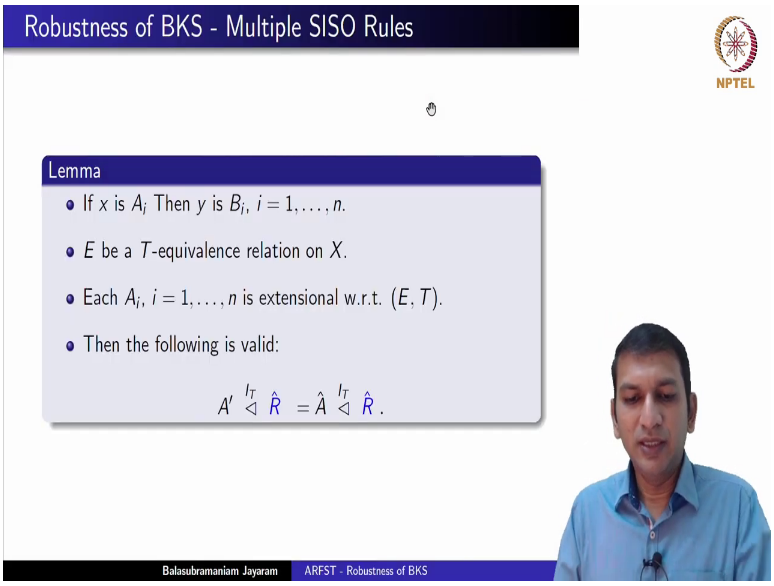

What happens in the multiple SISO rule case? Let us look at that. We have a set of if-then rules. If x is A i Then y is B i. And once again E is a T equivalence relation on X. Now, we insist that every antecedent A i is extensional with respect to this T equivalence relation mean. What we get is BKS in this setting is robust only if we consider R cap and not R check. Whereas, in the case of CRI we have to consider R check instead of R cap. Let us look at how to do this.

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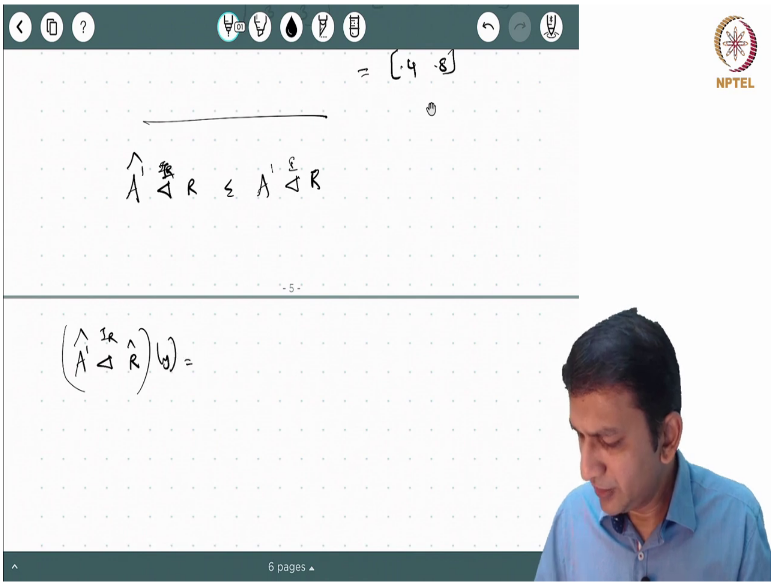

Note that if we have any R, any I, we know that this is less than or equal to A dash. So, what we need to prove is the other inequality. So, let us start with A dash cap delta I T R cap at a particular y this is equal to inf over x element of x A dash cap of x implies.

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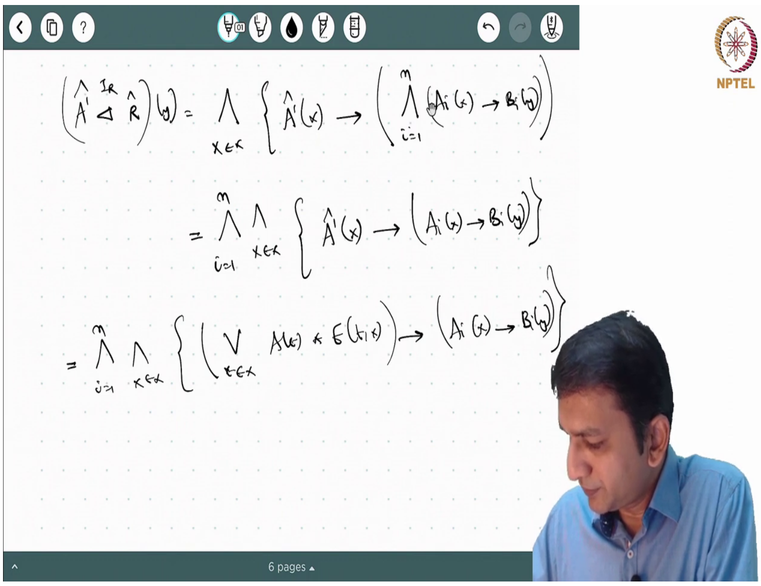

Now, note that R cap consists of many rules. So, it is inf over i is equal to 1 to n, A i of x implies B i y. Now, once again by distributivity, but this is infimum in the second component where it is non-decreasing. So, clearly this can be written like this pull it out. A dash cap of x implies A i.

Now, let us expand the extension hull of A dash x element of x, A t star E of t, x implies A of x implies B of y.

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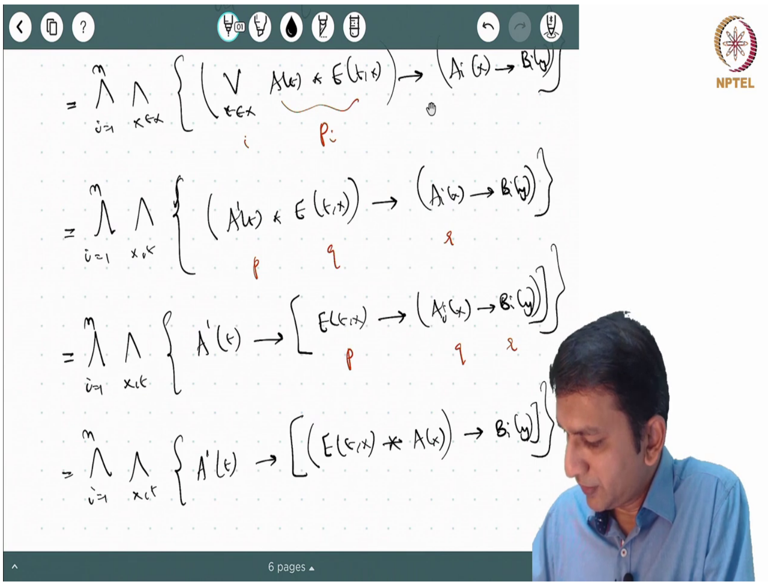

Now, once again considering this we taken this p i, I pull this out. This is i is equal to 1 to n x comma t, A of t star E of t, x implies A of x implies B of y. Well, now the law of importation comes into play, p star q implies r we know is p implies q implies r let us write it is equal to A dash of t implies E of t, x implies A i of x implies B i of y.

And once again you play the same trick, A dash of t implies taken this to be p, this to be q and r. We know that p implies q implies r is p star q implies r. So, we take that and write this as E of t, x star A of x implies B i of y, ok.

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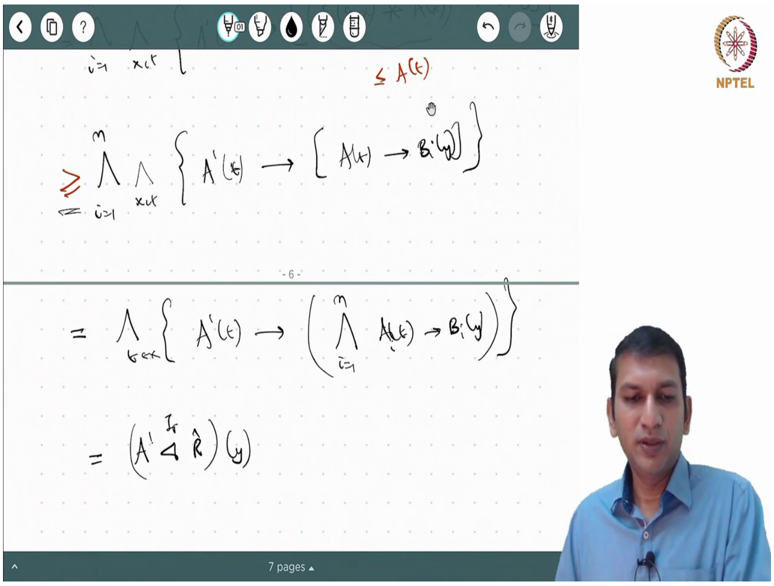

Now, note that this is less than or equal to A of t. However this appears in the first variable. So, when we append B i of y with an implication the inequality reverses, so this is greater than or equal to inf i is equal to 1 to n, inf of A dash of t implies A of t implies B i of y. Now, the inf can be pushed inside here. The second variable it is increasing. So, it does not matter.

And now you see that the free variable x has gone, so it is only over t element of x. A dash of t implies i is equal to 1 to n, A of A i of t implies B i of y. Here A is nothing but A dash delta I T R cap (Refer Time: 29:40) y. And this was the inequality that we needed to show. And we have shown that which means this result is valid.

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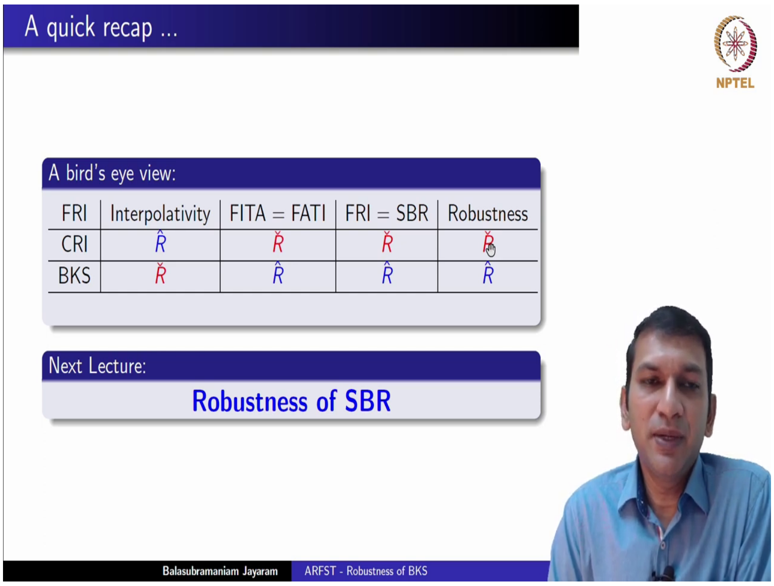

Well, a quick recap of the entire robustness that we have discussed with, with reference to the CRI in BKS in terms of some of the results that we have seen earlier too. If you consider these properties of an FRI interpolitivity, when is FITA equal to FATI, when can an FRI be seen as an SBR and when does it have a robustness, this is what we have. In the case of CRI, for interpolativity we want that R cap should actually be used.

So, we have seen that for interpolativity, in terms of fuzzy relational equations, that R cap should be a solution. Then, it is interpolative. Of course, there are many many other relations which can also give you interpolativity. We have seen that. But in the case of FITA being equal to FATI or when the CRI can be looked at as an SBR, when is it robust even in the multiple rule case, we see that it is R check that is playing the role. In the case of BKS, duality persists.

For interpolativity we need to check it with R check relation, whereas, for the other 3 properties of equivalence between FITA and FATI, and when this BKS can be looked at as an SBR or the fact that it will be robust, even in the case of multiple SISO rules, this happens only if we consider R check.

So far we have discussed the robustness of FRI mechanisms schemes, specifically CRI and BKS which are the two well-known ones. In the next lecture, we will look at discussing robustness of a similarity based reasoning scheme.

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The topics that we have covered in this lecture, once again, they can be found in this seminal work of Klawonn and Castro. So, the parts that are related to the Bandler Kohout of product, you could also refer to the work of Stepnicka and Jayaram. In the next lecture, we will meet to discuss the robustness of similarity based reasoning scheme.

Glad, you could join us for this lecture. Hope to see you soon again in the next lecture.

Thank you everyone.