Approximate Reasoning using Fuzzy Set Theory
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Lecture - 49
Robustness of CRI

Hello and welcome to the 2nd of the lectures in this week 10 of the course titled Approximate Reasoning using Fuzzy Set Theory, a course offered over the NPTEL platform.

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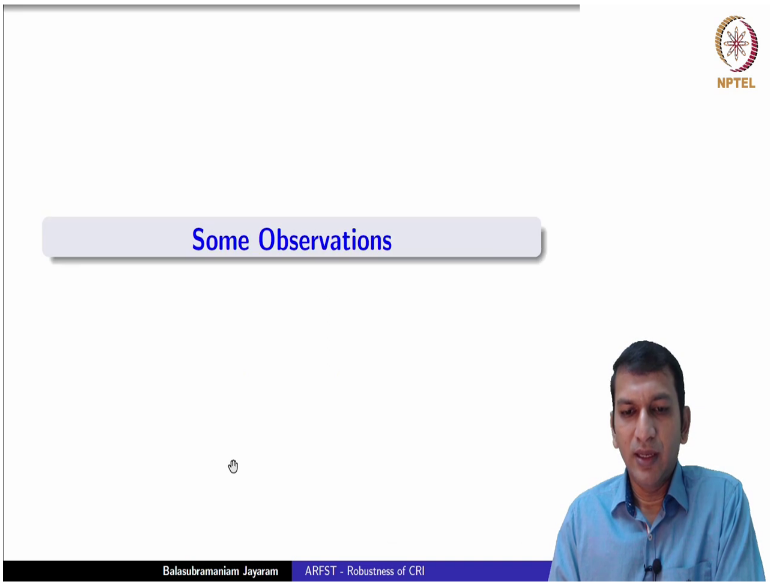

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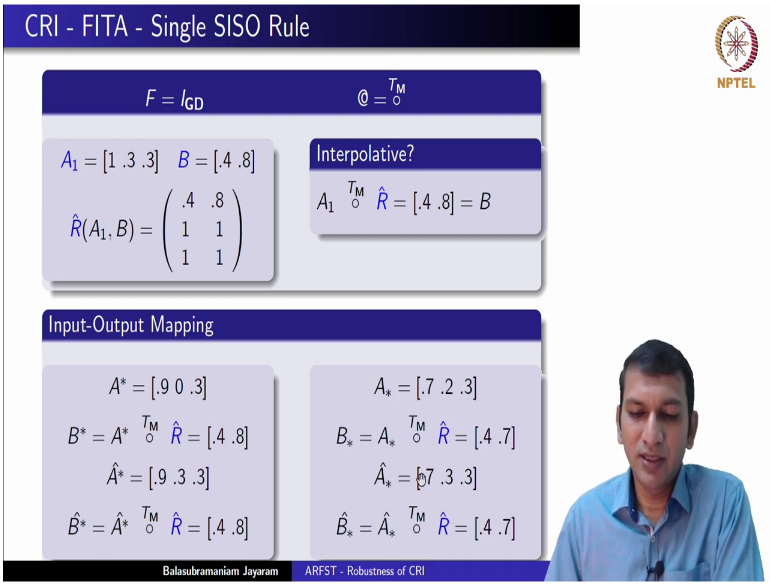

In the previous lecture, we made some observations; let us have a relook at them. We are looking at a single SISO rule in the CRI inference scheme. To obtain the relation from the rule, we are using the Gödel implication and for the composition we are using the sup min composition. If we take this rule where A 1 is the antecedent and B is the consequent.

And, if you use the Gödel implication obtain this relation, we saw that it was an interpolative system, it is an interpolative system. But, what was interesting was that if we took two different inputs, both of them seem to give identical inputs. Similarly, if you consider another pair of inputs, this A sub star and the second one, we saw that again we obtained the same in outputs.

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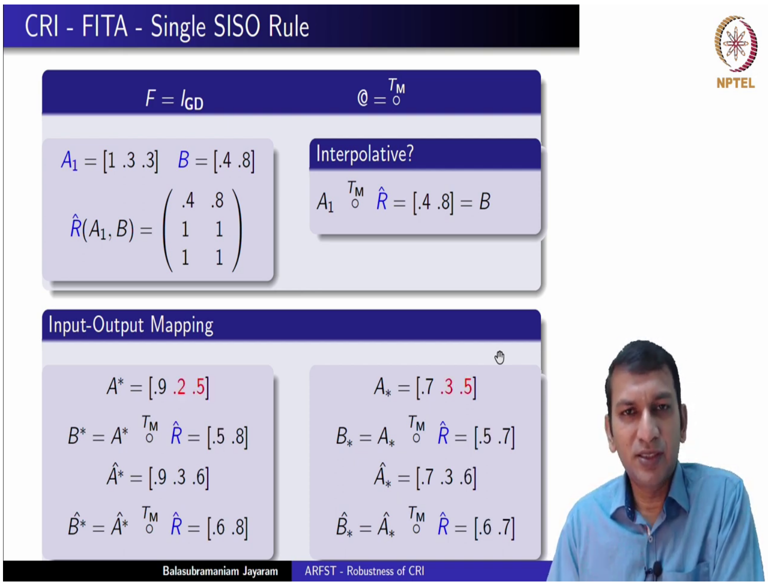

However, when we changed and considered two different pairs, it was no more true. We wanted to investigate if this were magic or if this were magic, then under what conditions this magic will hold. Towards discussing this, we introduced the concept of extensionality of a fuzzy set.

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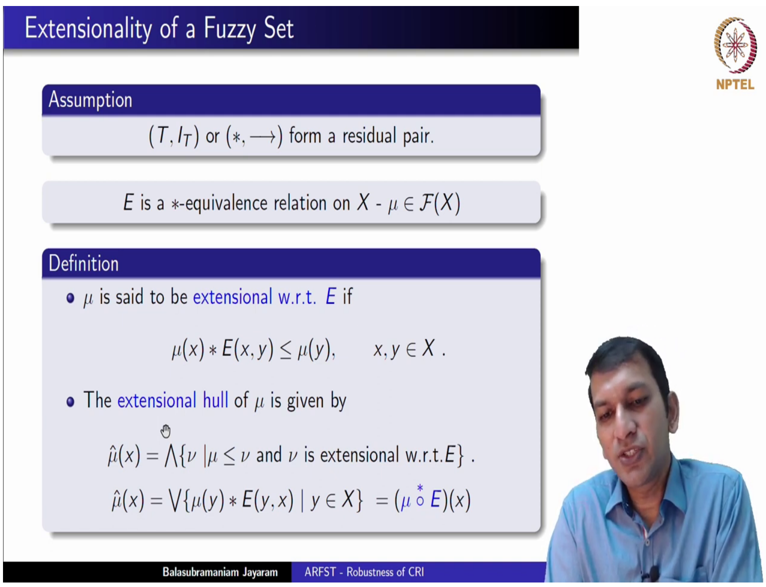

Once again, we are in the realms of residuated lattices; that means, we are going to consider left continuous T norms and the corresponding residual implication, the R implication. To start with, we have an equivalence relation, a star equivalence relation on X. And, if mu as a fuzzy set on X, we define these two concepts. We say mu is extensional with respect to E, if the following inequality is valid for every pair of x and y in X.

What does it essentially say? It says that if x and y are equal in some to some degree according to E, then mu x multiplied with that equality degree with respect to the star that we are considering should be less than or equal to the membership degree of y to the same concept mu.

So, an easy way to understand this is assume that E(x,y) is 1, then we are insisting the membership value of y and mu, the degree of belongingness of y and mu should at least be as much as the membership value of x in mu, the degree of belongingness of x in the concept mu.

The second concept that we saw was that of the extensional hull, not every fuzzy set may be extensional with respect to a given star equivalence relation E. So, we wanted a way of obtaining the extensional hull, making it extensional; towards that concept extensional hull was defined as follows. Given a mu, it is denoted as mu cap; what exactly is this extensional hull? It is the smallest extensional fuzzy set with respect to E which contains mu.

So, we take every nu which is extensional with respect to E which also contains mu and we take the smallest amount them. So, it is smallest extensional fuzzy set with respect to E containing mu. Well, this is a characterization description of what an extensional hull is, but how do we obtain it? Well, there is an easy way to obtain it; given mu and E all we need to do is essentially obtain the sup star composition of mu with E. This will give us the extensional hull.

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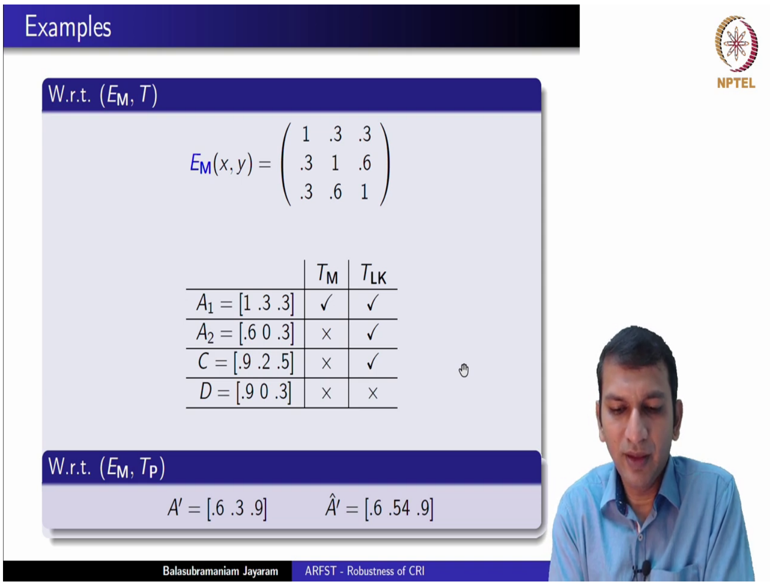

Let us look at one more example. This is the relation that we have seen in the previous lecture. So, we assume that x has 3 elements 0.3 0.6 and 0.9 and we have come up with this equality relation which is a which as a T equivalence relation. And, we know that if an if a relation is T equivalence with respect to min, then it is T equivalence with respect to any T norm. So, this is the T E M equivalence relation that we are considering.

Then, we ask the question, we took a couple of fuzzy sets and we asked the question are these fuzzy sets extensional with respect to this equality relation and the star that we are considering? For that we consider the minimum on the Lukasiewicz T norm and we have seen that this four examples, the first one is extensional with respect to the same E M and with respect to both min and Lukasiewicz.

This A 2 and C, they were only extensional with respect to the Lukasiewicz T norm whereas, D is not even extension with respect to Lukasiewicz T norm. Let us consider another example just so, that we refresh our memories. Let us take the fuzzy set A dash which is given by 0.6 0.3 and 0.9. This time we will consider the same equivalence relation E M, but we will try to see whether it is extensional with respect to the product E M.

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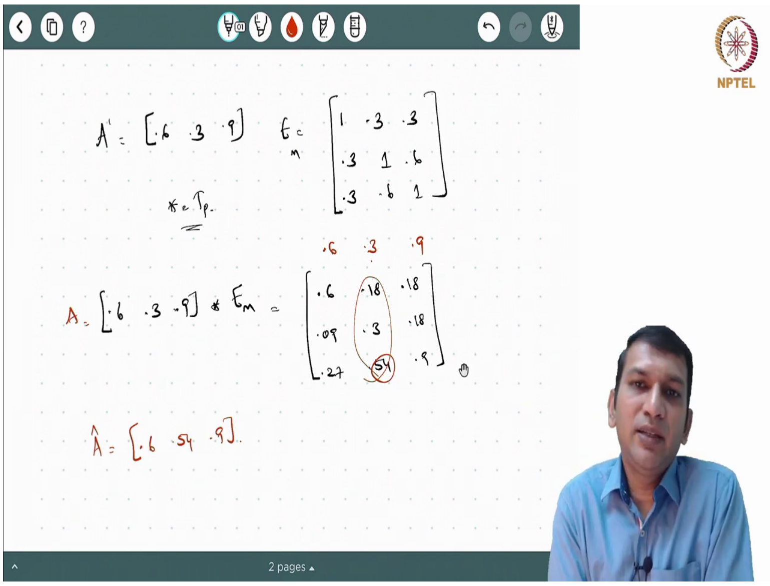

So, what we have is A dash is 0.6 0.3 0.9, E is given like this 1 0.3 0.3 0.3 0.3 1 0.6, 0.6, 1. We know that we have to use star to be T P, let us do the math. So, 0.6 0.3 0.9 star with E M. If you look at it, this is what we will get 0.6 into 1 is 0.6, 0.6 into 0.3 because now you are using product for the T norm 0.18 0.18 0.3 0.3 is 0.09 0.3 0.18 0.9 0.3 is 0.27 0.54 (Refer Time: 06:56).

Now, if you write this fuzzy set on top, we see that this fuzzy set is not extensional because 0.3 does not dominate rest of them. So, it is not extensional and if you are looking at the extensional hull of this A, then it is A cap which will be 0.6 0.54 is maximum this 0.9. So, this will be the extensional hull of this fuzzy set A with respect to E M and the product T norm that is what we have here.

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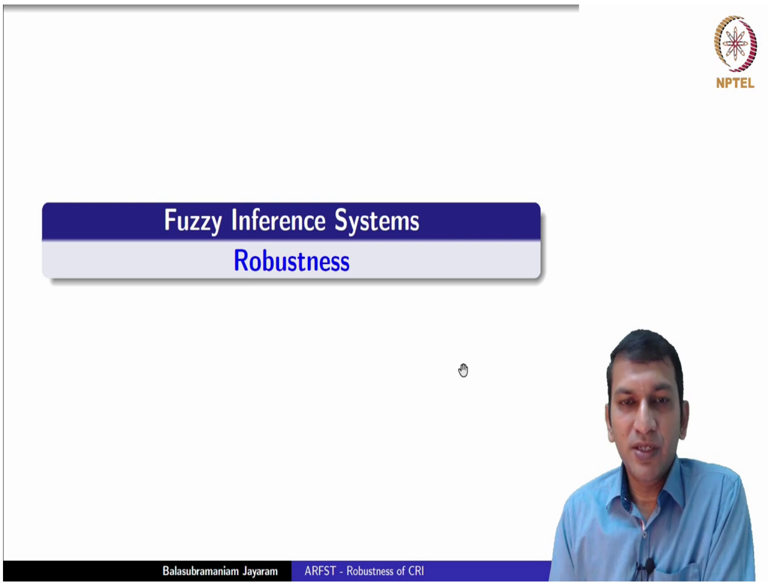

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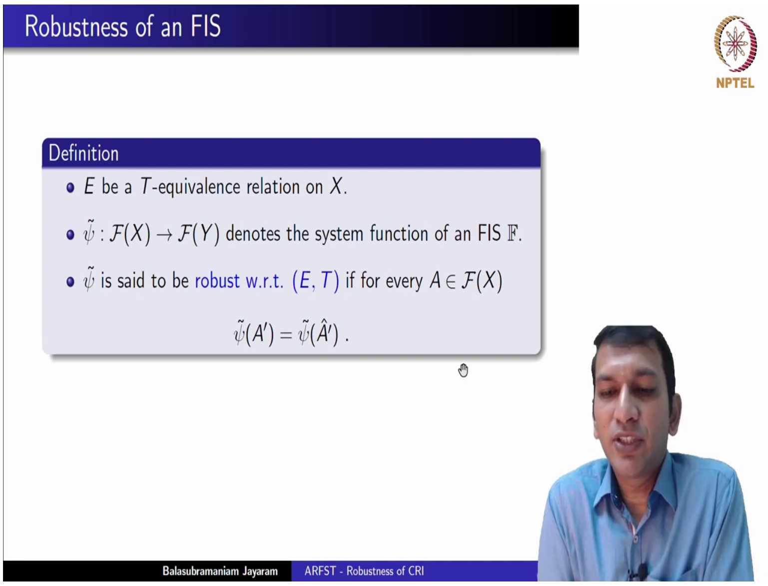

Well, we define robustness in the previous lecture. Given, a T equivalence relation on X and a mapping psi tilde from F of X to F of Y which denotes the system function of an FIS F, we said psi tilde is robust with respect to this T equivalence relation E, if for every A in F of X, for every fuzzy set A in F over X psi tilde of A dash is actually equal to the psi tilde of its extensional hull.

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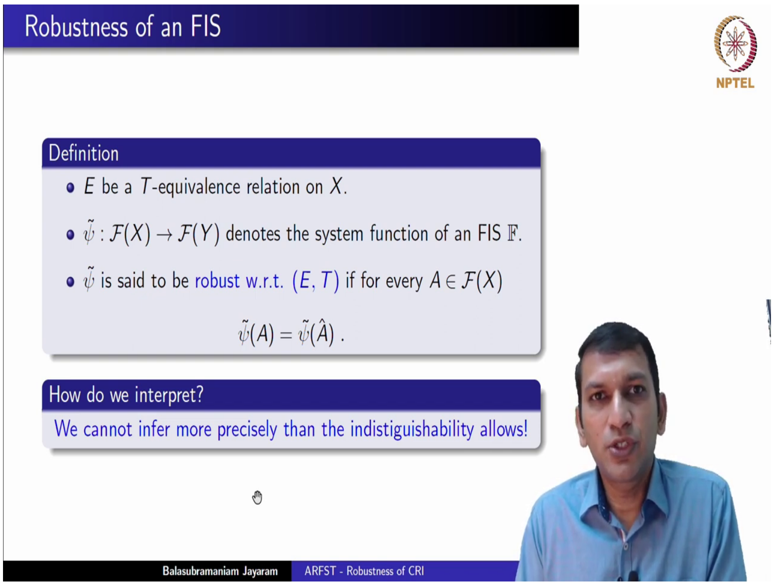

Now, we defined what is robustness in the previous lecture, we are given an equivalence relation on X, a T equivalence relation on X. And, if psi tilde denotes the mapping between F of X to F of Y which is the system function of an FIS, that we are considering; when we say psi tilde is robust with respect to this T equivalence relation E, if for every A in F of X psi tilde of A is actually equal to psi tilde of A cap; that means, the output for A and the output for its extensional hull are equal.

How do we interpret this? It essentially means we cannot infer more precisely than the indistinguishability allows. Now, equality also is in some sense indistinguishability, when we say E(x,y) is 1; that means, essentially x and y are indistinguishable.

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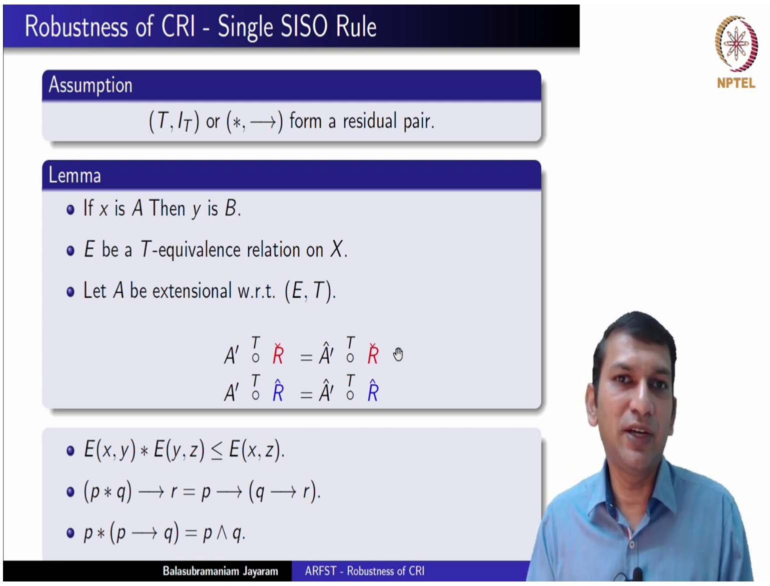

To study the robustness of CRI, let us begin with looking at a single SISO rule. Once again, we are in the realms of residuated lattices. Let us have a single SISO rule, if x is A then y is B and E is the given T equivalence relation on X. Now, we also insist that if A is extensional with respect to E, T; that means, the antecedent A is extensional with respect to the T equivalence relation E.

Then, it can be shown that for any A dash the sup T composition with R check and is equal to the sup T composition with R check of the extensional hull of A dash. This is not so, R check is essentially using f to be the corresponding T norm. Now, this is not true only for R check, it is valid even for R hat; that means, whether you obtain the relation between the antecedent and consequent of the single SISO rule.

That we are considering using the T norm for F or the corresponding residual implication for F, we see that for any input, the output that you obtain will be identical as if we were giving it the extensional hull of the input.

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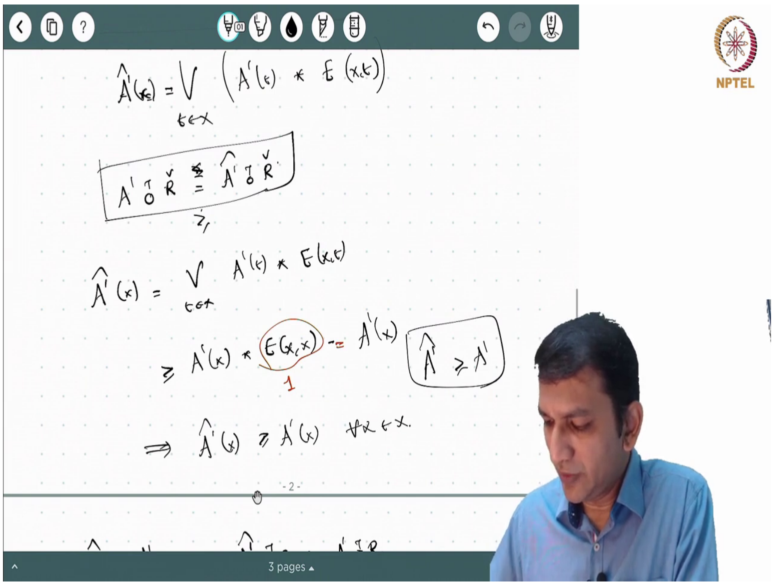

Now, how do we prove this? Note that given A dash A cap is given like this (Refer Time: 11:07) x A dash of t star (Refer Time: 11:15). So, this is how we have defined A dash cap. What we need to prove is A dash sup T R check is in fact, equal to A dash cap sup T R check, this is what we need to prove. Notice firstly, that for any given A, its extensional hull is greater than that.

This can be easily seen, if A dash cap of x is equal to sup T norm of T E M of x A dash of t star E of x norm t. Now, we know this is greater than or equal to when t is equal to x, A dash of x star E of x, x But, we know E of x, x is in fact, equal to 1. So, this quantity is in fact, 1 which means this is equal to A dash of x. So, this is what this implies is this for an arbitrary x so, A dash of x, this greater than or equal to A dash of x for all x; which means A dash cap is greater than or equal to A dash.

Now, what does it tell us? Since, A dash cap is greater than or equal to A dash, we know by the increasingness of sup t composition with respect to any R A dash circle T R is greater than or equal to A dash circle T R. So, if you want to prove this in equality already this is available for us, we only need to prove the reversing equality.

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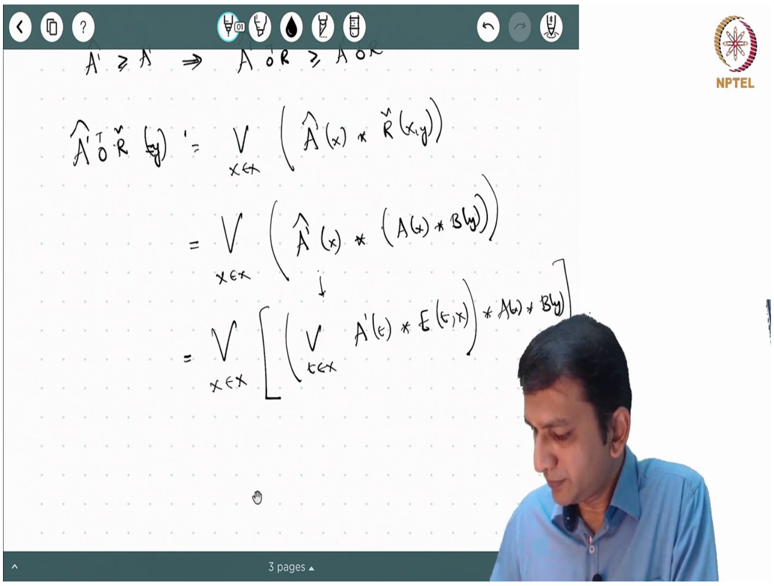

So, let us look at this, let us start with this A dash circle R check, just for to avoid cumbersome notation we are not writing the T on top, but it is there and of course, in the proof we will use the infix notation. So, now, this is nothing but at a particular y, this supremum over x element of x A dash cap of x star R check of x, y.

Now this is nothing but supremum over x element of x A dash check (Refer Time: 13:55) x star; now this is R check which means it is essentially A of x star B of y. So, this is equal to supremum over x, let us write A dash x A dash cap x the extensional of A dash x is supremum over t element of x A dash of t star E of t comma x star E of x star B of y.

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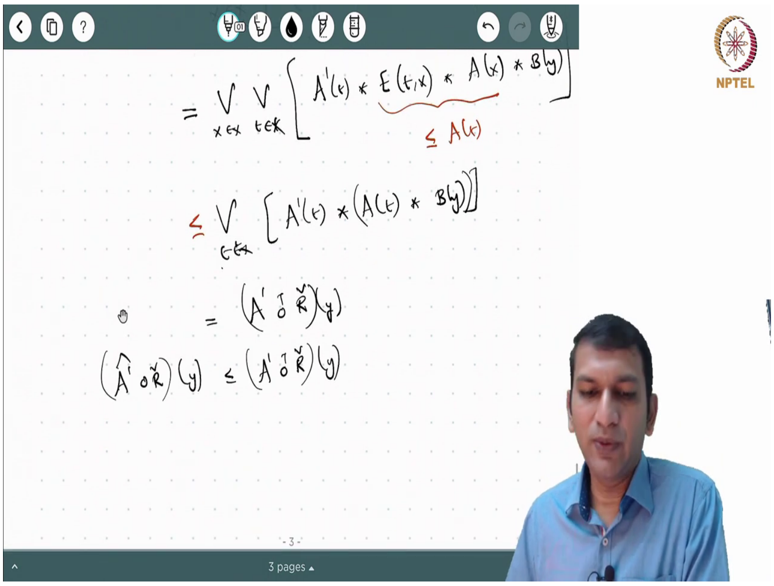

Now, we are in the realms of residuated lattices which means these are left continuous T norms. So, sup can be easily pulled out. What we have here now is supremum over x element of x supremum of t element of x A dash of t star E of t, x; since star is associative, we do not need to put brackets anywhere, A of x star B of y.

Now, notice this, this quantity here we know because A is extensional, you look at the assumption A is extensional, we know that this quantity is in fact, less than or equal to A of t because of extensionality. So that means, this entire thing is less than or equal to supremum over x comma t, A dash of t star A of t star B of y. Clearly, x is no more there, its independent of x. And, now if you look at it this is nothing but equal to A dash circle T R check of y.

Because it is varying over t, whether it is t or x it does not matter, it is a dummy variable there. So, this is essentially of R check at A B and this is what we have. So, over what we have proven is A dash cap of R check of y is less than or equal to A dash circle T R check of y. The other inequality is always valid. So, this inequality we have proven which means the first equality we have shown, let us also try to show the second inequality, second equality.

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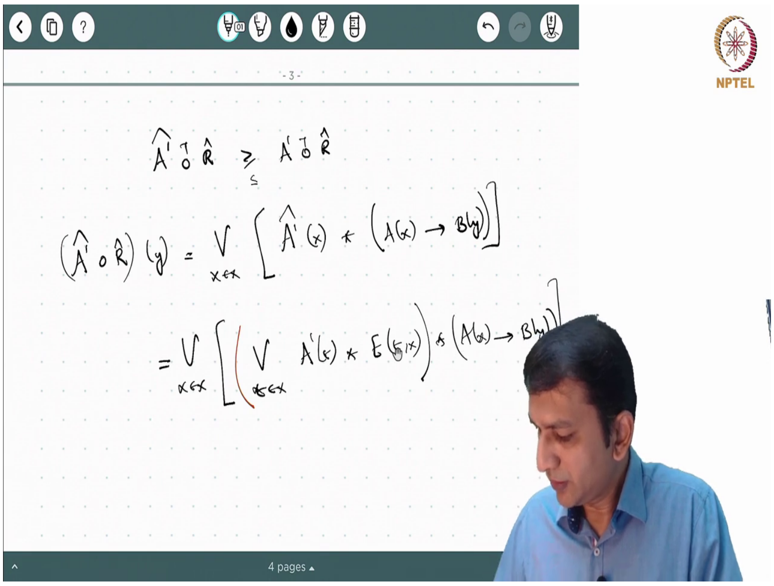

Now, once again it is clear that A dash cap circle T R cap is greater than or equal to A dash circle T R cap, because it is true for any R. Now, we need to show the other inequality. So, once again let us start with this A dash cap circle R cap at an arbitrary y, what we have is supremum over x element of x A dash cap of x star.

Now, notice that we are looking at R R cap, which means we are using the implication to relate the antecedent to the consequent of the rule. Now, this is the (Refer Time: 17:17) x (Refer Time: 17:19); let us t element of x A dash of t star E of t comma x star A of x implies B of y.

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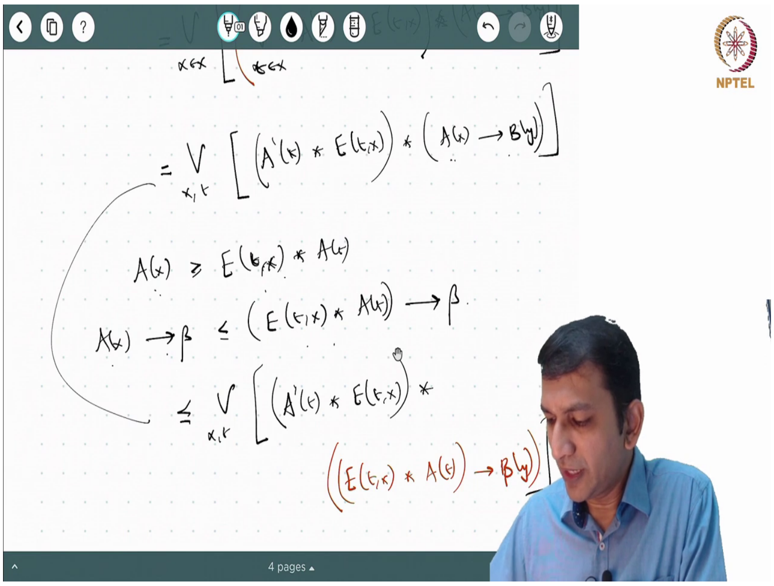

Once again, we can pull the t outside because the star is in fact, left continuous. So, supremum over x, t A dash of t star E of t, x star A of x implies B of y. Now, we know that A of x is in fact, extensional. So, A of x is greater than or equal to E of t, x star A t. So, quantity is greater than or equal to E of t, x comma A t star A t.

Now, we know that A of x implies any beta will be less than or equal to E of t, x star A of t implying beta, because we know that implication is non-increasing in the first variable; that means, it is monotonic in the decreasing sense. So, since this is greater than this, when you apply it in the first, when you take it in the first component; this is what you will have.

Now, substituting for A of x, this quantity here so, what we get is this entire thing is in fact, less than or equal to supremum over x comma t A dash of t star E of t comma x star E of comma x star A of t implying beta which is B of y. So, all we have done is taken A x, we have seen that because of extensionality we can write this for any t.

And, now we know that since implication is decreasing in the first variable, not in the strict sense of course, we can write this that A x implies B of y, can be written as we know that is smaller than E of t, x comma star at implying B of y. Now, this is what we have.

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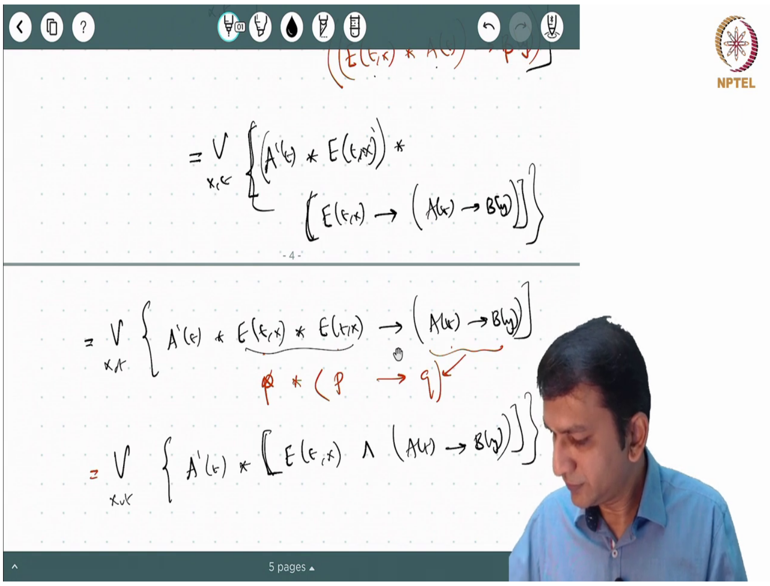

Now, you see that this is A star B implies beta. We know that E is star transitive and this is the law of importation that we have seen earlier too, because we are in the realm of residuated lattices, the star and the arrow the implication they enjoy this property. So, p star q implies r this in fact, equal to p implies q implies r. So, let us use this and what we get is supremum of x, t A dash of t star E of t, x star E of t, x implies E of t implies B of y.

Now, if you look at this star is associative. So, rewriting this along write it from here it is A dash of t star E of t, x star E of t, x implies A of T implies B of y. Now, look at this, this is alpha p star p implies q; this entire thing you can take it as q. So, it is p star p implies q. We know that p star p implies q is in fact, p meet q, where meet this is the usual lattice operation that is defined on the underlying (Refer Time: 22:22).

So, taking this as p, this as p and this as q, what we have is equal to supremum of x comma t A dash of t star. So, on these two, one only one will remain E of t comma x and A of t implies B of y. We are using the property of property that p star p implies q is in fact, equal to p meet q.

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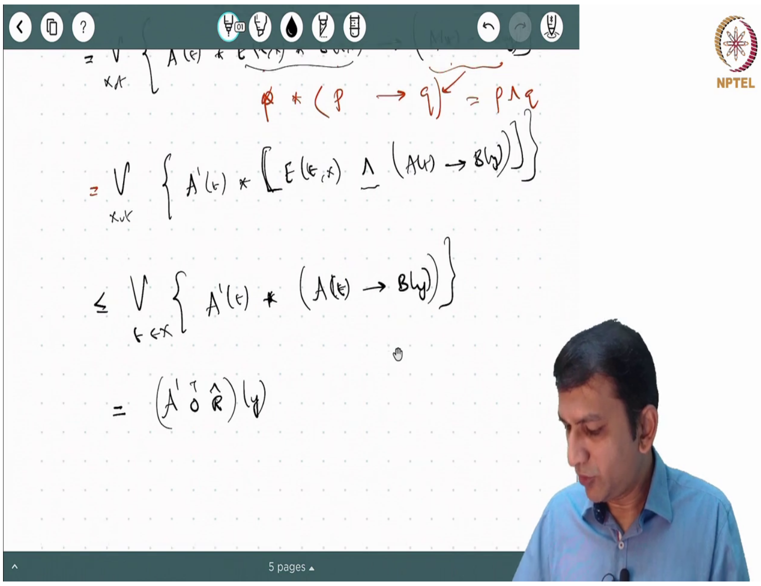

Note that this is the meet operation. So, it is smaller than either of both of the operations. So, we can write this as less than or equal to supremum of x comma t A dash of t star A of t implies B of y. Now, x does not appear anymore. So, just hold this and this is actually clearly A dash circle T R cap at y. So, what you have proven finally, is that this is less than.

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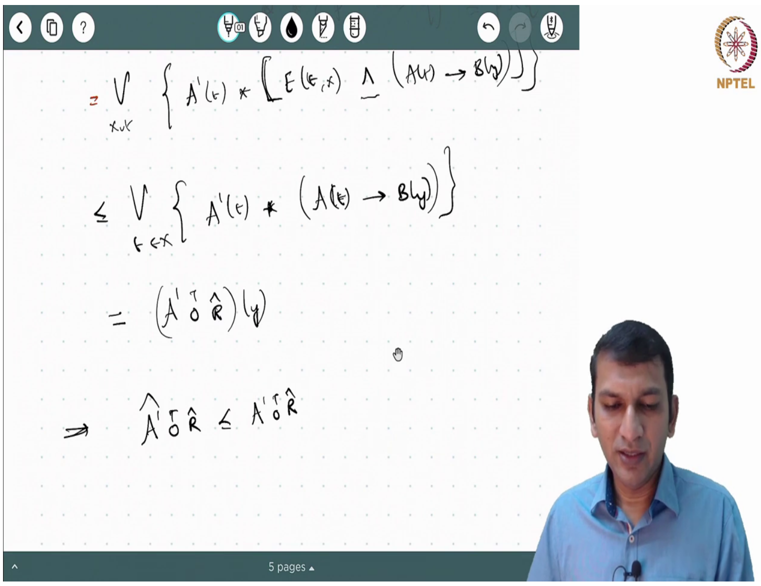

So, this implies A dash cap circle T R check is less than or equal to A dash circle T R check. The other inequality is always valid. So, we have proven this inequality which means we also get the second of the equality, the A dash composed with R cap is same as A dash cap being composed with R cap; that means, it shows that whether you use A dash or the extensional hull of A dash and use either R check or R cap for the relation their outputs are going to be identical.

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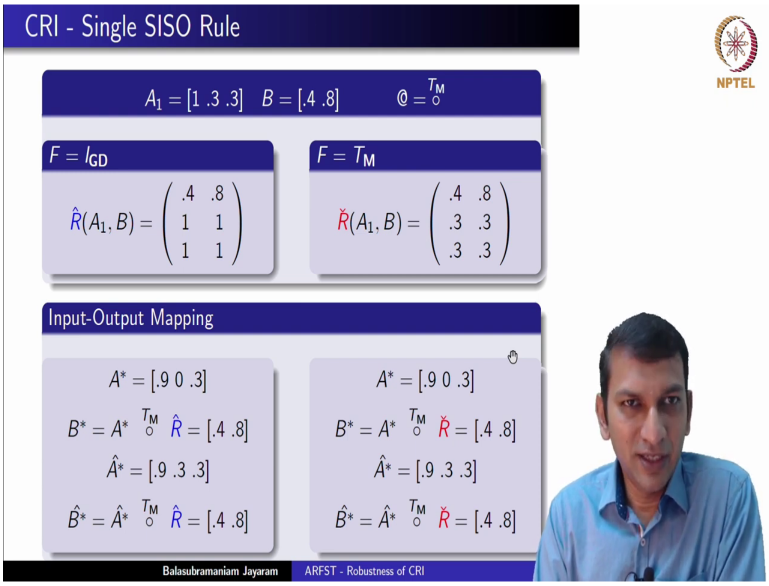

Now, let us look at the example that we have taken. We are considering a single SISO rule CRI and for the T norm we are considering the min T norm. We know that A 1, this 1.3 0.3 is in fact, extensional with respect to the E M equality relation that we have considered. So, let us consider the E M equality relation, let us use R cap with the Gödel implication. This is the relation we get and for R check, we use the same T norm, T M because these are related as the residual pair, this is what we obtained.

Now, if you see for this pair of inputs, we get identical outputs because for 0.9 0 0.3, the extensional hull is in fact, 0.9 0.3 0.3. Similarly, for this we know that this is the same, but when you compose it with R check, again we see that the outputs are identical. Clearly, this shows that this is an example validating our theorem there, just to illustrate that yes this is exactly what we get.

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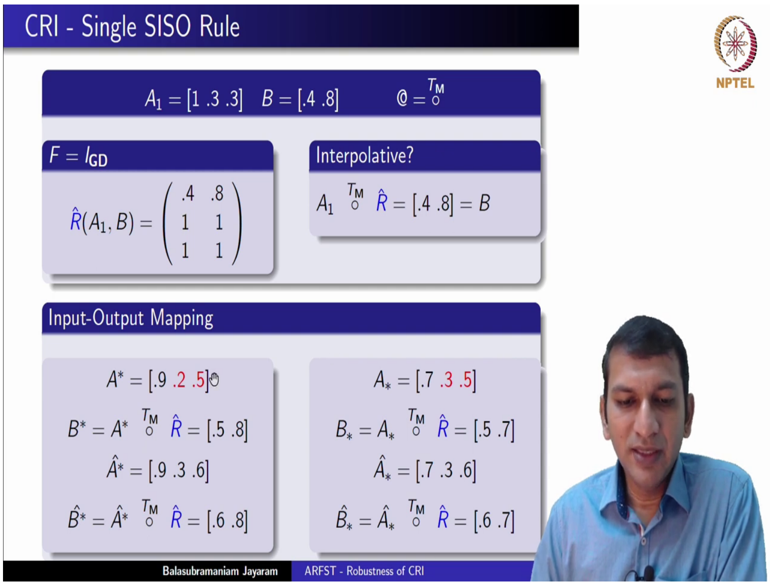

Now, why did it not happen in this case? Because, this A star and this A star cap even though they denoted like that, these two sets, these two fuzzy sets are not related to each other by extensionality. If you look at 0.9 0.2 0.5 of course, it is not extensional, but its extensional hull is not actually equal to 0.9 0.3 0.6.

Similarly, if you consider 0.7 0.3 0.5, it is not extensional with respect to min; however, its extensional hull is not equal to 0.7 0.3 0.6. So, since these pairs each of them were not related to each other in terms of extensionality, the outputs that we obtained are also different.

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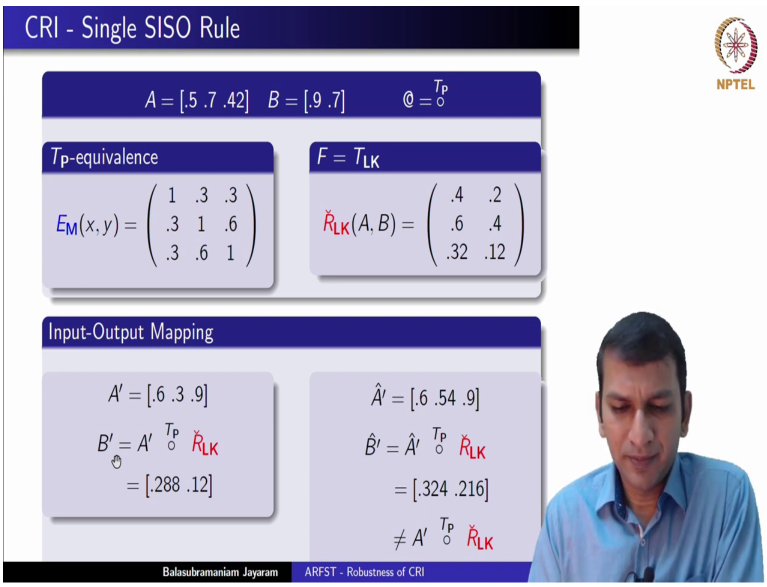

Now, to see that if you take arbitrary operations, this result may no more be valid. Let us consider a single SISO rule A, B where this is the antecedent and now for the composition we consider sup product composition. Since, E M is T transitive with respect to min, we know that it is also T P transitive and T P is a continuous T norm. So, it will lead to a residuated lattice structure.

So, this is also a T P equivalence. Now, it is easy to verify that this A in fact, is extensional with respect to this E M and T P, let us verify that.

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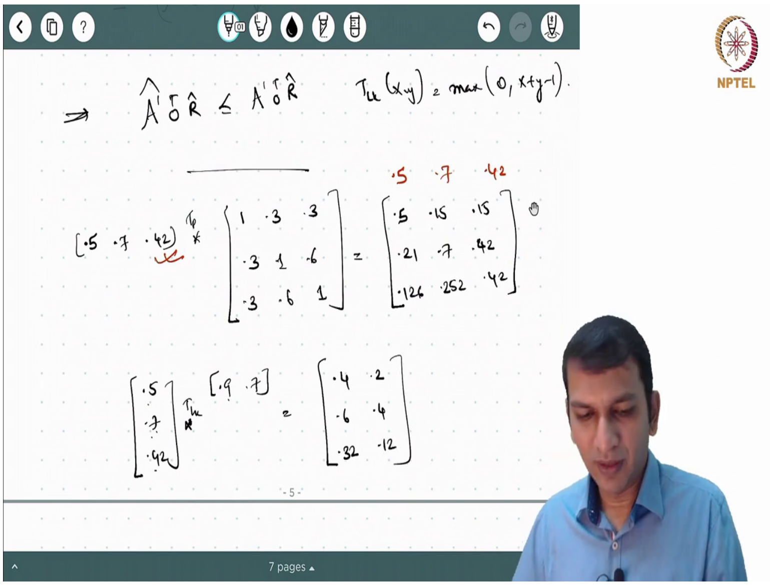

So, we have 0.7 0.5 0.7 and 0.42 star T P E M, please remember E M is 1 0.3 0.3 0.3 0.3 1 0.6 0.6 1. Now, we are using T P product 0.5 0.25 0.15 0.21 0.7 0.7 (Refer Time: 27:45) 0.42 0.3 0.126 252. So, clearly if you write the set here 0.5 0.7 0.42, you see that these values are dominating the corresponding column. So that means, this set is in fact, extensional with respect to E M and the T P, T norm. Now however, for the relation let us use the Lukasiewicz T norm instead of the product T norm.

So, now when you do the math 0.5 and 0.9 is 0.4, 0.5 and 0.7 is 1.2 minus 1, it is 0.2. This is the relation that we get, we can quickly work this out; 0.5 0.7 0.2 and what we have is this 0.9 0.7 (Refer Time: 28:55). We are using Lukasiewicz T norm here, obtain the relation 0.5 0.9.

Please recall, Lukasiewicz T norm is T LK of x, y is maxima of 0 comma x plus y minus 1, 0.5 0.9 is 1.4 minus 1 which is 0.4, 0.5 0.7 is 1.2 so, it is 0.2, 0.7 0.9 is 1.6 so, it is 0.6, 0.7 0.7

is 0.14 minus 1, 0.42 1 0.9 will give you 0.32, 0.42 and 0.7 will give you 0.12. So, that is exactly the relation that we have got now. Now, let us take A dash, this is the example that we worked a few slides earlier.

We know that it is not extensional with respect to the product T norm. However, if you compose this using sup product composition with R LK, it is easy to see that this is the value that you would get. However, if you consider corresponding extensional fuzzy set which we have found out a little while earlier A dash cap, this is the A dash cap.

If we compose it with this R check LK, we see that the output obtained from A dash cap and A dash they are not actually the same. So, what we have as conditions there are required if you want the output to be identical whether you give the input or its extensional hull. This is just to illustrate the sufficiency conditions that we have there to ensure outputs are identical when the inputs, the input pairs is related, the input pair is related to each other using extensionality.

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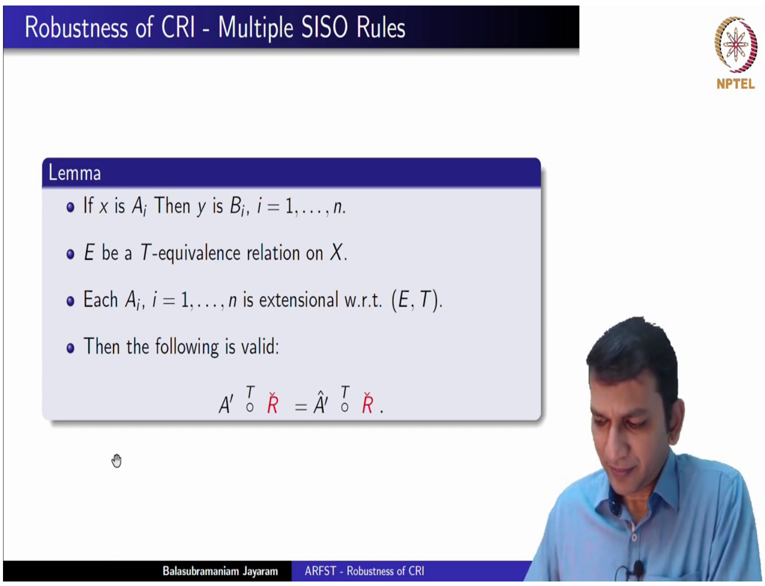

Well, what about the multiple SISO rule case? Let us consider it, we have n such rules, if x is A i then y is B i. Once again, let us consider in E which is a T equivalence relation. Now, we insist that each of the A i's is extensional with respect to this T equivalence relation E.

Then what we can prove is that whether you give the input as A dash or its extensional hull as long as you compose it with R check relation, where the relation comes through the T norm by relating the antecedents to the corresponding consequence using the T norm star, that we are considering; then we see that we outputs are in fact, identical.

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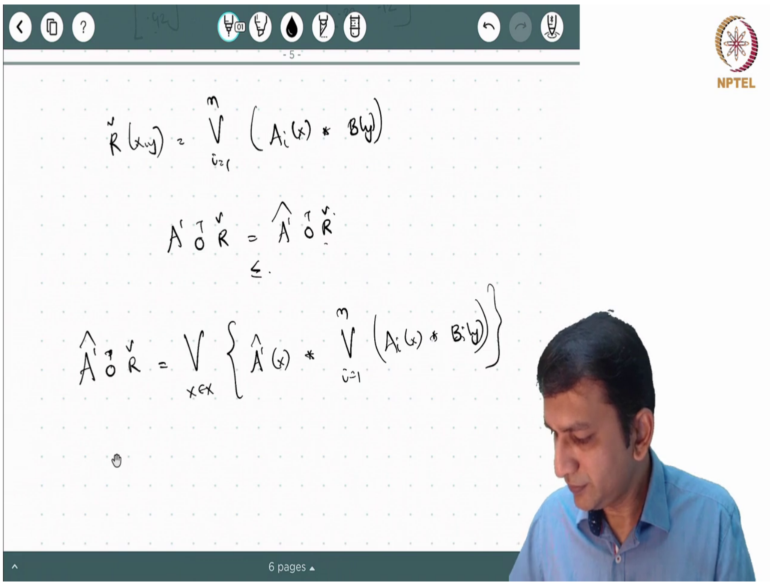

Let us once again prove this. We need to prove now, note that R check is there are n rules A i of x star B y for any particular x y is what we have. And, what we need to prove is A dash circle T of R check is in fact, equal to A dash cap circle T of R check. We know that this is always true because, it does not matter what R is. So, essentially we need to only prove other inequality. So, let us look at that. Once again supremum over x element of x A dash cap of x star, R check we know is this A i of x star B i of y.

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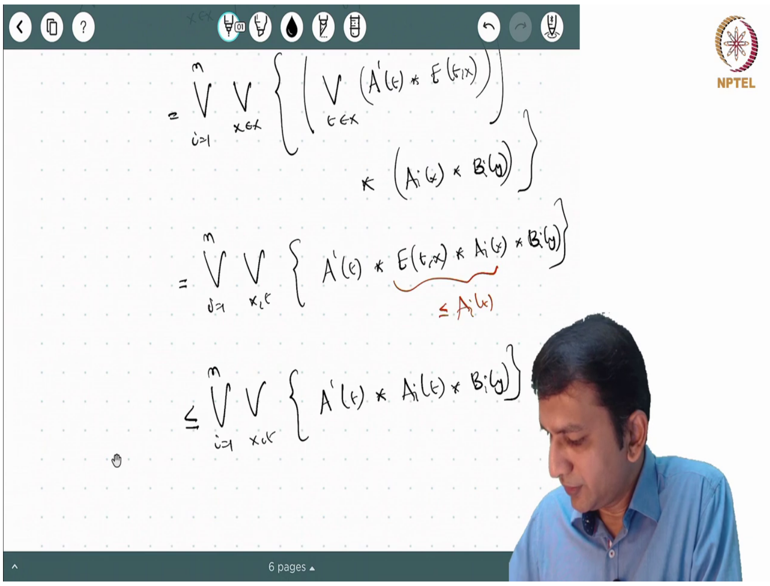

Now, once again because star is left continuous, we can easily pull out the supremum. So, we see that this is 1 to n. So, x this unfurl A dash cap, this is supremum of t element of x A dash of t star E of t, x star A i of x star B i of y. Once again, because we are in the realms of residuated lattices, we can pull out t outside supremum of x comma t A dash of t star E of t, x start A i of x start B i of y.

Notice that A i is extensional with respect to E. So, now, this we know is less than or equal to A i of t. So, once again making use of this fact, we see that this is less than or equal to so, i is equal to 1 to n. So, x comma t A dash of t star A i of t star B i of y.

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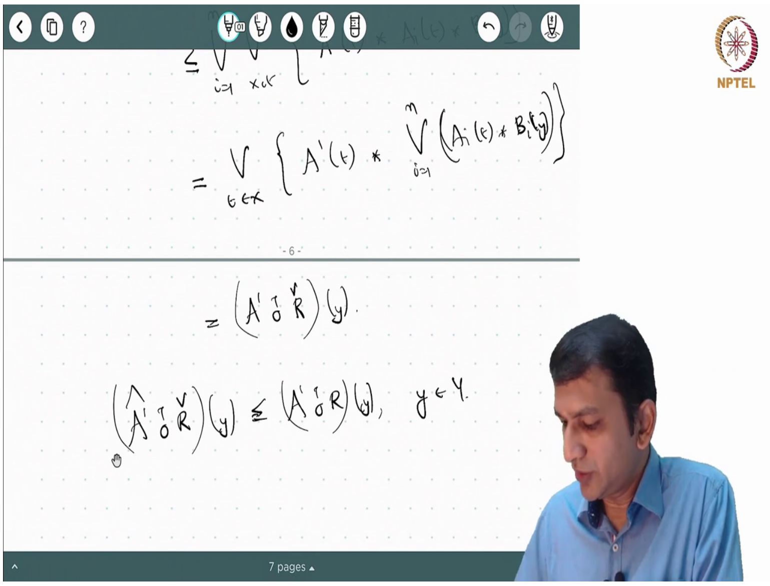

Let us push back supremum, now you see that it is all in terms of t only. So, x does not play a role, A dash of t star i is 1 to n A i of t star B i of y which is essentially A dash circle T R check of y. So, what we have shown is that A dash cap circle T R check even in the multiple case is less than or equal to A dash circle T R at y is for any y element of Y.

So that means, this inequality holds, the other inequality always holds which means we have this. Notice that, this does not happen if you consider R cap. So, R check seems to be the relation that we need to use, if you want to ensure extensionality, if you want to ensure robustness.

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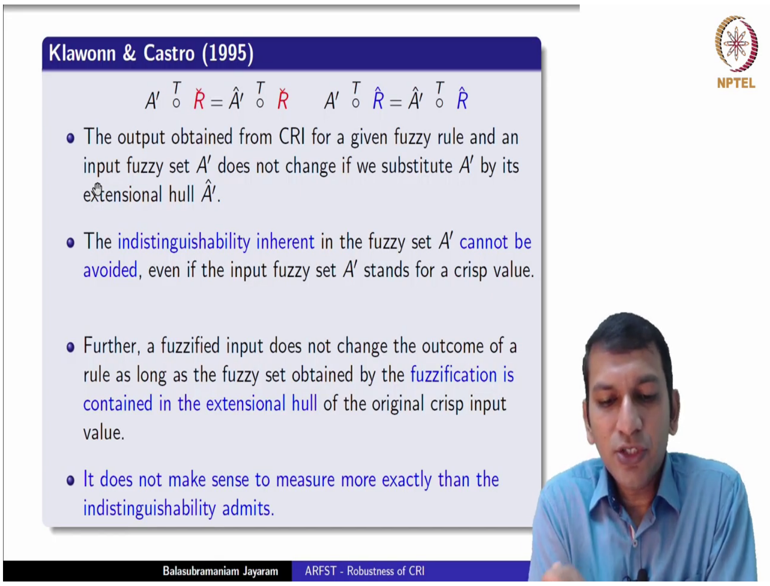

Well, we define robustness; we interpreted it as saying that we cannot measure more precisely than what the indistinguishability allows. Let us listen from the authors, the original authors who proposed this in their seminal work; the authors of Frank Klawonn, Juan Luis Castro. These are the two equalities that we have. Now, this is what they have written.

The output obtained from CRI for a given fuzzy rule and an input fuzzy set A dash does not change if you substitute A dash by its extensional hull which is clear. The indistinguishability inherent in the fuzzy set A dash cannot be avoided, even if the input fuzzy set A dash stands for crisp value.

So, it does not matter in the case of CRI, whether you give an input which is crisp input, which essentially you for single means you can fuzzify it using a singleton fuzzifier, the output that you get will still be that of the extensional hull of the crisp value. The indistinguishable inherent there cannot be avoided. They go on further to say, even if you give a fuzzified input, it does not change as long as the fuzzified version of the input remains within the extensional hull of the original fuzzy set.

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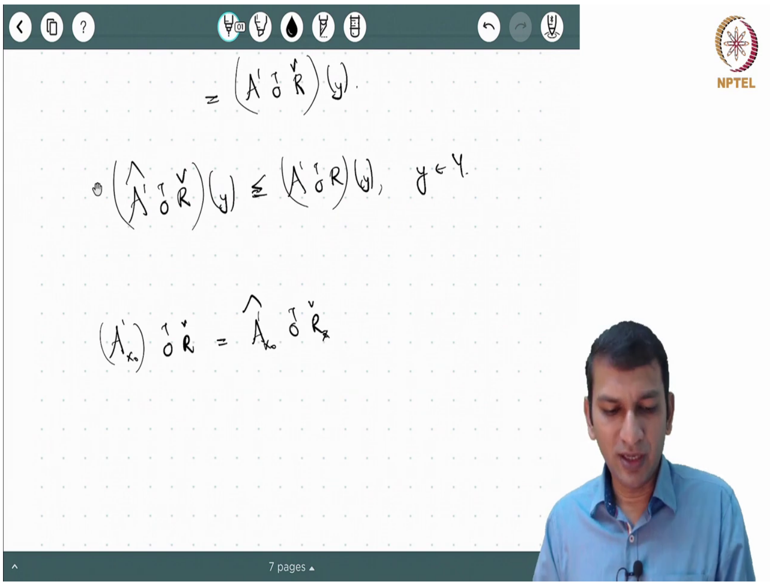

So, what they mean is you can give an A dash, it could also be a singleton A dash, but if you composite this with R check or R cap, this will still be the same as whatever you obtain from the corresponding extensional hull. Finally, they say it does not make sense to measure more exactly than the indistinguishability admits.

So, you do not have to only depend on giving crisp values or a particular fuzzification, particular fuzzifier name of use because, we understand that the underlying equality relation comes into play when we are using a CRI.

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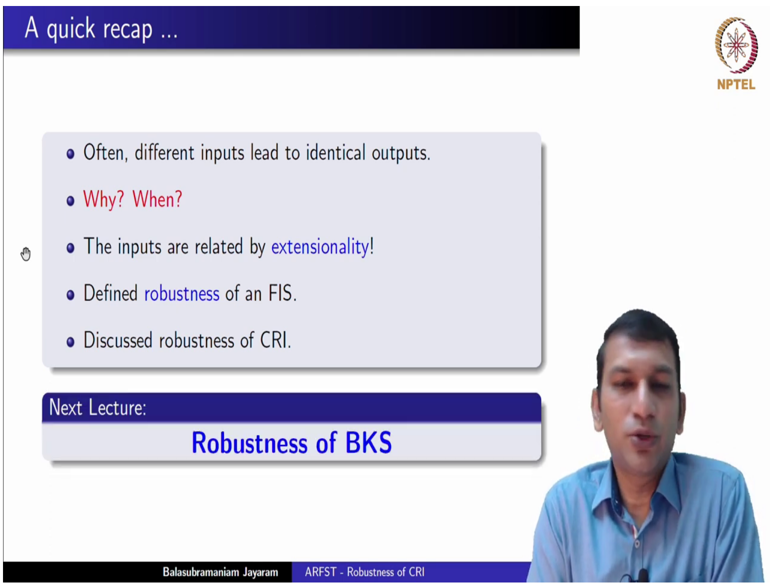

A quick recap, in the previous lecture we have seen often that different inputs lead to identical outputs. We ask the question why does it happen and if it can happen often when does it happen? We have now seen that this happens at least in the case of CRI when the inputs are related by extensionality.

We define robustness of an fuzzy inference system as that system function which respects this extensionality, which respects the equality relation that underlies the space. And, in this lecture we discussed the robustness of CRI. In the next lecture of course, we will look into the robustness of BKS, the Bandler-Kohout Subproduct.

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Once again, this is the seminal work of Klawonn and Castro, wherein they are discussed the robustness of a CRI inference mechanism. In the next lecture, we will meet to discuss the robustness of BKS inference scheme. Glad you could join us today. Hope to see you soon in the next lecture again.

Thank you again.