Approximate Reasoning using Fuzzy Set Theory

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Lecture - 05
Fuzzy Sets - Some Important Notions

Hello and welcome to this lecture dealing with Some Important Notions on Fuzzy Sets, offered under the course titled Approximate Reasoning using Fuzzy Set Theory, a course offered through the NPTEL platform.

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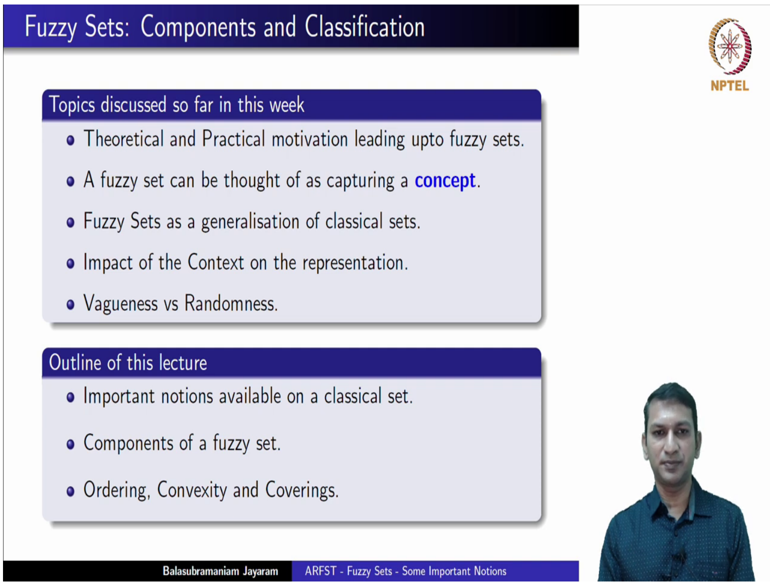

So, far this week we have seen some theoretical and practical motivations that lead up to the introduction of fuzzy sets. We have also understood that a fuzzy set can be thought of as capturing a concept. We have also seen fuzzy sets as a generalisation of classical sets and have come up with a useful and an appropriate representation of fuzzy sets.

And importantly we have seen the impact that context can have on the representation of fuzzy sets themselves. And finally, we have seen how fuzziness deals with vagueness while probability deals with randomness.

In this lecture, we will look at an appropriate generalisation of many of the concepts or attributes we associate with the classical set. For instant, given us classical set we could talk about whether it is bounded or not. Similarly, if you are given two sets classical sets, we could discuss whether one is contained in another.

So, some important components of fuzzy sets that is those attributes which are associated with fuzzy sets which will be useful and helpful later on for many purposes including classifying them. We will have a look at a few of them which are relevant for the course and the content of this course.

Specifically, we will look at ordering between fuzzy sets, the convexity of a fuzzy set, and fuzzy coverings over a space.

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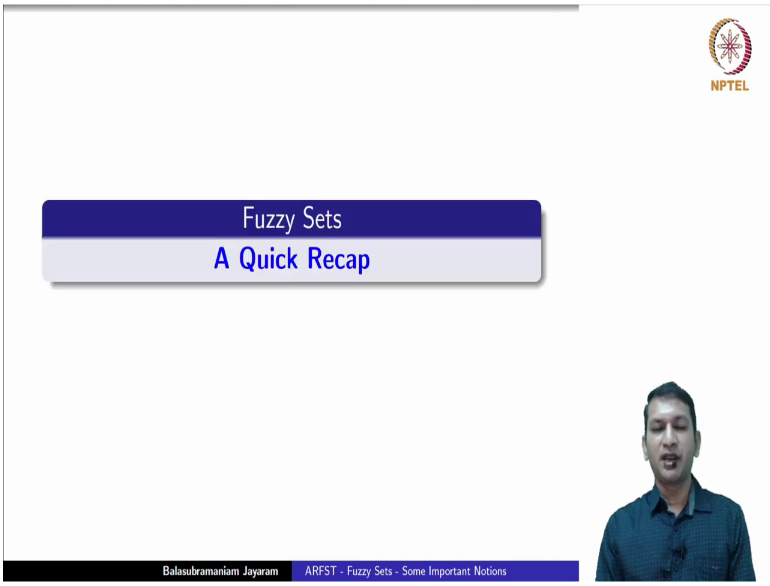

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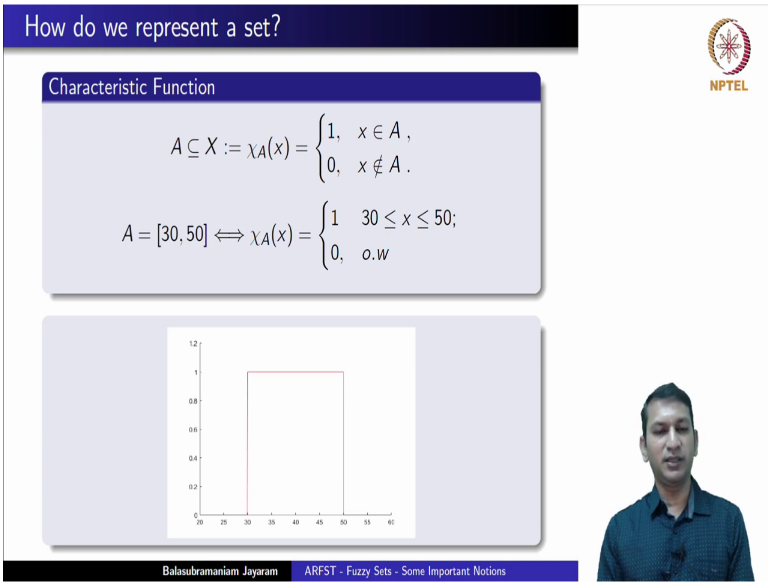

Let us have a quick recap. So, we know that a classical set can also be represented by its characteristic function. So, if we consider this interval, it can also be written through its characteristic function in this way and a graphical representation of that would look like this.

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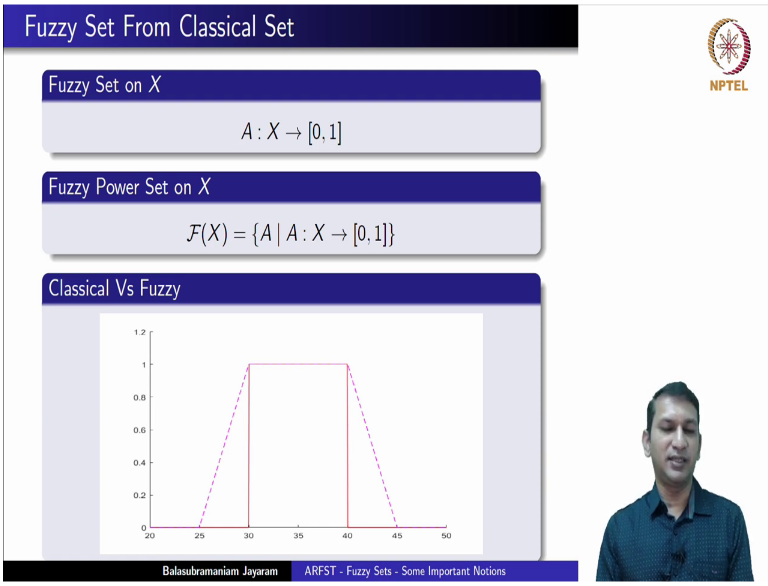

A fuzzy set is a function from an underlying domain x to the unit interval [0, 1]. We will denote by the script F of X, the set of all fuzzy sets that you could define on the underlying domain X. So, in a sense graphically a classical set might look like a rectangle, whereas a fuzzy set tries to dilate it.

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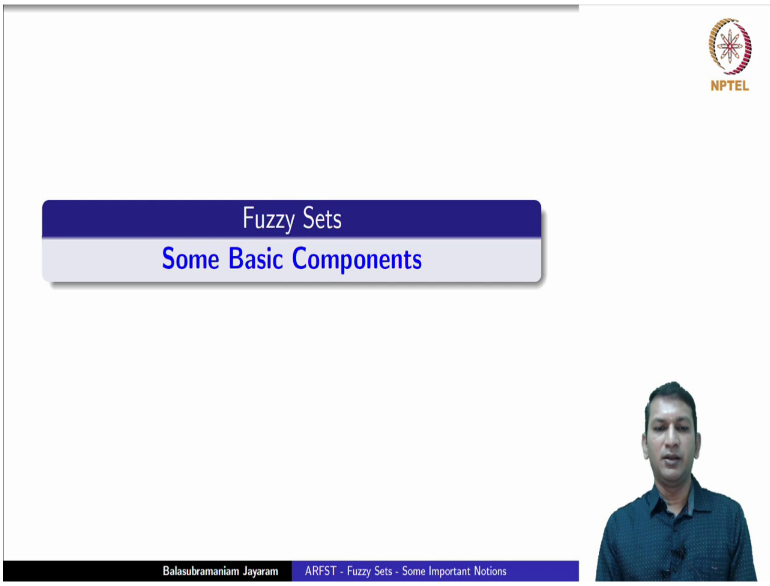

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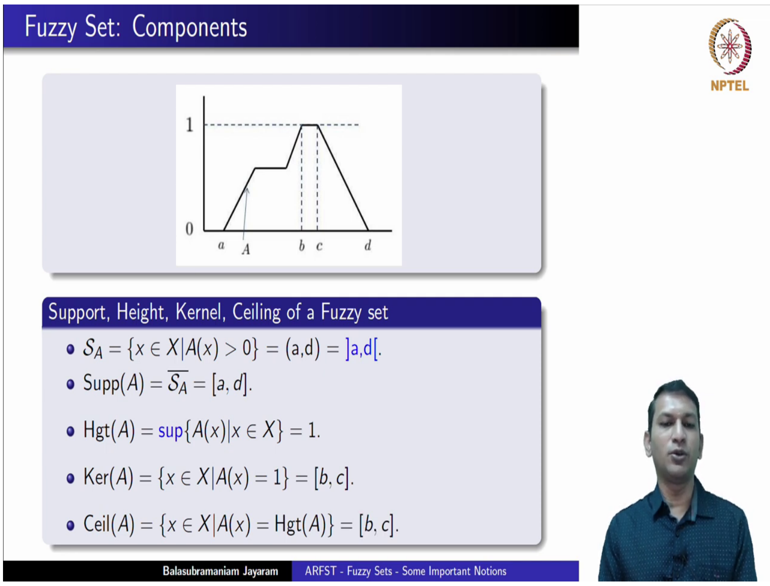

Let us look at this particular fuzzy set. What are the main or major attributes that we associate with a given fuzzy set?

Given a fuzzy set A, we can associate this set S_A with it where it contains all those points from x which have a non-zero membership value, that is all those x in X such that A of x is strictly greater than 0. Now, given the set S_A, in this case if you look at this figure then we are looking at a set which is whose membership values for any point below a is 0 and up to a is 0 and beyond d is 0 and at d also it is 0.

Now, given this fuzzy set A, S_A would look like the open interval (a, d); that means, all those points between a and d, but not including a and d themselves. This is one notation for writing open interval, (a, d). In books, you might often find this kind of a notation also. So, as to feel comfortable with both notations in this lecture I would use both of them. But later on when you see either of these, either in these lectures or in a book, I hope you will all feel comfortable with this notation.

Now, what is the support of a fuzzy set? Fuzzy the support of A fuzzy set is nothing but the closure of this set S_A. Now, what do we mean by closure? Well, it would mean to discuss some topological concepts. But for the present case let us look at this a, d as an open interval. Let us restrict ourselves to a domain which is nothing but the real number line. And in this case for this particular case where S_A is given by the open interval (a, d), we could consider the closure of S_A as also being able to add the end points.

In a later lecture or whenever the situation warrants or demands, we will try to look into these concepts from a more mathematical and deeper rigorous perspective. What is height of a fuzzy set A? It is nothing but the maximum membership value attained by any of the points from the domain X. Mathematically, we could write it as supremum of all the membership values that are taken over the domain X.

In this particular case, for this particular fuzzy set that we have on the screen it would mean 1 because you see that every point between b and c inclusive of b and c attain the values 1. And you should also note that we have used Supp, the concept of supremum instead of the maximum there is a subtle difference between supremum and maximum.

Once again we will not look into it currently, but in one of the later lectures when we deal with partially ordered sets and lattices we will look into this concept a little deeper for sure. For the moment, for this particular fuzzy set you could consider supremum to be equivalent just to the maximum. So, once again height of a fuzzy set is the maximum membership value that any point on the domain attains.

We can also talk about the kernel of a fuzzy set. The kernel of a fuzzy set is essentially picking out those points on the domain X, wherein they attain the membership value 1. So, in this particular case, you see that every point between b and c including b and c attain the values 1.

What is ceiling of A? Ceiling of A is set of all those points in X which attain the maximum membership value that they can attain in the fuzzy set A. So, essentially picking out all those points X whose membership value is equal to the height of A in this case. Once again it turns out to be the closed interval b, c.

A few points to be highlighted this kernel and ceiling may not always be same, unless the height of A is 1. Note that kernel ceiling and the support they are all subsets of X. So, in that sense they are classical sets whereas, height of A is just a single value in 0, 1 because it talks about the maximum membership value a fuzzy set can attain.

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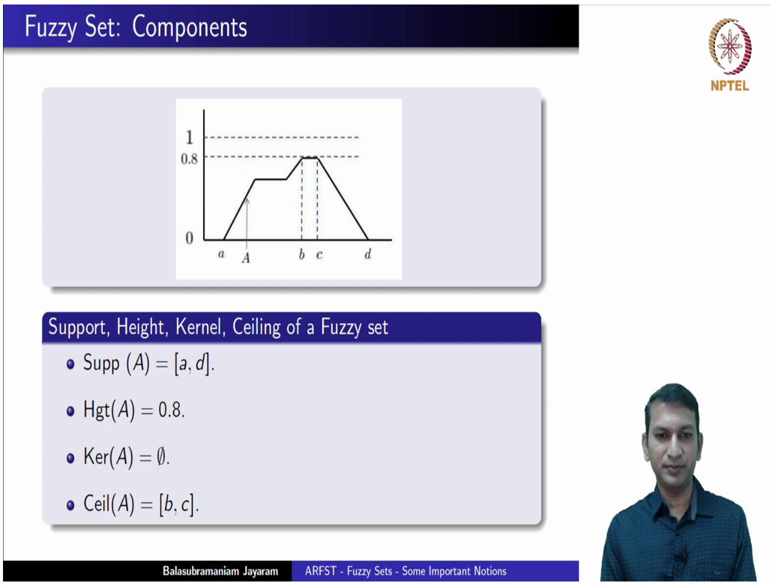

Let us look at another example. For this particular fuzzy set, if we were to find out what its support height and other attributes are, it would look like this. Support of A would still remain a, d, remember, all those points which take non-zero membership you take that set and you take the closure of it. In this case as we have seen it is adding the end points to the interval.

The height of here now we have seen that by definition it is the maximum membership value attained by any point from the domain or specifically from the support of the fuzzy set and you are here you will see it is actually not 1, but 0.8. Now, what about the kernel of A? Remember, the kernel of a fuzzy set a set of all those points which attain the value 1, in which sense you could think of them as prototypical elements belonging to that concept which is represented by A.

In this case, as you will see it is empty. The ceiling of A is set of all those points which actually attain the height of the corresponding fuzzy set. So, in this case, you will see that that remains to be the closed interval [b, c], ok.

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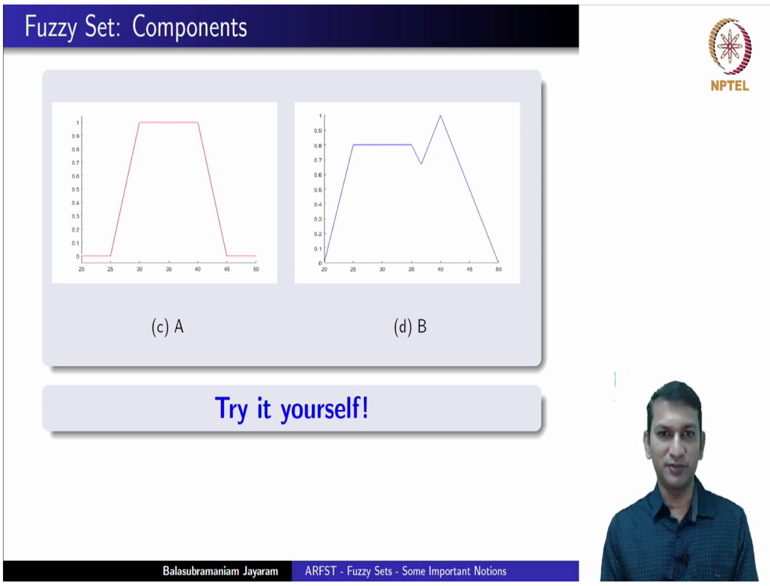

So, let us consider this these two fuzzy sets, a and b and let us try these things by ourselves. Take a moment. Please work out what the support, height, kernel and ceiling, for these two fuzzy sets could be, and within a few minutes we will compare notes. Please try it out yourself.

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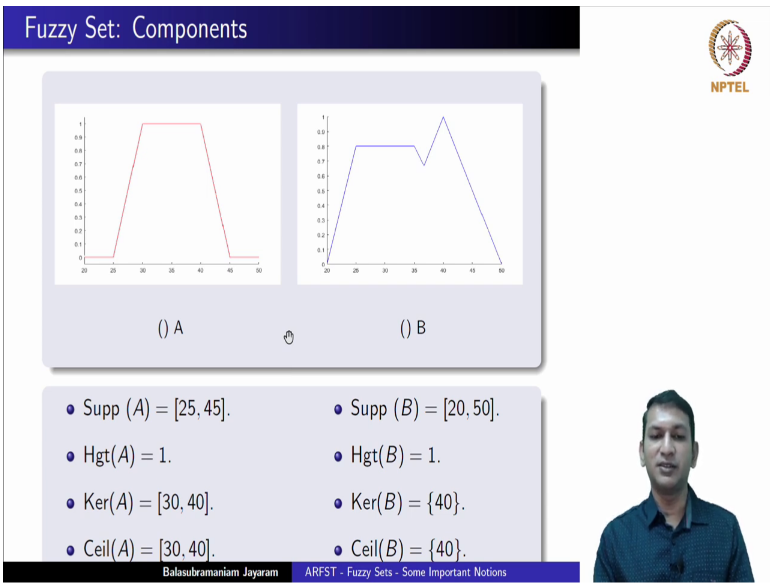

Well, I hope you have had enough time to work out these attributes. Let us look at the support of A. Support of A is clearly between 25 and 45. Note that we are looking at the closure of the points which will have non-zero membership to the set A. Once again this will turn out to be the closed interval 25 to 45. The height of A in this case there are points, which attain the membership value 1. So, the height of A is 1 here.

If you look at the kernel, once again as you could probably see it is between 30 and 40 and so would be the ceiling of A. If you look at the fuzzy set B, the support of B is almost the entire domain that has been shown here between 20 and 50. The height of B once again is 1. The kernel of B in this case is actually just a singleton set at the point 40. And so will be the ceiling of B. We hope you got all of you have got these values, right.

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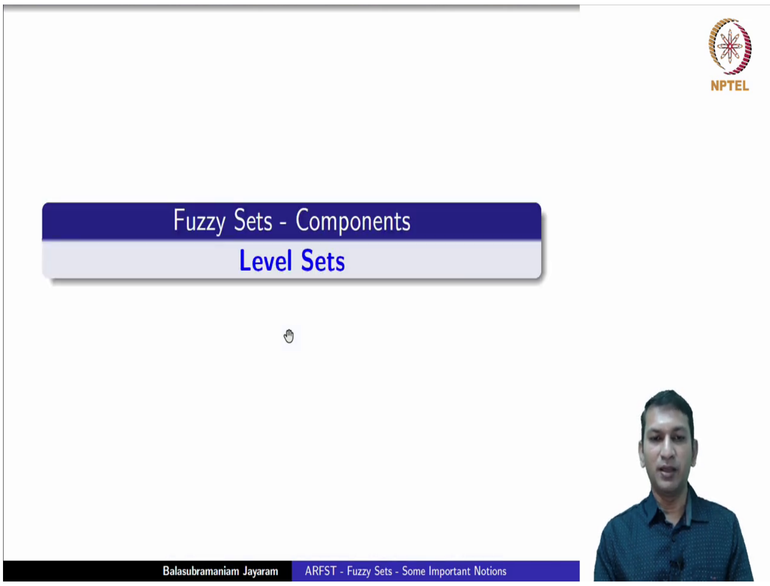

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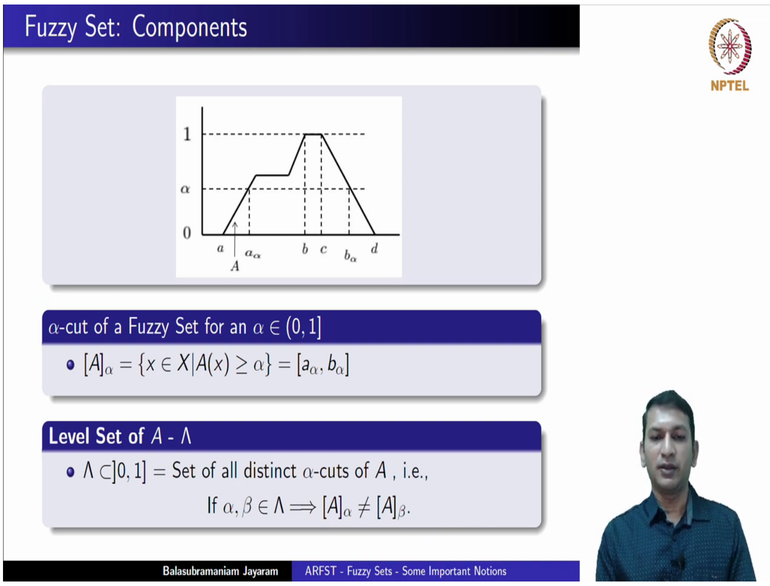

Let us look at another very important concept that of level sets. Consider this fuzzy set. We discuss the notion of an alpha cut for a fuzzy set and these alphas vary between the open 0, close to 1 interval. And you would immediately see why we need to restrict this interval once you have seen the definition of an alpha cut.

So, given a fuzzy set A, by this notation we will denote an alpha cut of A. What is an alpha cut of A? It is a set of all those points in x such that its membership value is greater than or equal to alpha. So, immediately you should see that the alpha cut of A is actually a classical subset of the domain X. It consists of all those points which take a membership value of greater than or equal to alpha in the concept represented by the fuzzy set A.

So, if we look at this particular fuzzy set that is flashed on your screen then we will see that it is actually the closed interval a alpha b alpha. Remember, we are considering those points whose membership values are greater than equal to alpha. Now, what is the level set of a fuzzy set A? It is typically denoted by lambda.

It is a proper subset of open 0, 1 and it is the set of all distinct alpha cuts of A. What do we mean by that? That is if you put an alpha or beta inside lambda only if for each of those values that you put, the corresponding alpha cuts are actually distinct. Remember, lambda consists of the alphas coming from the interval open 0 close to 1, in that sense you are picking out unique values of alpha for which the corresponding alpha cuts are distinct.

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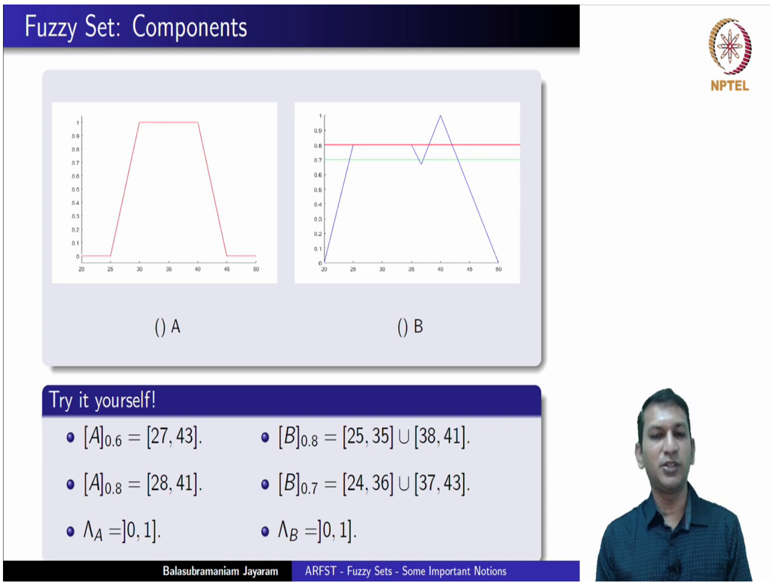

Let us look at some examples. Once again let us look at these two fuzzy sets A and B. Let us try on our own. So, let us look at the value 0.6 and the corresponding alpha cut for this. So, essentially we are looking at taking this value 0.6 and seeing over all the elements whose membership values are greater than or equal to 0.6. From this figure, we hope that you will be able to make this out.

Once again, take a couple of moments and then we will get back with the answers and at that time we will compare notes. We hope you have had enough time to evaluate these attributes. Let us compare notes. To help us in this let us have these two lines indicating the alpha cuts for B at both 0.7 and 0.8.

Now, it is clear from these graphs, the alpha cut at 0.6 for A is 27 to 43, the interval 27 to 43. The alpha cut of A at 0.8 would look like somewhere between 28 and 41. And the level set, as you can see this fuzzy set is a continuous function and so it would take all the values between 0 and 1 inclusive of 1 and by definition we leave out 0.

Now, if you look at the alpha cut at 0.8 for B, it is clear that it consists of two intervals 25 to 35 and 38 to 41. Similarly, the alpha cut of B at 0.7 also is a union of two intervals. Once again the level set consists of all the points between 0 and 1, inclusive of 1, but not 0.

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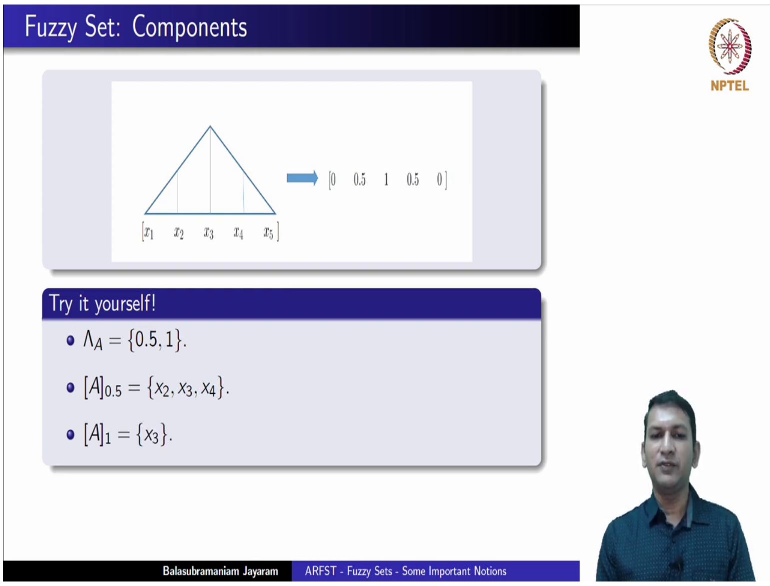

Let us look at the same concepts for a discrete fuzzy set. That means, a fuzzy set which is defined over a domain which has only finite number of points. So, let us try this yourself; perhaps this is going to be easy for you. So, if you look at the fuzzy set A that we have on the screen, the level set consists of all those alphas greater than 0, which will lead to distinct alpha cuts.

It is clear from the representation of fuzzy set written as a vector, that there are only two distinct values greater than 1, which means the level set would be both 0.5 and 1. The corresponding alpha cut for 0.5 would be all those points which take a membership value greater than or equal to 0.5, which in this case clearly translates to taking the points x_2, x_3 and x_4. And in the case of level set of one it essentially finds the kernel of the corresponding fuzzy set. In this case, it is just the point x 3.

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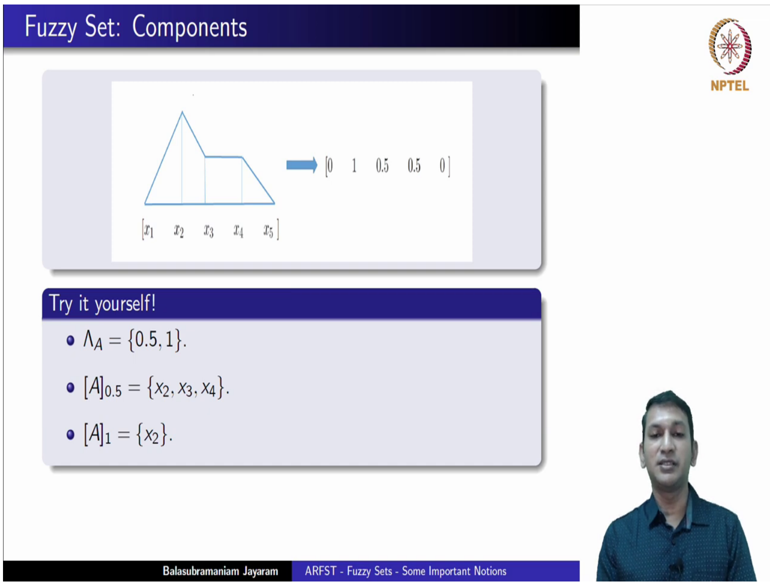

If you consider another fuzzy set, let us take a moment to find out what the level set of A is. I am sure you would have got this to be the set with 0.5 and 1. Immediately, the alpha cut of A at 0.5 would be once again the points x_2, x_3, and x_4. However, the kernel of this fuzzy set would differ from the previous one. It would attain what we call normality, the point at which it takes the value one at the point x 2.

Well, in the rest of the lecture we will see the important role that this concept of alpha cut and level sets would play.

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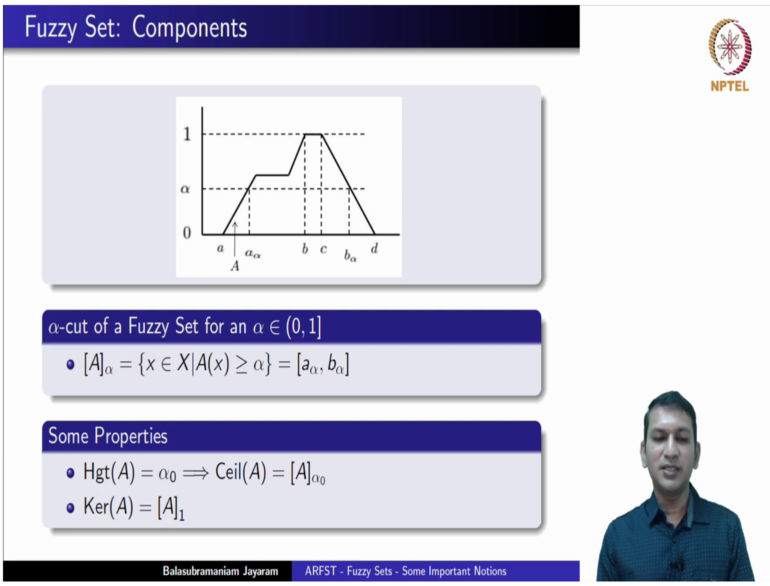

So, we know the definition of an alpha cut is the set of all those points in the domain X which take the value of membership to be greater than or equal to alpha. Let us look at some interesting theoretical properties that we can claim from them. Clearly, the height of A if it is to be some alpha naught, then we know that the ceiling of A is nothing but the corresponding alpha cut of A. The kernel of A is nothing but the alpha cut at the value alpha is equal to 1.

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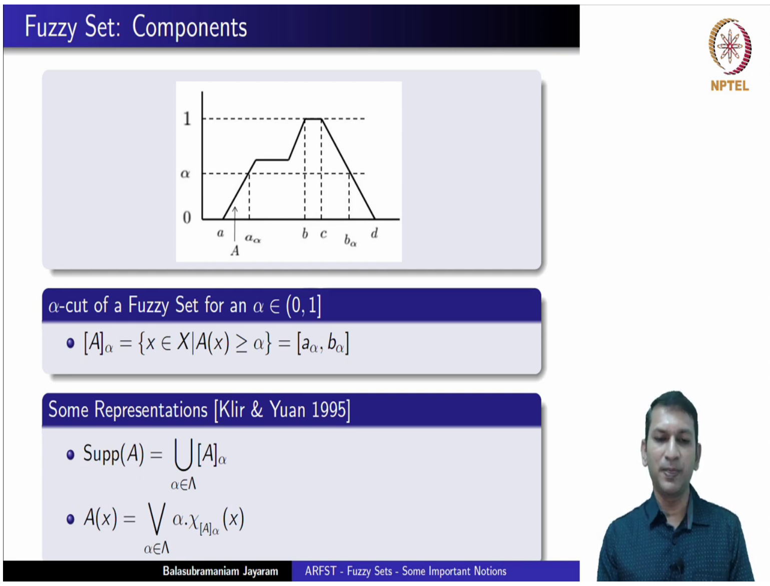

These are pretty simple properties. However, we can even give some very useful representations with this concept of an alpha cut. Once again, if you look at support of A, it can be written as the union of all the alpha cuts for the alphas coming from the level sets. Remember, the level set will only consider will only consists of alphas which lead to distinct alpha cuts. So, you can look at this as some kind of resolution formula for the support of A.

We could also represent the fuzzy set itself in this form where the supremum here, if the lambda is uncountably infinite set then you look at it as supremum, otherwise the symbol v there translates to just a simple max if lambda is discrete finite. We will not look into this, these two representations in detail, but you can find more details about this in the book by George Klir and Bo Yuan.

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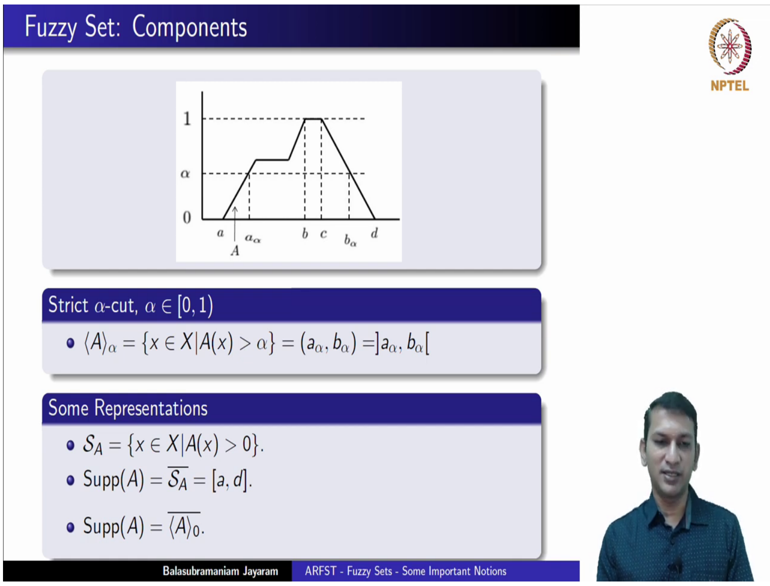

Now, we can also talk about strict alpha cut. That means, all those points which take a membership value strictly greater than alpha. So, in that sense, you will see that alpha now can belong to the entire unit interval including 0, but excluding 1. So, in this case, you the strict alpha cut of A at alpha would look like a open interval a alpha, b alpha. Once again, using this strict alpha cut, we can write support of A as simply the closure of the strict alpha cut at 0.

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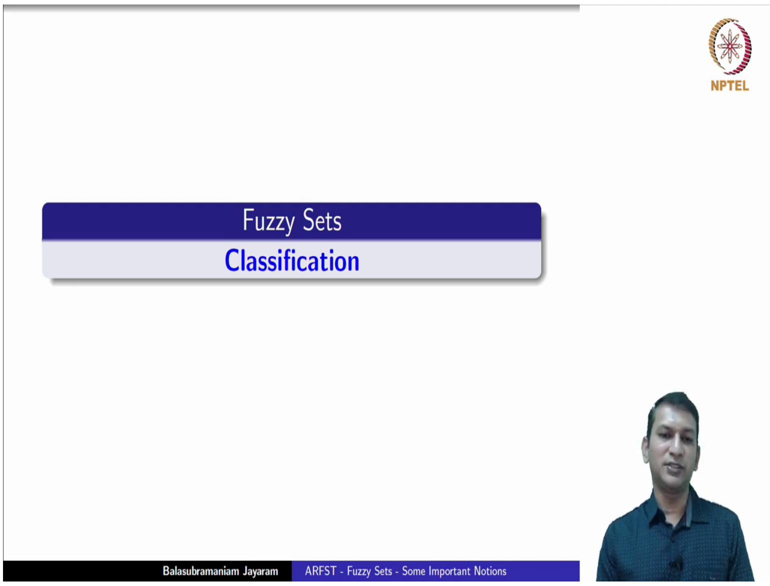

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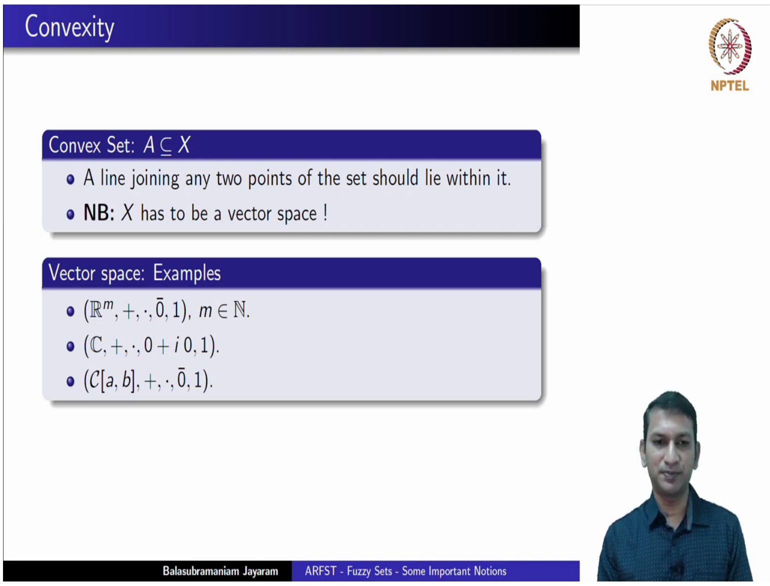

Let us look at some interesting classifications of fuzzy sets. We know what a convex set is in the classical case. A set A, a subset of X is called convex if any line joining the two points contained in the set lies within it. That means if you take two points within the set and draw a straight line connecting them then the entire line should lie within the set.

However, remember this means the underlying set X from which you pick up this A should originally be a vector space. We all know what a vector space is. Some examples of vector space, so that you would readily understand, what is the structure that we are looking at.

So, any finite dimensional Euclidean space is a vector space. This is nothing but the Cartesian product of real line taken m times. The usual complex plane is also a vector space. If you consider the set of all functions defined over the closed interval a, b, continuous functions closed defined over the closed interval a, b, they also form a vector space.

But largely in this course for us we probably would be only talking about fuzzy sets defined on R or R power m and so we normally will have a vector space readily available on the domain and so the fuzzy set that we will discuss about we can discuss the corresponding convexity of such fuzzy sets or not.

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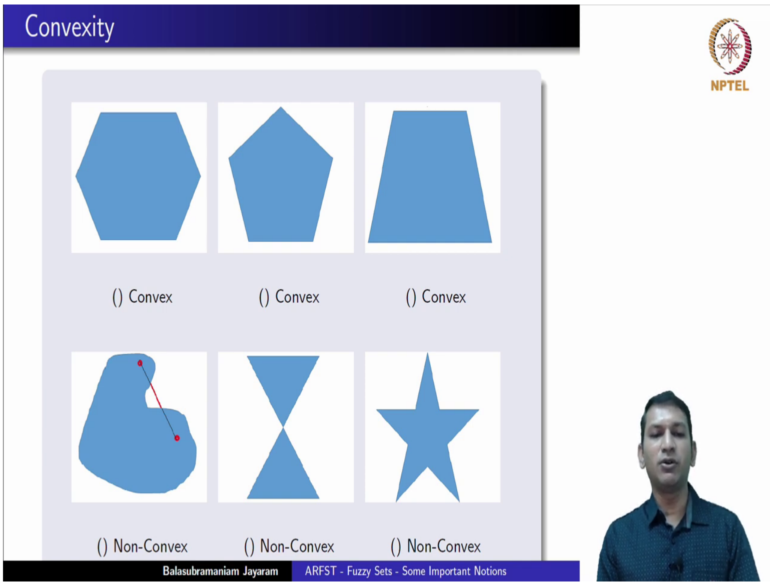

Let us look at some examples of convex and non-convex sets. So, clearly these are some sets that these geometrical shapes. When you consider them we know that these are convex sets. Some examples of non-convex sets are these. And why are these non-convex? At least for instance, if you take this these two points from this set and join them with a straight line you see the middle part highlighted in red does not belong to the set itself.

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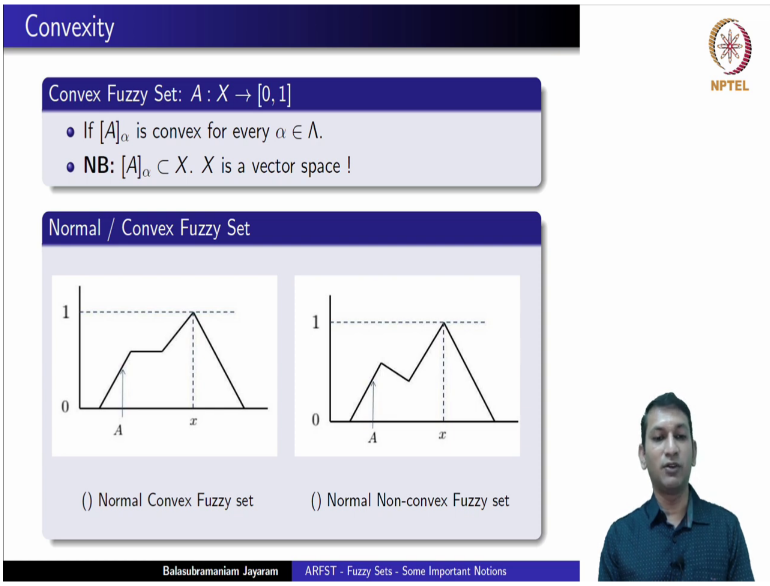

How do we generalize this concept to that of fuzzy set? What is the convex fuzzy set? Remember fuzzy set is a function A from X to [0, 1]. We say a fuzzy set is convex if for every alpha coming from the level set the corresponding alpha cut is also convex. Remember alpha cut is subset of the domain X. So, we can talk about convexity of alpha cut which is subset of X. Once again note that being a subset of X when you want to talk about convexity of this subset X has to be a vector space.

Now, let us look at some examples of convex fuzzy sets. If you look at this, this is a convex fuzzy set. This is a non-convex fuzzy set. Now, why do we say that the figure on the left is a convex fuzzy set? Because if you take any alpha cut, you will see for this value of alpha the alpha cut is the closed interval from this point to this point, which is convex set because it is an interval, so is the case here.

And you will see that pretty much every alpha that you can consider from the level set all of those alpha cuts will be convex subset of X. And this is an example of a normal fuzzy set. As you know normal means there is some value there is some point in the domain X, which assumes the maximum membership value of 1.

In this case, we have a normal fuzzy set. There is a point which attains the value 1, but this is not convex because there exists some alpha for whom the alpha cut looks like this. As you can clearly see there is an inter, it is a union of two intervals and hence it is not convex.

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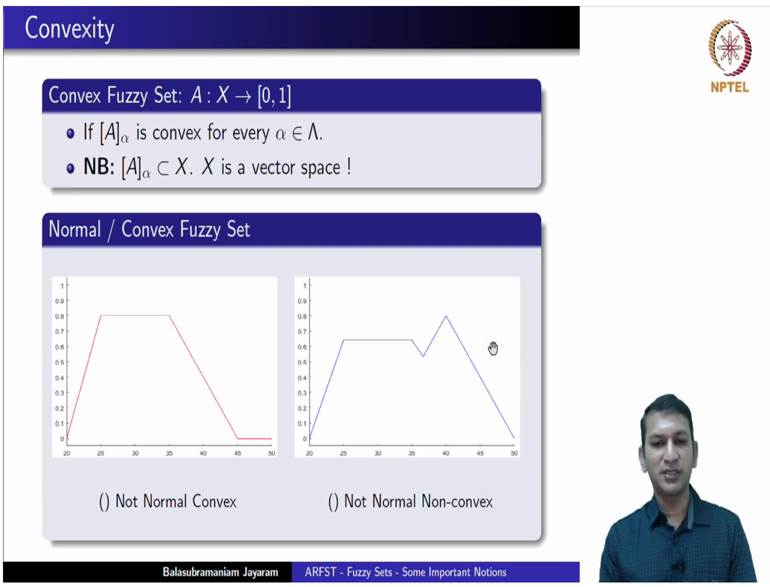

To show that normality has nothing to do with convexity that these are two independent concepts, this is a convex fuzzy set, but which is not normal and whereas, what we have here is set fuzzy set which is neither normal nor convex.

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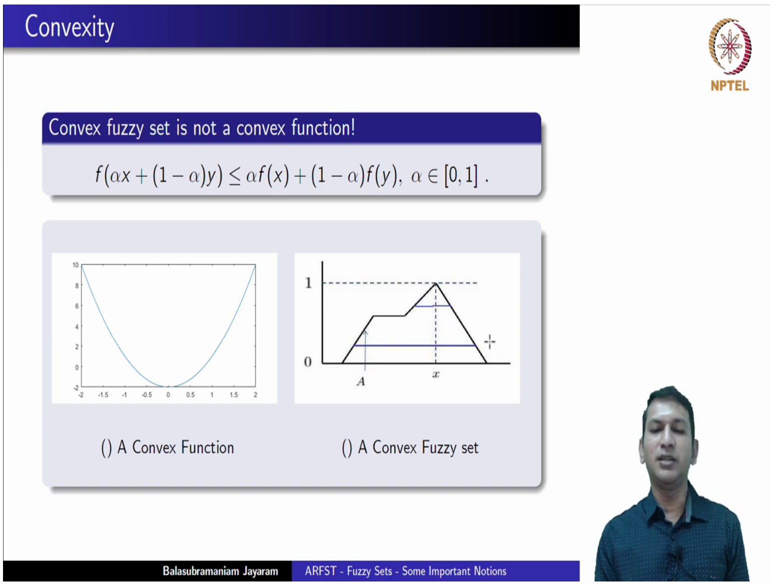

We know a convex fuzzy set is also function on a domain, which is a vector space to the unit interval 0, 1. So, it immediately begs the question is a convex fuzzy set also a convex function. Now, a function is said to be convex if this inequality is held for every alpha between 0 and 1. Now, geometrically what does it mean take two points x and y on its domain and consider the function values at x and y. Join them by a straight line. If the chord on the curve lies above the curve itself, on the corresponding sub-domain then we say that this is a convex function.

However, you will see this is no more true for a convex fuzzy set. For instance, if you take up a this particular alpha and consider the corresponding alpha cut. Now, if you join the values the membership values at these two points x and y by a straight line you will see the graph over this particular sub interval of this fuzzy set lies actually above the chord joining these two points.

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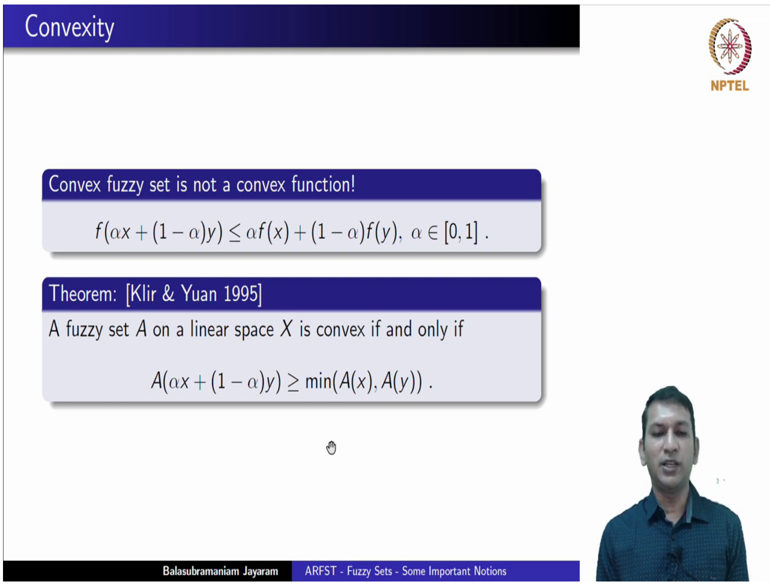

However, we still are able to find a relation of this type if a fuzzy set A is defined on a linear space X; we say it is convex if such an inequality is valid. Once again alpha comes from 0, 1. What does this inequality say? It says that if you take two points x and y and draw a straight line joining them and consider any point on this straight line, you will see the membership value of that point will always be greater than or equal to the minimum of the membership values at the end points x and y.

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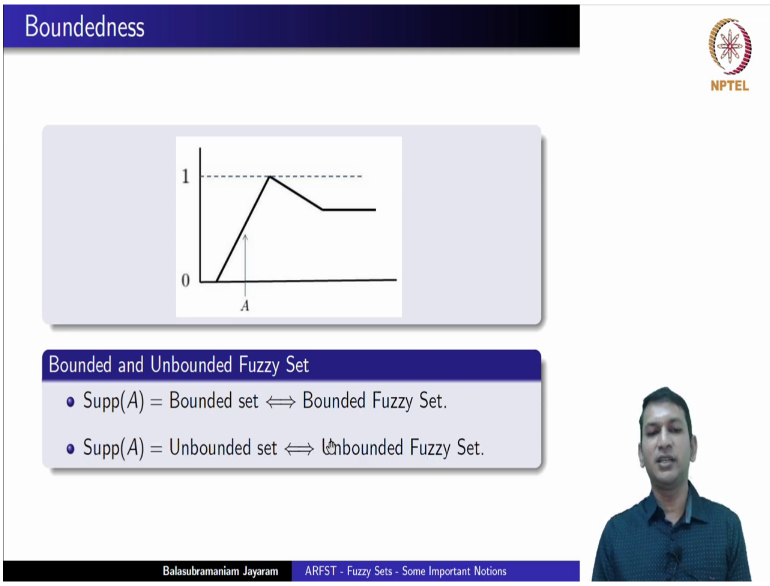

Given a classical set we could talk about whether it is bounded or not. Similar concepts can also be discussed in the case of fuzzy set. So, we say a fuzzy set is bounded, if its support is bounded and we say it is unbounded, if its support is not bounded.

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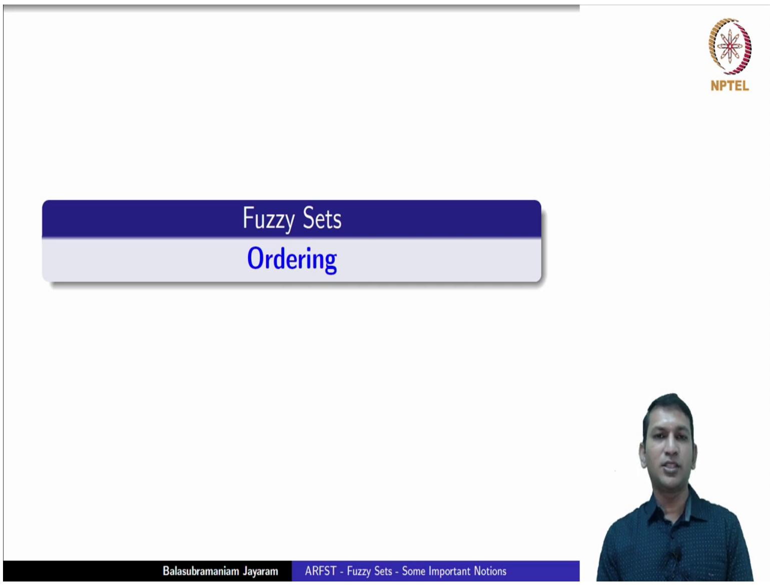

Well, let us look at another attribute that you could discuss in the case of classical sets, that of ordering between a given pair of sets.

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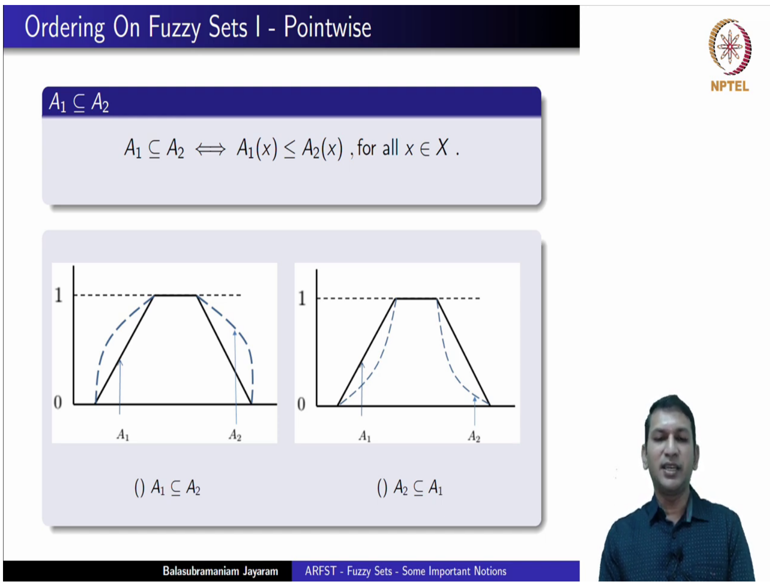

A fuzzy set as we know is a function from the domain to the unit interval 0, 1. So, the first kind of ordering that you could immediately think of is the point wise order. So, given two fuzzy sets A_1 and A_2, we say that A_1 is contained in A_2 if and only if the corresponding membership values that A_1 takes is smaller than the values taken by A_2. Now, if you think about it this is exactly the way you could define subset hood even in the case of classical sets when A_1 and A_2 are interpreted in terms of their characteristic functions.

Let us look at some simple graphical illustrations. Given these two fuzzy sets A_1 and A_2, it is clear the membership value of A_1 at every x on its support is smaller than the values that the same point take at over A_2. In this case the roles are reversed; A_2 is contained in A_1.

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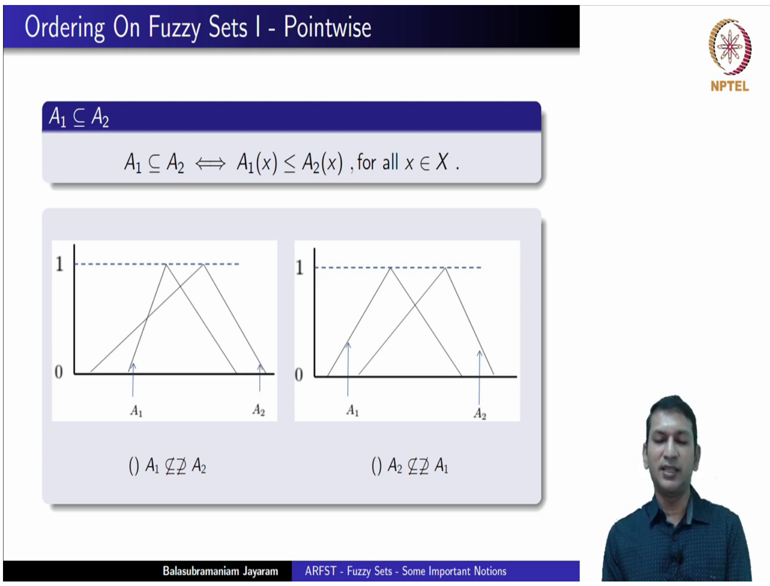

Now, if you consider this pair of fuzzy sets, it is clear neither A_1 is contained in A_2 nor is A_2 contained in A_1 and so is the case with this pair of fuzzy sets on the right of your screen. However, you suspect that there is some kind of a regularity between A_1 and A_2, the pair of fuzzy sets A_1 and A_2 on the right of your screen and you would like to still say that there is some kind of an ordering between these two.

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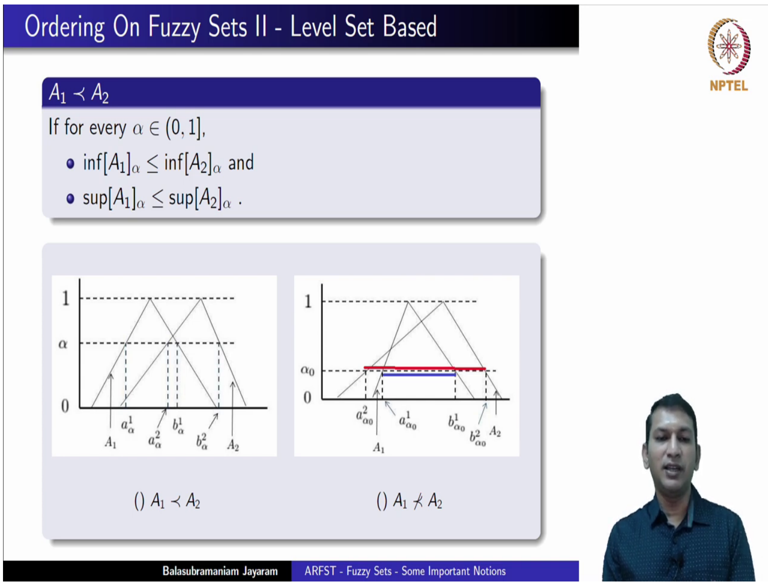

(Refer Time: 29:06) a new ordering on given pair of fuzzy sets was proposed. We say two fuzzy sets A_1 and A_2 are relatable under this ordering which is represented by the symbol. If for every alpha if we take the corresponding alpha cuts from A_1 and A_2, these two inequalities should hold.

We will look at deciphering these inequalities with the help of some graphical illustrations. Consider these two pairs of fuzzy sets. In the pair of fuzzy sets that you are on the left, let us consider for a particular alpha the corresponding alpha cuts. So, this is the alpha cut for a alpha; that means, all those points that lie under this red line which is essentially the interval a alphA_1, b alphA_1 and for the same alpha the alpha cut that you obtain for from A_2 is the interval a alphA_2, b alphA_2.

Clearly you see that a alphA_1 is smaller than a alphA_2, so is b alphA_1 smaller than b alphA_2. As was mentioned earlier for the current context, we can assume infimum and supremum to be the corresponding minimum and the maximum. So, in that sense, they actually become the left end points and the right end points of the corresponding alpha cuts since we are dealing with intervals.

Now, look at the pair of fuzzy sets on the right. Now, if you take this particular alpha naught you will see that the alpha cut for A_2 is given by a alpha naught 2 and b alpha naught 2 and that of A_1 is a alpha naught 1, b alpha naught 1. The interval found by these two points as the end points. While b alpha naught 1 is less than or equal to b alpha naught 2, we see that a alpha naught 1 is not less than or equal to a alpha naught 2.

Remember, the definition says for every alpha the above inequality should be valid. In this case we have found at least 1 alpha naught where in this inequality fails, this pair of inequalities, not both of them hold which means these two fuzzy sets are not orderable under this definition. Once again note that for the pair of fuzzy sets you have on the left, the above inequalities are valid for every alpha in the open 0 closed 1 interval.

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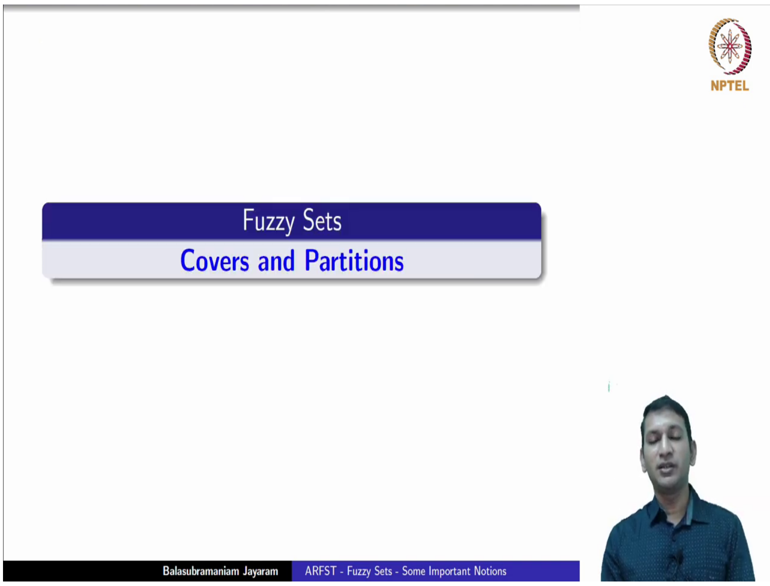

Finally, let us look at these important notions of covers and partitions.

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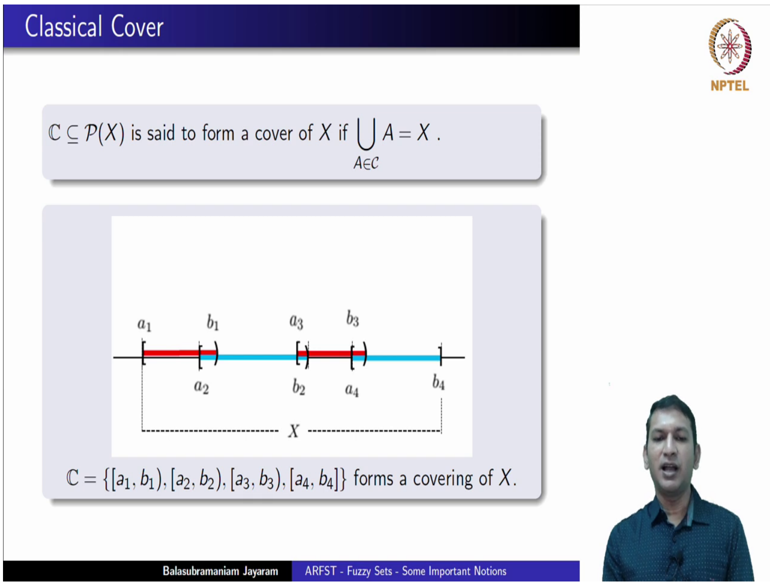

Consider a classical set X and let us pick up some subsets of X. We say it forms a cover if its union contains X or is equal to X. For instance, consider X to be the interval A_1 to b 4 and you will immediately see that these 4 subsets of X, which are sub intervals of X actually cover the entire domain X.

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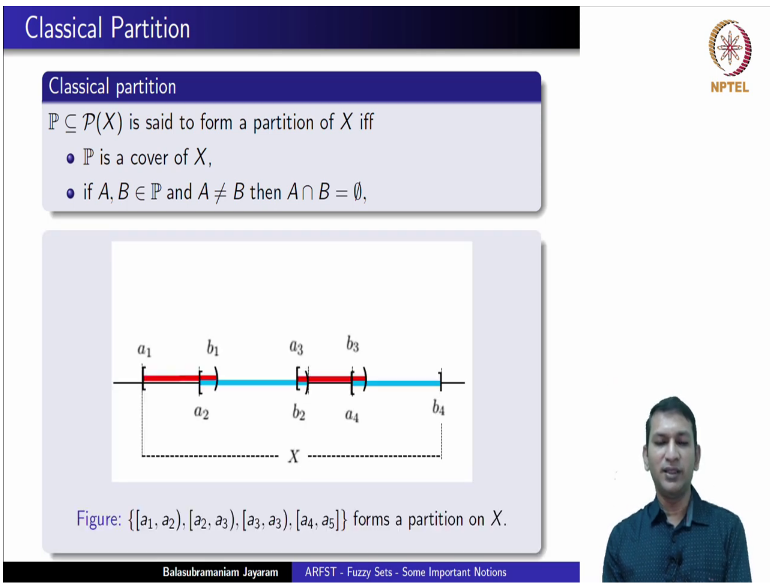

The partition of X differs from that of a cover, in the sense that it should be a cover of course, but the pieces in the partition, the subsets that you consider should also be disjoint.

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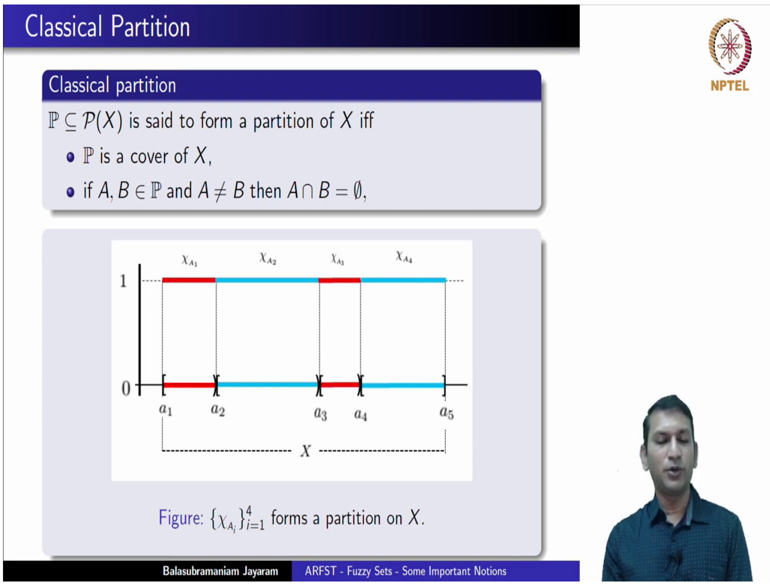

For instance, while this forms only a cover, we see that this collection of sets which are sub intervals of X, they actually form a partition. If you were to plot them in terms of the characteristic functions, you would see that this is how they would appear. Taking you from this, we could talk about a fuzzy partition of X.

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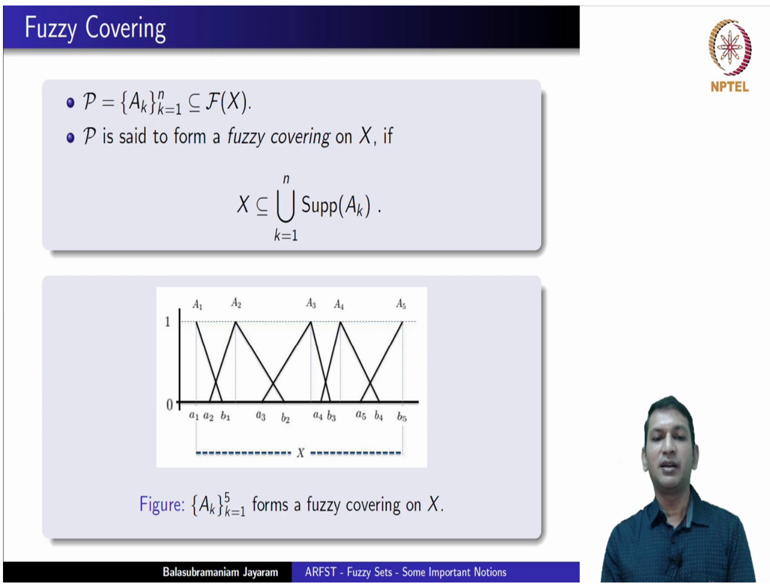

Let us start with a classical set X and consider a collection of fuzzy sets on X. We say this collection of fuzzy sets is a fuzzy covering on X, if you consider the corresponding supports of each of these fuzzy sets and take their union it should contain X. Note that a fuzzy covering on X is a collection of fuzzy sets on X, while x is classical set, the collection of sets is that of fuzzy sets on X. Consider this X and you will see that if you take these 5 fuzzy sets, they actually form a fuzzy covering on X.

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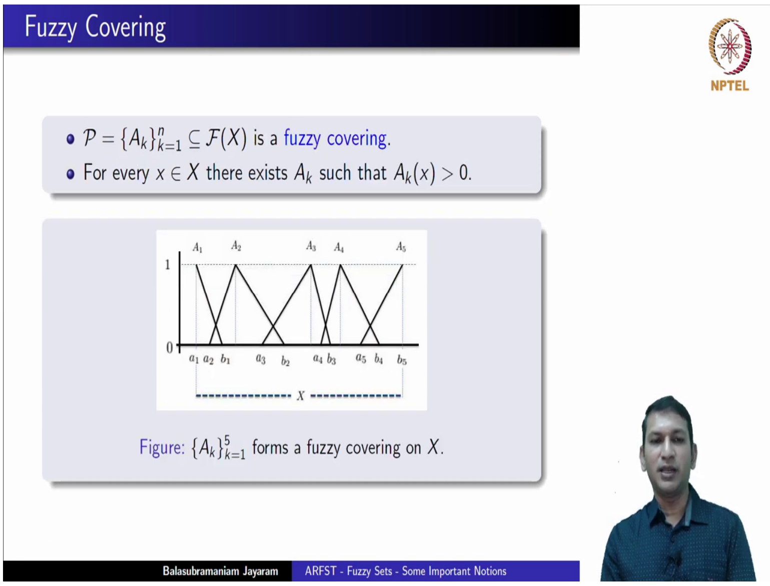

Note that in looking at it another way, a fuzzy covering on X means you have a collection of fuzzy sets such that every point in X has a non-zero membership value to one of these fuzzy sets in the collection. So, you will see that for this particular X, a collection of 5 fuzzy set that we have considered they actually satisfy this property. Any X you take there exists some fuzzy set to which it belongs to a degree greater than 0.

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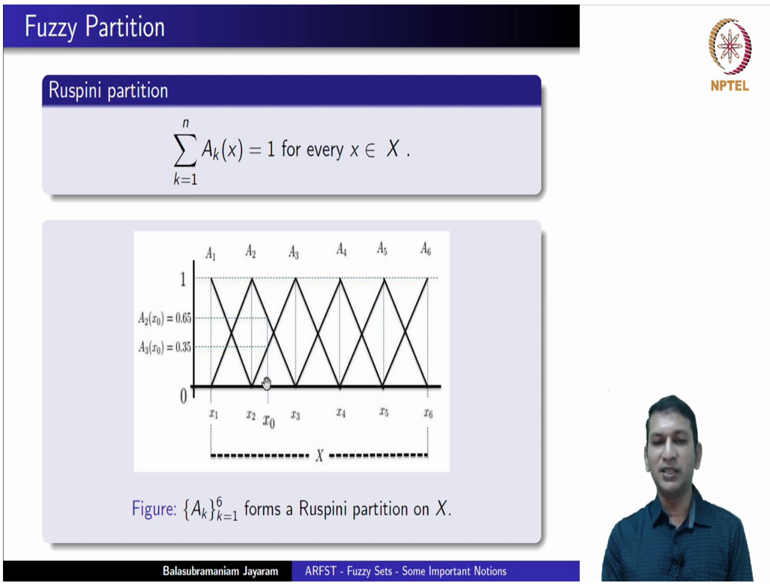

There are notions of fuzzy partition that have been defined very rigorously, but it would need us to look into some concepts which you have not introduced so far. So, for the moment we will only restrict ourselves to looking at what is often or popularly known as Ruspini partition.

Ruspini partition is once again a collection of fuzzy sets on X, which forms a cover along with it we ask for this particular property. What does this property say? If a collection of fuzzy sets is a cover we know that for any x in X, there exists a fuzzy set in this collection which has a non-zero membership to this one. In this case, we say that for every X, if you pick up the corresponding membership values that X has to each of these fuzzy sets in the collection and sum them up it should sum up to 1.

For instance, consider this domain X, which is the interval between x 1 and x 6, you can easily see that this collection of fuzzy sets will form a Ruspini partition. You take any point and find out the membership values of that point to each of these 6 fuzzy sets, add them all up and it will turn out to be 1.

For instance, take this point x naught, you will see that it only belongs to both A_2 and A 3, but not to any other fuzzy set in this collection and it belongs to A 3 to the extent 0.35 and it belongs to A_2 to the degree 0.65 whose summation turns out to be 1. The concept of Ruspini partition will turn out to be extremely important especially when we are discussing few of the desirable properties of fuzzy inference system.

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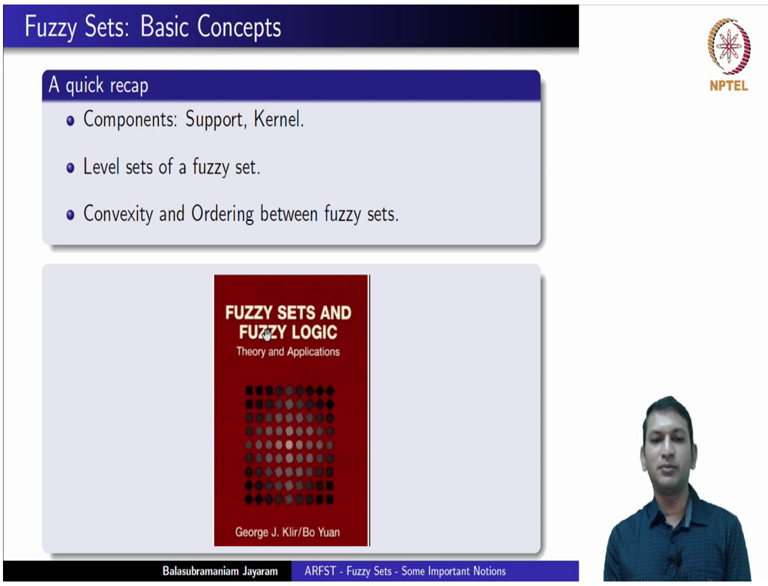

So, in this lecture, we have looked at some basic concepts relating related to fuzzy sets. We have seen some components of a fuzzy set, the support, the kernel. We have seen a very important attribute of fuzzy set that can be characterized in terms of level sets, and also the convexity and ordering between fuzzy sets in the concept of covers and partition.

Now, some of the results that we have seen which have been displayed during the course of this lecture, you could get more details about them from this excellent book by George Klir and Bo Yuan.

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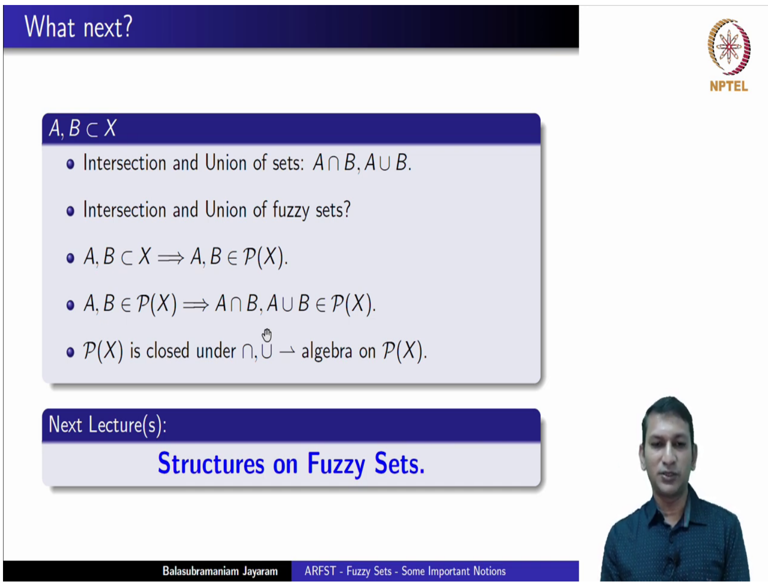

Now, the question is what next? Let us consider a pair of sets, classical sets, subsets of X we could talk about at their intersection and union. Now, it makes a natural question can we talk about intersection and union of fuzzy sets. Note that if you consider two subsets of X, they can also be considered as just elements of the power set.

And interestingly if we take these two elements of power set and if you apply the intersection and union operation, you will find that you will get a subset of X which in a sense is again an element of the power set of X. In that sense, you will see that the set P of X the power set of X is closed under both intersection and union, and which means you could talk about some nice algebraic structures on P of X.

Now, once you have managed to define intersection and union on fuzzy sets, the next question is what structure would you give on the set of all fuzzy sets on X? This will form most of what we will deal with in the lectures that will be delivered during second week of this course. So, in week 2 of this course, we will largely concentrate on the different structures that you could have on fuzzy sets.

Thank you. See you soon.