Approximate Reasoning using Fuzzy Set Theory
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Lecture - 48
Extensionality of a Fuzzy set

Hello and welcome to the first of the lectures in this week 10 of the course titled Approximate Reasoning using Fuzzy Set Theory, a course offered over the NPTEL platform. In this lecture, we will look at a concept termed as the Extensionality of a Fuzzy set.

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Before moving to this concept, let us make some interesting observations.

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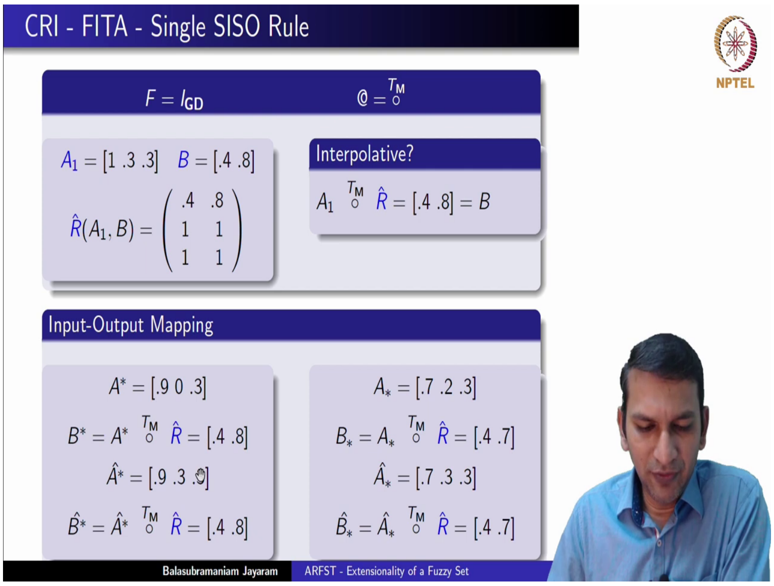

Let us consider a single SISO rule, Single Input Single Output rule and let us use this in the CRI inference mechanism. Of course, in this case either you if you use the FITA or the FATI inference strategy, both are same because we have only a single rule.

So, let us assume that we have a rule A implies B and since it is CRI we have to use a sup T composition for that we have taken the minimum sup min composition and the operation F is required to obtain the relation from the rule for which we have taken the Godel implication.

Let us assume the rule is A 1 implies B A 1 is given as this fuzzy set 1 0.3 0.3 and B is 0.4 0.8. Now, we know that we obtain the rule using this implication F, the Godel implication, so, quickly how do we do this?

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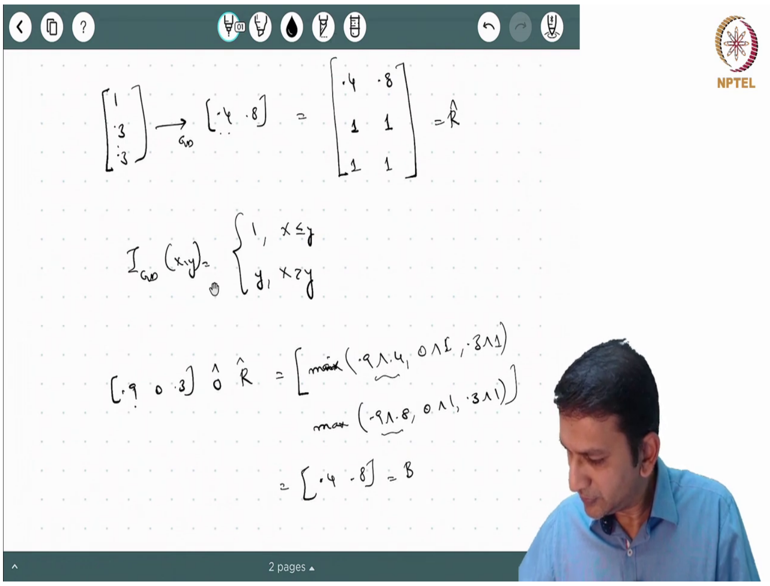

We have this 1 0.3 0.3 is the Godel implication 0.4 0.8. Please recall the Godel implication is this one if x is less than or equal to y and y if x is greater than y. So, put together 1 and 0.4 0.4 because it is a neutral implication 0.3 is smaller than 0.4 which means 1.3 smaller than 0.81. So, we get this.

So, you see here that is how we have got the relation from the rule. Now, the first interesting question to see is, is this interpolative? Well, we know this because minimum and we are using the corresponding residual implication and there is only one single input here rule here which is also normal fuzzy set. So, perhaps you suspect it will be interpolative and as can be verified yes, indeed it is interpolative.

Now, what is interesting is this. Let us look at the input output mapping for some arbitrary fuzzy sets. Let us give this as the input to the system; that means, given A star we are going to compose it with R cap and see what the output that we get is now clearly if this is the input 0.9 0 0.3 composite sup min with R cap which is this. Clearly, we are doing sup min composition the first component is minimum of infact maximum of min of 0.9 and 0.4, 0 and 1, 0.3 and 1.

So, max of this and max of 0 0.9 and 0.8 0 and 1 0.3 1 clearly this will turn out to be this quantity here which is 0.4 and this quantity here which is 0.8. So, this is actually equal to B. So, the point is not that it is equal to B, the point is this is the output we are getting well.

Now, what we will do is we will give another input which is slightly different. So, you see in the middle component it is 0.3 and if you do the same calculation we still continue to get the output as the fuzzy set 0.4 0.8. Now, let us change the input to something else 0.7 0.2 0.3 and if you give this as the input to the system the output is 0.4 0.7 we are not discussing interpolative here because this is not actually a one.

But, what is interesting is if you slightly modify this instead of 0.2 if you use 0.3, we still obtain the same output 0.4 0.7. So, even though A star and A star cap are different we obtain the same outputs from it and once again A sub star and A sub star cap even though they are different we obtain the same output 0.4 0.7. Now, you might suspect is it it is probably because we have an interpolative system.

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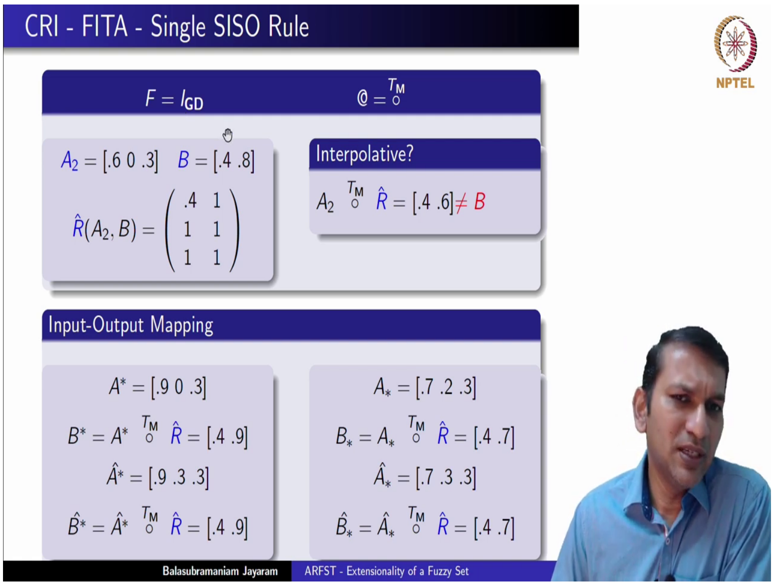

Well, let us look at another system we will retain the same consequent beam, but we will change the antecedent to this fuzzy set A 2 which is given as 0.6 0 0.3. Once again using the Godel implication we see that this is the relation that we would get. Is this interpolative? Well, if you work out we will see the output is 0.4 0.6 which is not equal to B. So, this system is not interpolative.

Let us give the same input a star which is 0.9 0 0.3 to the system and look at what we get as the output of course, it is not the same as B as we got earlier. However, as we said the point is not what exactly is the output that we get. The point is about if we give two different inputs are we getting the same output and we see here by giving this pair of inputs we actually get the same output.

Now, what about the other pair of inputs? Once again we see that for this input 0.7 0.2 0.3 0.4 0.7 is the output and so, is the case when the input is 0.7 0.3 and 0.3. Well, now the system is not interpolative. So, what could be playing the role of ensuring that for these inputs we get the same output?

Once again we know the power of resituated lattices. So, we may suspect ok, we have the minimum T norm here and we use the corresponding R implication which is the Godel implication, perhaps this pair is playing the role can be.

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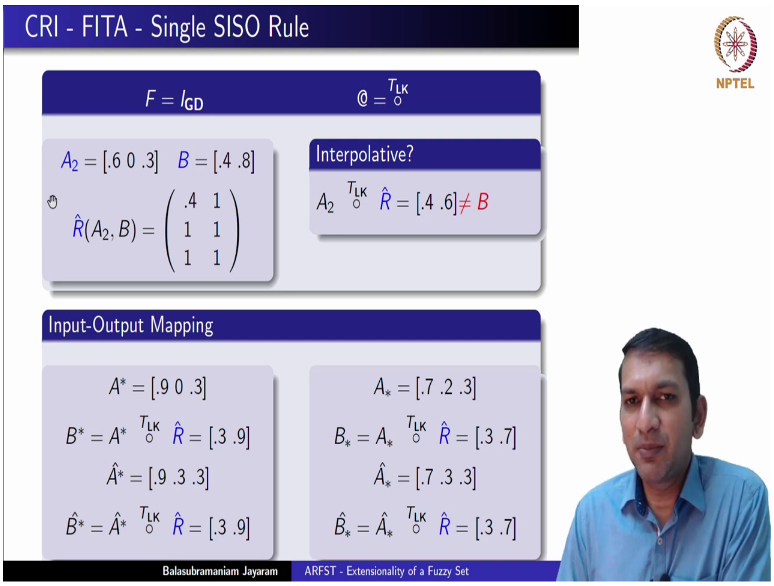

So, let us put that theory also to test. So, let us change the sup t composition from sup min to sup Lukasiewicz e norms composition and, but still use the same Godel implication and once again we are going to use the same antecedent A 2, ok. Even with this changed composition it is not interpolative. Let us give the same inputs.

Quite interestingly we find that for this pair of inputs once again the outputs are same and for this pair of inputs again the outputs are identical. The next thing that you would think is perhaps it is the R cap relation that is playing a role.

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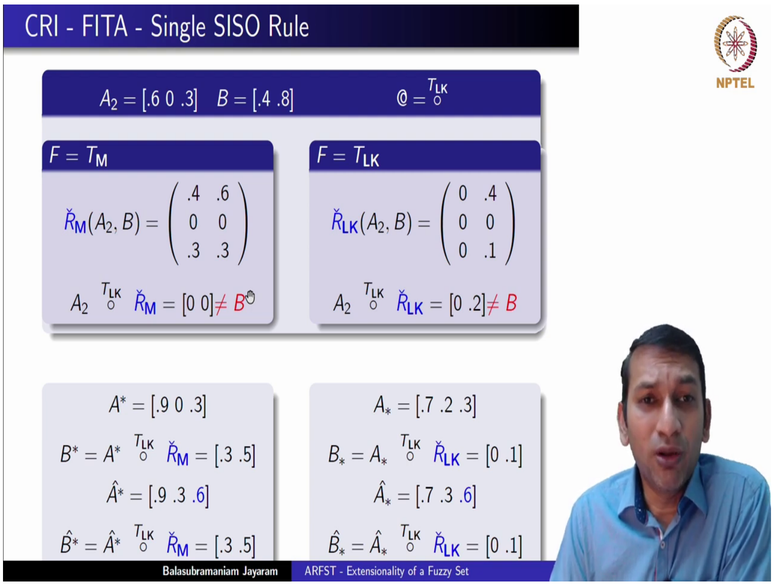

So, once again let us put this theory to test keep the composition to be sup Lukasiewicz same fuzzy sets, but let us use the minimum T norm to obtain the relation. So, we are looking at it as positive rule and if you give A 2 to this we do not get B. So, it is not interpolative, why only T_M why not also T_L K.

So, let us use the Lukasiewicz T norm also to obtain the relation and once again, we see that it is not interpolative. So, there is no relation in terms of the residual operation between the composition operation that we are using and their operation F that we use to obtain the relation from them.

Now, we will see we in fact, increased the distance so to speak between A star and A star cap, but we still seem to be getting the same identical output. What about for the other pair of fuzzy sets? Once again instead of taking to be 0.3 0.3 we have pushed it even further to 0.6 and what we see is that the outputs from these two inputs are in fact, identical.

Now, is this magic? Will this happen always? Is this an artifact of the operation that we are employing or is this an artifact of the numbers that we have considered or is there some insightful gain in knowledge looking into this a little deeper. Also the question is will this always happen.

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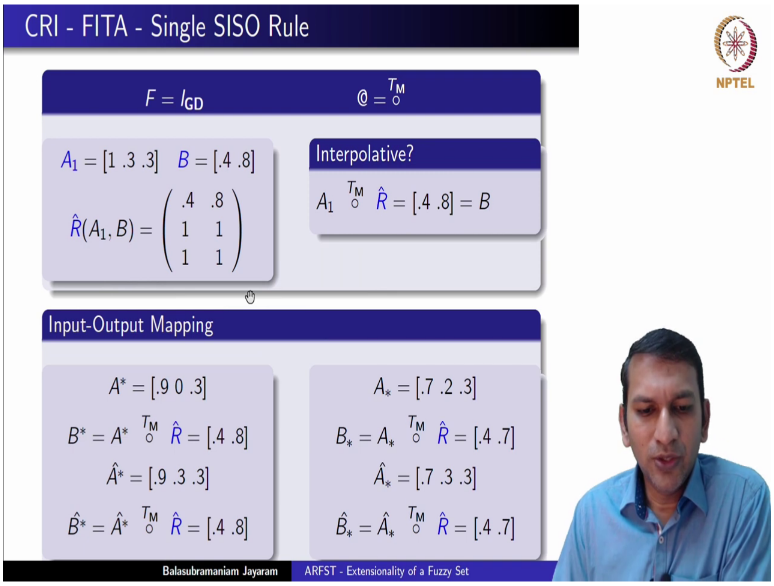

Now, let us look at this was the first system that we considered interpolative and we gave these as the inputs these two pairs of fuzzy sets as inputs and we got identical outputs for each of the pairs.

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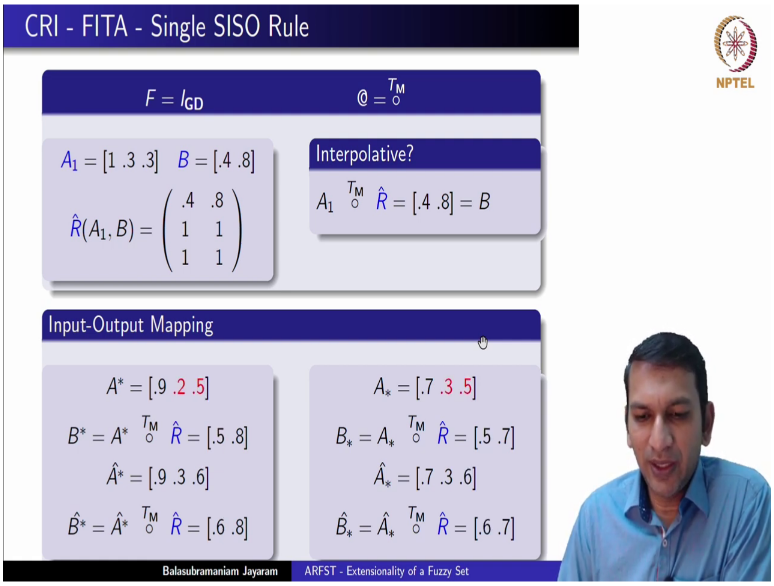

Now, what we will do is we will slightly tweak this, we give this as the input 0.9 0.2 0.5 this is the output and we have only modified it slightly. We have not gone further ahead, but we will see we see that the outputs are different here shows the case here. The we only changed one component from 0.5 to 0.6 and we immediately see that even for the original interpolative system that we considered the outputs are different.

So, clearly there is something else at play here and this is what we want to understand because remember we are not looking at this from the fuzzy relational equations point of view you know, we are looking at it as having a system. And, given an input we are obtaining an output and we are only interested in seeing there is a phenomenon that is happening where two different inputs are giving us identical outputs and this seems to happen even when we consider different operations different antecedents.

So, it would be gainful to look into this little deeper.

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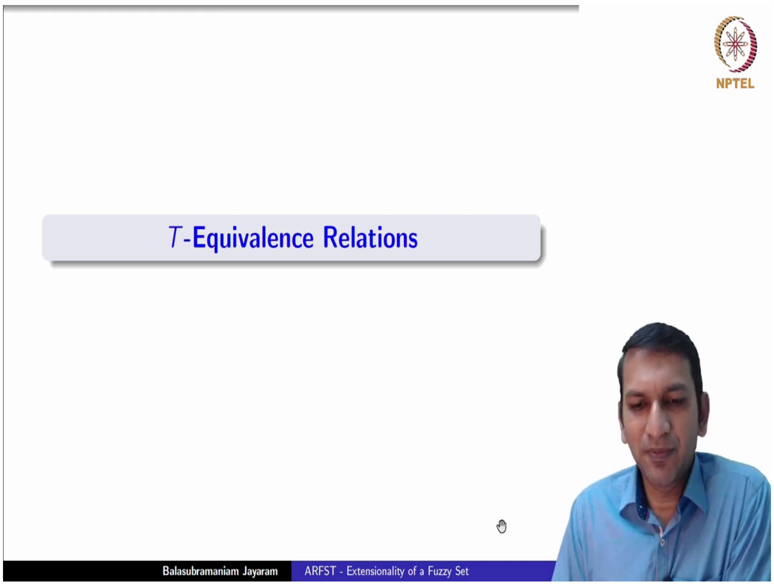

Now, towards that end let us look at the concept of extensionality of a fuzzy set, for that let us revisit what a T – Equivalence relation is.

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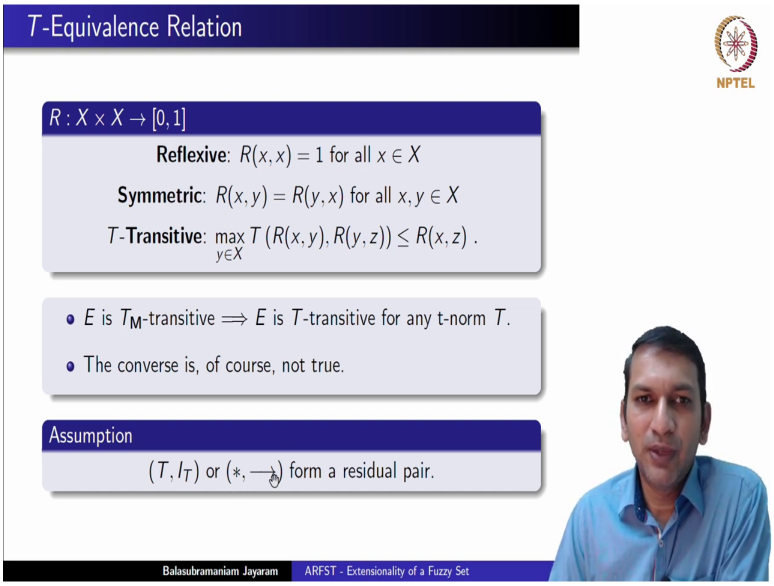

A T – equivalence relation is a binary relation on x a fuzzy binary relation on X which is reflexive symmetric and T – transitive. Now, we know that if the relation is T_M – transitive then it is also T – transitive for any t-norm because for instance if it is T_M transitive.

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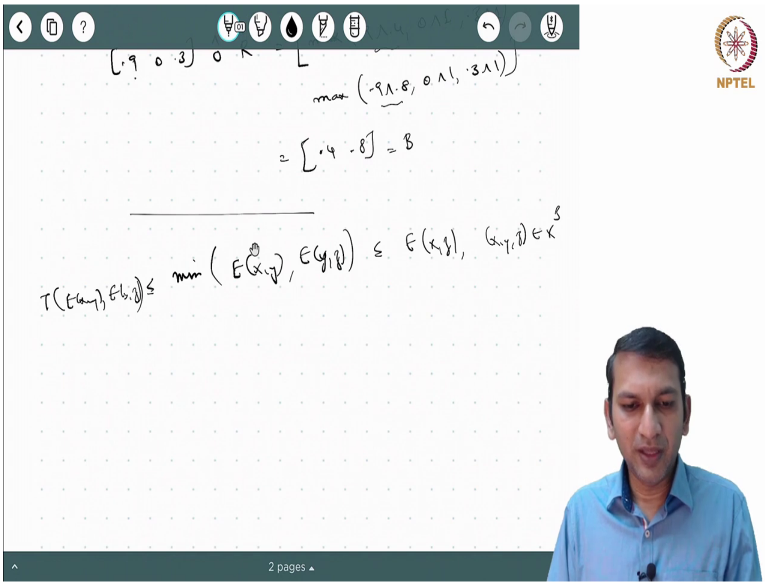

And, what we have is min of E of x, y, E of y, z is less than or equal to E of x, z for all x y, z element of x cube, but we know minimum is the largest t norm. So, any other t-norm if we use by monotonicity it will be smaller than this which means any E which is T_M transitive is also T-transitive for any other t-norm. Of course, the converse is not true, we can find examples.

Now, for the rest of the lecture once again you know for this week also we will assume that when we discuss the case of CRI or BKS; that means, as long as we are in the realms of fuzzy relational inference we will consider the operations to come from the corresponding resituated lattice. That means the T and I_T are related as t-norm and the corresponding residual implication and often we will also use the in fixed notation to make the notation simpler.

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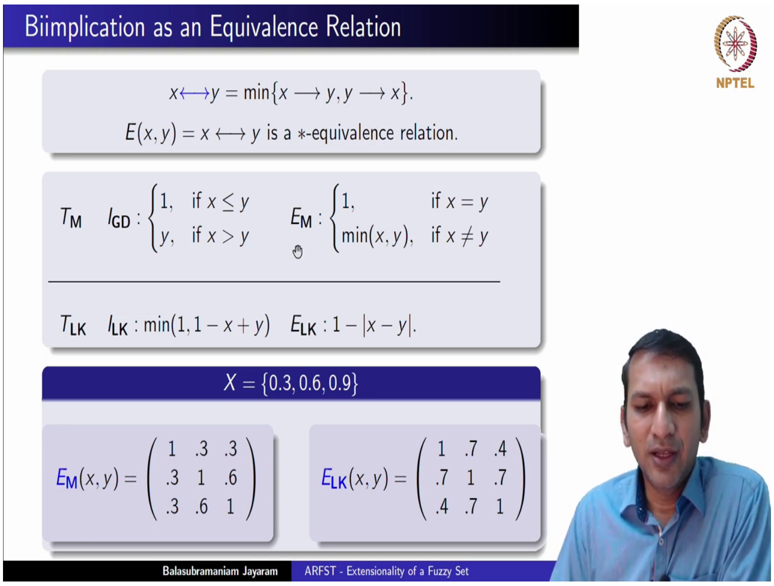

Now, we have seen in all the lectures previous in the last week that biimplication can also be seen as an equivalence relation. What is a biimplication? x biimplication ys minimum of x implies y and y implies x. In the case, this implies implication comes from a resituated lattice structure we know that E of x, y defined as x double implies y or y implies y is in fact, reflexive symmetric and star transitive where star and implication are coming from the resituated lattice structure it is a star equivalence relation.

Now, recall if we consider the minimum t norm the corresponding R implication is the Godel implication and using this definition of bi residuum we get that the E M. The E M means M min transitive fuzzy relation E M is given like this if we consider the Lukasiewicz t-norm the corresponding Lukasiewicz implication the bi residuum of that is essentially this.

So, getting the equivalence relation from that would look like this E LK would be one minus mod x minus y mod of x minus y. So, for the moment let us consider that x consist of these three values 0.3, 0.6 and 0.9 and the equality relation the equivalence relation E M on this using this formula can be obtained for the set X which consist of these three values. Clearly 0.3 0.3 is 1.3 0.6 minimum of them is 0.3 0.3 0.9 is 0.3, 0.6 0.9 is 0.6.

So, this equivalence relation is easy to obtain. For the same set x if you obtain the equivalence relation e l k this is how it would look like 0.3 0.3 is 1, 0.3 0.6 is 1 minus the difference is 0.3. So, that is 1 minus 0.7 1 minus 0.3 which is 0.7 and when we will put 0.3 and 0.9 here the difference is 0.6. So, 1 minus 0.6 is 0.4. So, these are the corresponding min equivalence and Lukasiewicz equivalence relation that we can get on this set X using these two formal.

Of course, there are many many other equivalence relations which can be min transitive. For example, we have seen that E M, it is min transitive which means it will also be E LK Lukasiewicz transitive product transitive every t-norm in fact, it will be T-transitive with respect to every t-norm.

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Now, let us look at what is extensionality of a fuzzy set.

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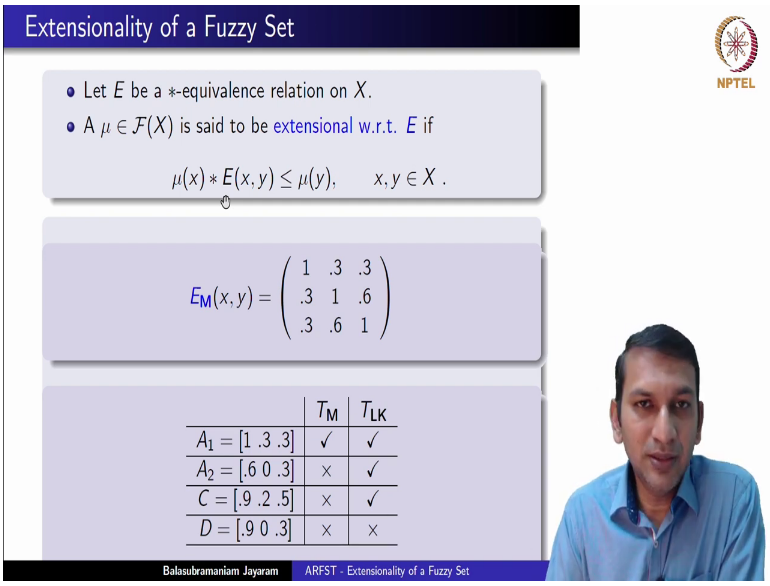

Let us be given a star equivalence relation on X means E is reflexive, symmetric, and star transitive if we take a fuzzy set mu on X it is said to be extensional with respect to E if it satisfies this particular property. What does this say? If we take an x and y, mu of x star E of x, y should be less than or equal to mu of y.

Remember E is an equivalence relation in some sense equality relation or similarity relation we have also seen that it is strongly reflexive; that means, E of x, y as obtained if it is obtained from the bi residuum operation. We know that E is strongly reflexive; that means, E of x, y is 1 if and only if x is equal to 1 is equal to y.

So, in the general case x and y can be different, but let us assume that E of x, y is actually equal to 1; that means, with respect to this equality relation x and y are in fact, indistinguishable for that equality relation they are indistinguishable. So, now, all it says is if x and y are indistinguishable, then what should be the value of the membership value of y in this fuzzy set, in this concept mu.

It says it should be at least as much as x and clearly because of the symmetry of E you could also write it as mu of x star E of y comma x less than or equal to mu of x and if x, y E of x, y is 1 then E of y, x is 1 we see that mu x is in fact, equal to mu y. So, this is one way of ensuring that if x and y are indistinguishable with respect to the similarity relation or the equivalence relation E, then they should have the same membership value in that concept represented by mu.

So, this is the main idea behind defining extensionality like this. So, let us look at the relation E M which we know is the min transitive fuzzy relation, and let us look at some fuzzy sets and see whether they are extensional with respect to this relation. We will see whether they are extensional with respect to remember this E can be any equality relation we are only going to talk about E this mu being extension with respect to E and this star because E is star equivalent first.

And, since this is T M equivalent so, this will be t equivalent for any t-norm. So, we could consider the extensionality of a fuzzy set with respect to this relation with any t-norm in particular we are going to consider the min t-norm and the Lukasiewicz t-norm. Let us consider the first antecedent that we took A 1 which is 1 0.3 0.3 and ask the question is this fuzzy set extensional with respect to E and the t-norm T M.

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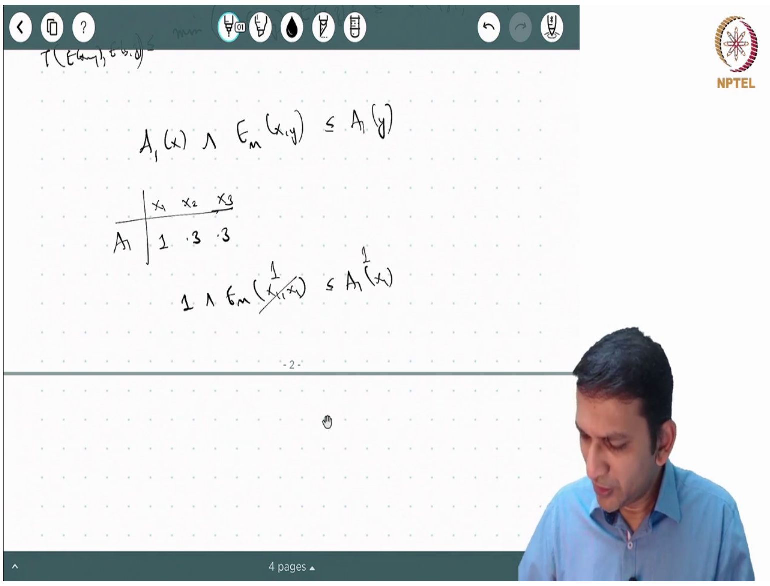

So, what does this mean? If A 1 is given we need to show A 1 of x star here the star is min E m of x y should be less than or equal to A 1 of y and this should happen for every one of them. Now, what is A 1? So, now, we have x to be x 1 x 2 x 3 A 1 is 1 0.3 0.3. So, now, if we take A 1 of x then it is 1 and E M of x 1 x 1 should be less than or equal to A 1 of x 1 clearly this will happen because x 1 x 1 is 1 this is 1. So, the for the diagonal elements we do not need to check.

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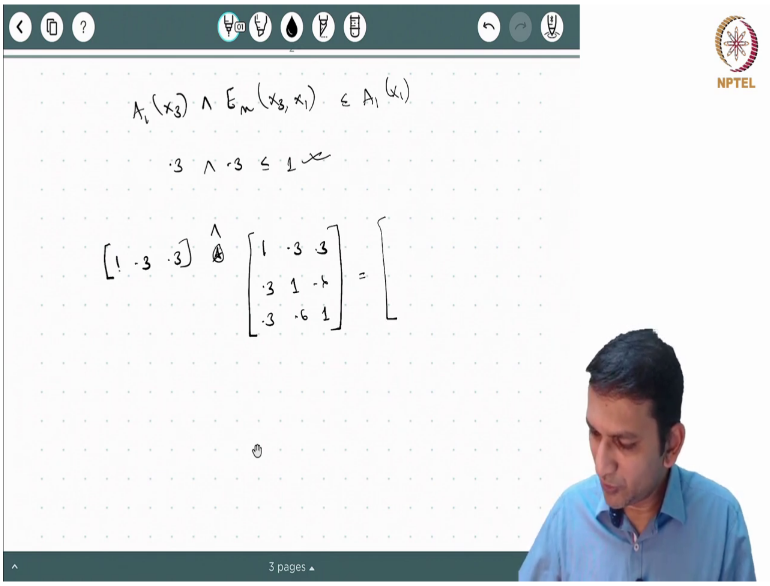

So, let us arbitrarily take A 1 of x 3 and E M of x 3 comma x 1 this should be less than or equal to A 1 of x 1. Now, A 1 of x 3 is 0.3 and what is E M of x 3 x 1. So, this is x 3 x 1 it is also 0.3 this should be less than or equal to A 1 of x 1 which is 1. So, this is true, but we need to check this for all possibilities.

So, the easiest way is to write something like this, let us write A 1 star in which case here it is this and let us put E M here which is 1 0.3 0.3 0.3 0.3 1 minus 6 minus 6 1. And, let us say take each of this and multiply the corresponding value.

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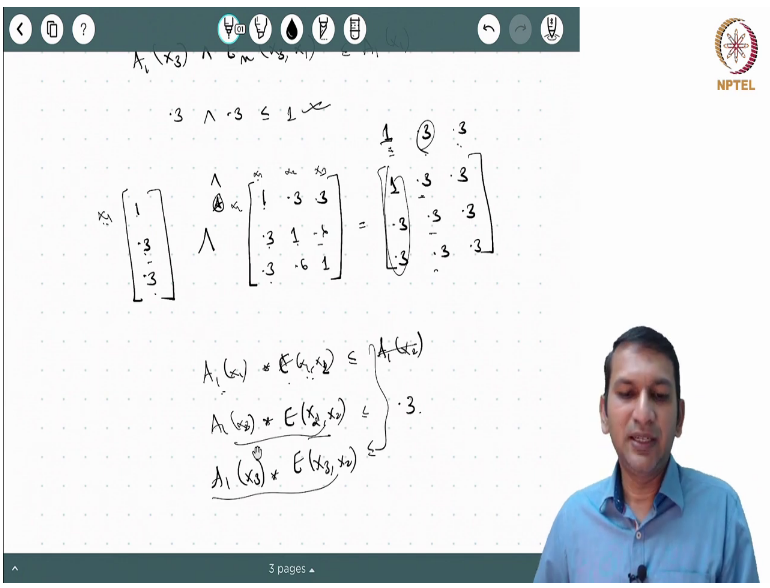

So, perhaps it is easier to write it as column vector 1 0.3 0.3. So, now, all we are going to do is take 1 and multiply with this because this x 1 for us. So, this is x 1 this x 1 x 2 x 3. So, x 1 into E of x 1 x 1 x 1 into E of x 1 x 2 and see whether it is less than equal to the corresponding A 1 of x 1 or x 2 or whatever they write here. So, this if we take it as this matrix so, now, we are losing minimum here 1 and 1 is 1, 0.3 0.3, 0.3 and 0.3 is 0.3, 0.3 and 1 is 0.3, 0.3 and 0.6 is 0.3.

Similarly, 0.3 0.3 0.3 0.3. Now, what do we need? We simply need that if we write A 1 here itself, we need that the max of this should be smaller than this. So, for each one this remember what is this? Let us fix this is A 1 of x 2; what we want is it should be smaller than A 1 x 1 star E x 1 x 2 x 1 x 1 A 1 x 2 star E x 2 x 2 and A 1 x 3 star E x 3 x 2. So, it should be smaller than all of them and what is this value? This value is 0.3.

Now, now what is this A 1 x 1 star A x 1 it is essentially sorry, this is x 2 x 1 x 2. So, this is this value this value is this entire thing is here and this entire thing is 0.3. So, it is easy to just find the matrix and look at if you write the correspond the same set on top a fuzzy set it is easy to see that this element dominates each of these elements dominates the column; that means, it is bigger than all the elements in the column which means this set is in fact, T_M transitive extensional.

So, given A 1 this is extensional with respect to min E M and min; that means, it is also going to be extensional with respect to the Lukasiewicz, it can be check. Now, let us consider A 2. So, we need to do the same thing again.

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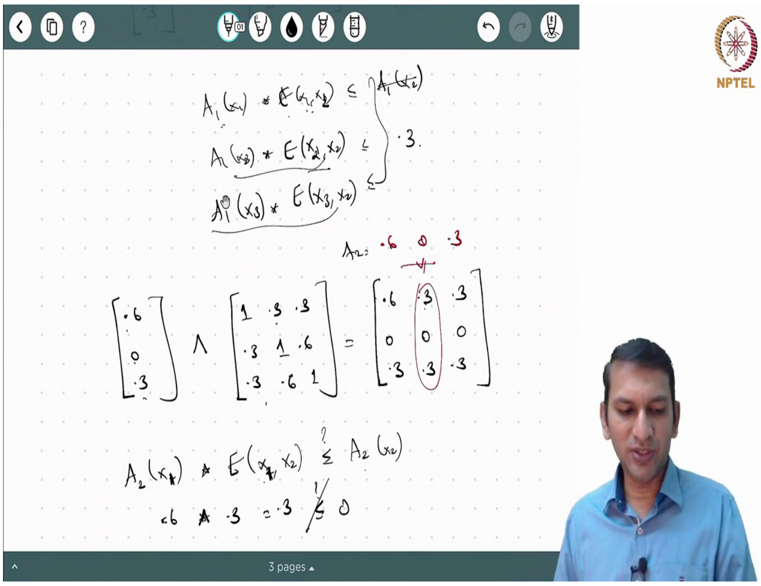

Let us write A 2 like this 0.6 0 0.3 and we consider min. So, 0.6 min 1 is 0.6 0.3 0.3 0 0 0 0.3 and this row 0.3 0.3 0.3. Now, if we write a A 2 here 0.6 0 0.3 it can easily be seen that max of this is 0.3 they are all not smaller than this. Note that simply take like this A 2 of x 2 star E of x 2 x 2 is it less than or equal to A 2 of x 2 and A 2 of x 2 0 star.

Yes, we can just a 2 of x 1 x 1 x 2 A 2 of x 1 is 0.6 star E x 1 x 2 is 0.3 and the star is minimum. So, this is 0.3 is it less than or equal to A 2 x 2? A 2 x 2 0, it is not. So, this is not extensional. So, this fuzzy set is not extensional with respect to min t norm. So, it is not extensional with respect to E M and min t-norm let us look at what happens whether it is extensional with respect to the Lukasiewicz t-norm. So, once again we write it like this 0.6 0 0.3.

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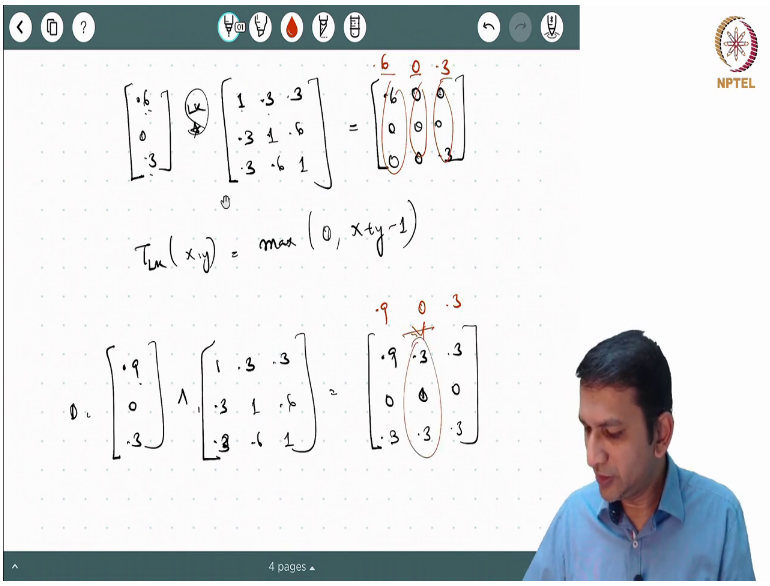

Now, star is Lukasiewicz t norm E M is same 0.6 and 1, now note that we are going to use the Lukasiewicz t norm what is the Lukasiewicz t-norm it says T LK of x, y is max of 0 comma x plus y minus 1. Now, 0.6 and 1 is 0.6, 0.6 and 0.3 0.6 plus 0.3 is 0.9 which is minus 1 is negative, so, it is 0. Once again 0.6 0.3 will be 0, 0 will always be 0, 0.3 and 0.3 is 0.6 minus 1 is minus 0.4 which means it is 0, 0.3 and 0.6 is 0.9 which is less than 1 means 0 and finally, 0.3 and 0.3 is 0.3 and 1 is 0.3.

So, now you see here this is what we get. If we write out the corresponding A 2 here 0.6 0 0.3. We see that this is dominated by 0.6, this is dominated by 0, this is dominated by 0.3 which means this set is in fact, this fuzzy set is extensional with respect to E M and the Lukasiewicz t-norm.

Now, if we consider this fuzzy set it can be easily verified it is not min extensional with respect to E M and min however, it is extensional with respect to E M and T LK. Now, let us consider the last of them which is 0.9 0 0.3 and consider whether it is transitive with extensional with respect to E M and the min operation. So, what we have is 0.9 0 0.3 is equal to what we want to do is take here 1 0.3 0.3 0.3 0.3 1 minus 6 0.6 1. So, we see that we are essentially taking min operation it is 0.9 0.3 0.3 0 0 0 0.3 0.3 0.3.

So, now if we write out the corresponding 0.9 0 0.3 we see that this is not less than this. So, this is not extensional with respect to T_M. Is it extensional with respect to the Lukasiewicz t-norm? Let us look at that also. So, all we are going to do is change this into the Lukasiewicz t-norm.

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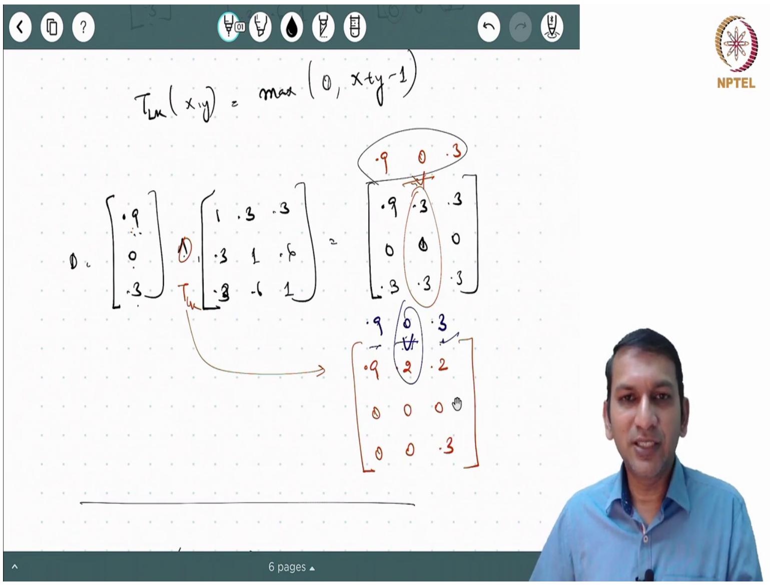

So, allow me to write it here in this case 0.9 and 1 will be 0.9 0.9 and 0.3 will be 0.9 plus 0.3 minus 1 1 0.2 minus 1 which is 0.2. So, this is also 0.2 0 will give you 0 throughout 0.3 0.3 will be 0, 0.3 0.6 will be 0, this is 0.3. So, now what we have here is if we write 0.9 0 0.3 you see that this is dominating it; this is not dominating it; 0 is not greater than this; this is dominating ok, but 0 is not greater than 0.2 which means this is neither extensional with respect to E M and min nor is it extensional with respect to E M and the Lukasiewicz t-norm. So, this is what we have here.

Now, the next question is if a fuzzy set is not extensional with respect to a given equivalence relation, can we make it extensional?

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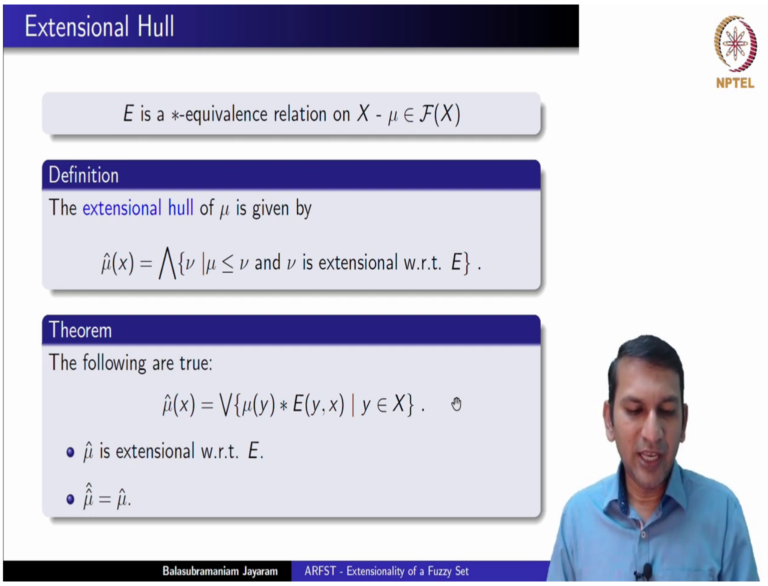

Well, let us begin with a star equivalence relation on X and let mu be given fuzzy set. The extensional hull of mu is given like this. What do we do here? We take all those nu which are greater than mu and nu is also extensional with respect to E and the star that we have fixed among all these nus we actually pick the smallest one.

So, put another way the extensional hull mu which we of mu which we denote as mu cap is the smallest extensional fuzzy set nu which contains mu. So, first of all it should be extensional with respect to E, it should contain min and it should be the smallest one. So, the infimum of all of them, so, the smallest extensional fuzzy set with respect to E which contains mu.

Now, this is something like an external characterization of how to obtain an extensional hull if when you are given a mu.

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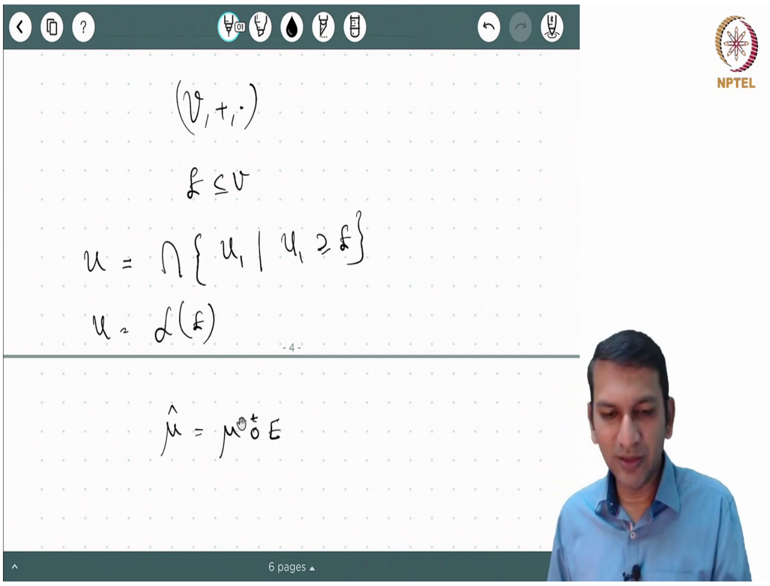

For instance, we know that if we have a vector space V and if we are given a subset E, we can ask the question what is the subspace that E can generate what is the subspace that E can generate. There are two ways to denote this. For instance, the external characterization is the subspace let us denote this subspace obtained from u as u from E as u we can say this is all those subspaces u 1 such that u 1 contains u and take the intersection of them.

You know that arbitrary intersections of subspaces is an is a subspace. So, this is consider all the subspaces u 1 containing E and take the intersection. Another is to say that u is equal to L of E; that means, it is essentially the linear span of E. Essentially what we are doing is taking different vectors from E multiplying them with different scalars from the underlying field and adding them up. And, all possibilities that we consider and put them inside and that is how we grow it.

So, either externally we take all the bigger ones and find the smallest one or you grow it from inside. So, in that sense this definition is the external characterization of what an extensional hull is, but how do we find all the set all the extensional fuzzy sets with respect to E that contain mu and then to obtain the infimum.

So, this is where the following definition comes in handy. It says mu cap given a mu is nothing but given like this and the supremum of mu y star E y, x for every y. So, essentially if you look at it another way so, simply put as mu cap is nothing but mu circle star E that you are considering.

Let us go ahead and prove that this mu cap is in fact, the extensional hull.

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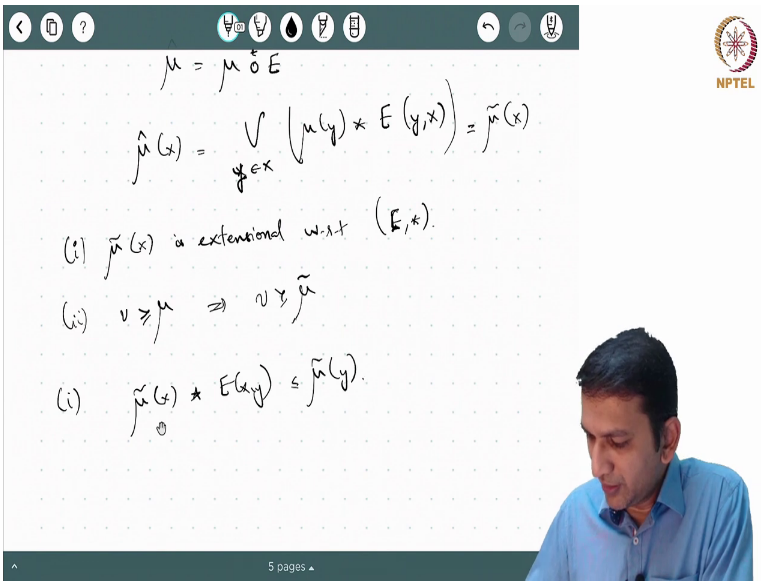

So, what we are given is mu cap of x can be written as supremum x element of x y element of x ok mu y star E y comma x that is what is mu cap of x. So, to show that this mu cap is given like this so, let us denote the right hand side as mu tilde of x, you need to show two things. Firstly, that mu tilde x is extensional with respect to and write it as E comma star this is the first thing that we need to show.

The second thing that we need to show is that if nu is some other extensional fuzzy set and contains mu and we need to show that mu is actually contain nu contains mu tilde. So, we want to show this is the this is the smallest mu contains mu tilde. So, mu tilde is essentially the smallest extensional set with respect to E which contains mu, ok.

Let us prove one. To prove one means we need to show mu tilde of x star E of x, y is less than or equal to mu tilde of y.

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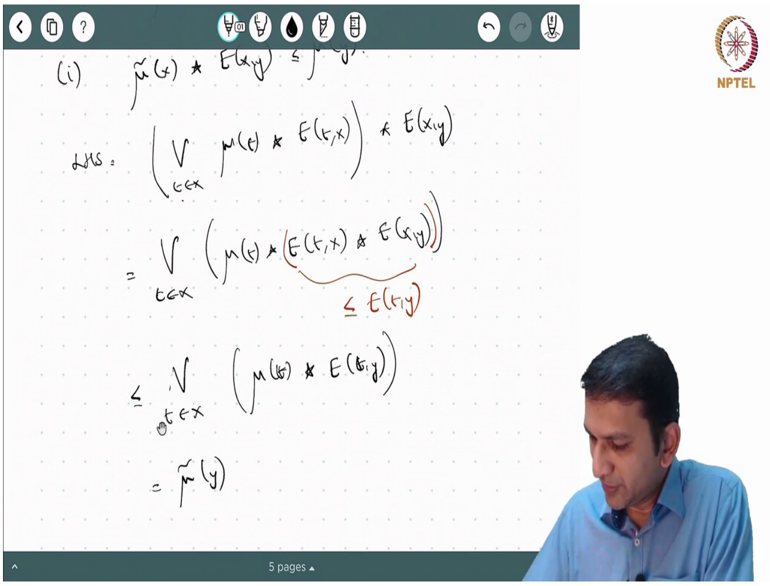

Let us look at the LHS. This is mu tilde of x as supremum over t element of x mu of t star E of t x star E of x, y. Now, this is supremum over t and this is star. We know that star comes star is a t-norm it is coming from a resituated lattice which means it is in fact, left continuous; that means, it is supremum preserving. So, we can simply push this inside what we have is supremum of t element of x mu of t star E of t, x star E of x, y.

Now, because of associativity we in fact, we do not need to put any brackets, but if you consider this fellow, we know that E is a star equivalence relation; that means, this term is in fact, less than or equal to E of t, y. So, overall more than what happens is this less than or equal to t element of x mu of t star E of t, y.

Now, this is essentially the definition of mu tilde y. Note that what is mu tilde of y mu tilde of x supremum over y element of x mu y star E y, x. So, this is essentially the definition of mu tilde of y and what we have shown is that this quantity is in fact, less than mu tilde of y. So, mu tilde is extensional. So, that is the first point we have shown.

So, let us also show the second point.

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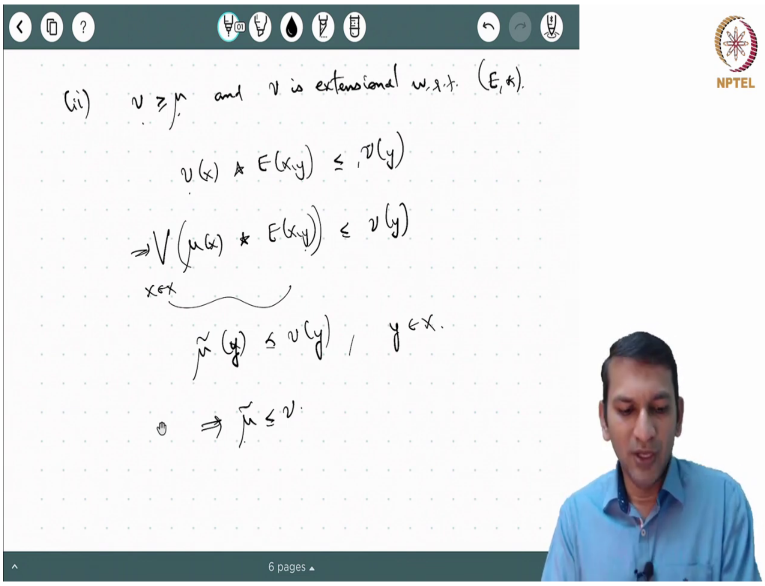

So, what are we given we are given mu is greater than or equal to mu and nu is extensional with respect to E comma star. So, now nu as extensional means nu of x star E of x, y is in fact, less than or equal to nu of y, but we know nu is greater than mu. So, this implies mu of x star E of x y is less than or equal to nu of y now this happens for every x.

So, which means we can put sup over x here this is still true, but this quantity is essentially a mu tilde of x mu tilde of y because y is fixed. So, what we have shown is mu tilde is in fact, smaller than nu of y for any arbitrary y in x. So, which implies mu tilde is less than or equal to nu note that mu was originally extensional we have shown that mu tilde is extensional and we have shown that if there exist any other extensional fuzzy set nu which contains mu then it also contains mu tilde.

In that sense, by the definition of mu cap we see that mu tilde is in fact, equal to mu cap. So, what we have managed to do is that we have given an internal characterization representation of mu cap and this is how we obtain the corresponding extensional fuzzy set. So, let us work this out.

We have seen that note here. So, we saw that 0.9 0 0.3 with respect to E M and min it was not transitive because it is 0 is the thing sorry, it was not extensional. Now, let us find the extensional of this with respect to the same relation.

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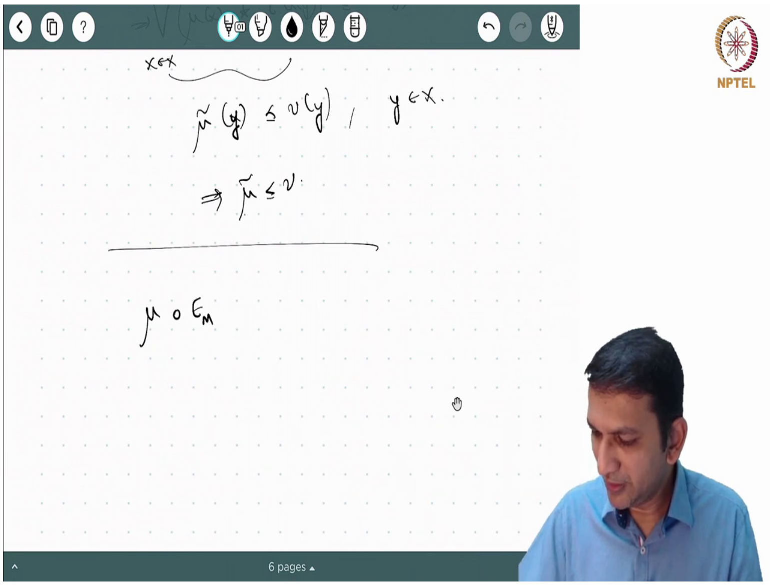

So, all we need to do is take mu circle E M.

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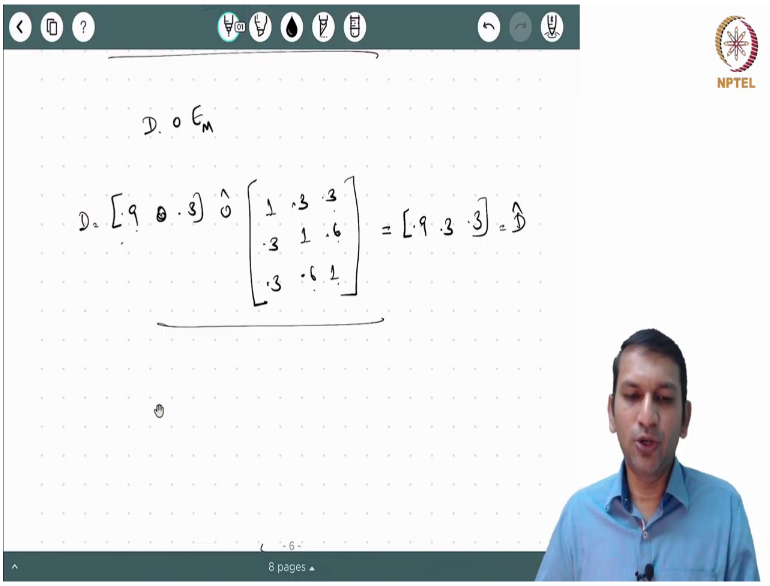

And, here it is mu is in fact, what we have considered as D. So, 0.9 0 0.3 circle min 1. Now, this will be equal to sup min 0.9 0.3 0 0.3 0.3 0.3 0.3 0.3 0 0.3 which is 0.3. So, if consider this as D, then this as our D cap. So, now, we have this mu cap. We have shown that this mu cap is extensional with respect to E and in fact, mu double cap is in fact, equal to mu cap. Now, what does it mean? If you take the extensional hull of extensional hull, then clearly it is equal to the original extensional hull.

Now, let us come back to the observations that we have made. Now, this is an interpolative system. We saw that for these two inputs the outputs were same. Now, we see something familiar. We have seen D to be 0.9 0 0.3 and that is A star and A star cap is 0.9 0.3 0.3 which is essentially the extensional hull of A star. And, we see that whether we gave this a star or it is extensional hull we seem to be obtaining the identical outputs.

Once again it can be verified that 0.7 0.2 0.3 is actually not extensional with respect to this R cap and minimum t-norm, but it is extensional hull is in fact, 0.7 0.3 0.3. And, once again we see that we get identical outputs and when we look this look at this at these two fuzzy sets we can once again see that they are not related to each other as extensional hulls. So, are these two they are not related to each other as extensional hulls and we get this as different outputs.

Now, this gives us some insight in showing that perhaps it is extensionality of fuzzy sets which is playing a role here. We will look into this a little deeper and that is how we will go up to the concept of robustness in the next lecture.

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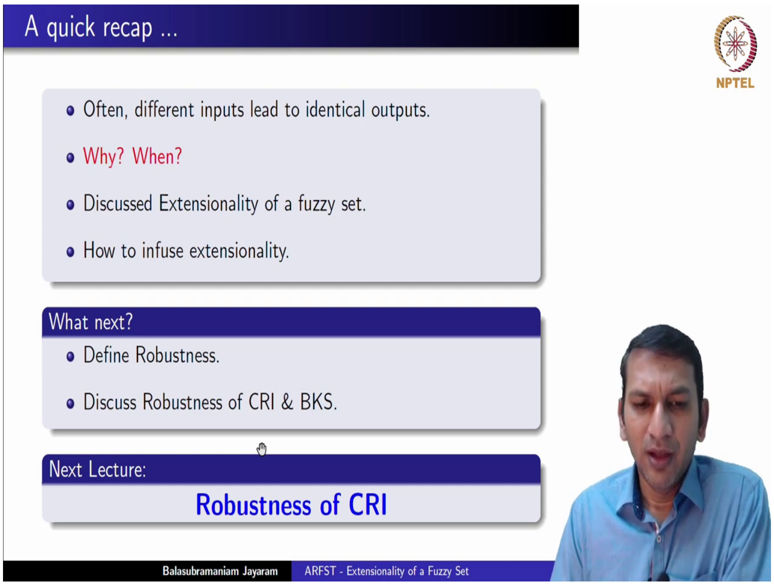

So, a quick recap of what we have done in this lecture. We have seen that often different inputs can lead to identical outputs. We ask the question why. We have to answer it and we will also have to answer when they will be identical. This is something that we will take up in the next lecture itself.

Towards this end we discussed the concept of extensionality of a fuzzy set with respect to a equivalence relation and the operation star and in the case a fuzzy set is not extensional, we have seen how to infuse extensionality, we have defined what is an extension hull and we have also seen some examples.

Going forward, we will define what is robustness of an FRI as we understand and in particular we will look at the robustness of both CRI and BKS. In the next lecture, we will concentrate on the Robustness of the Compositional Rule of Inference.

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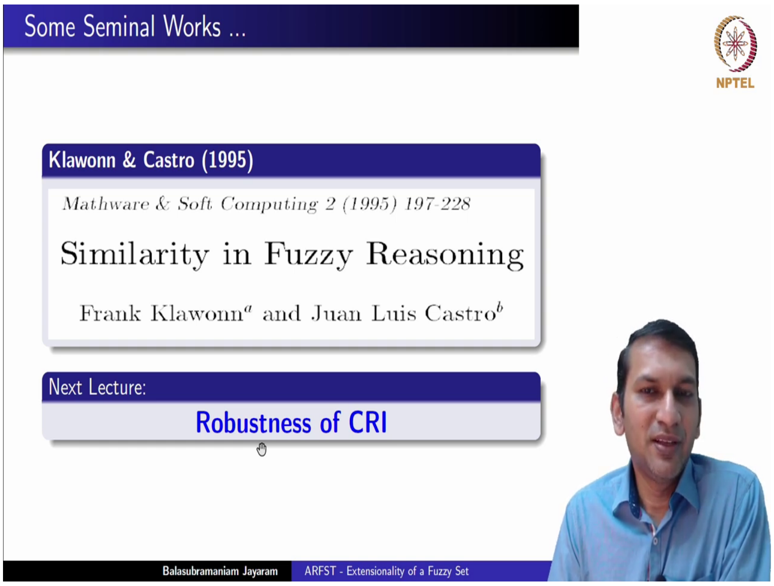

The topics covered in this lecture are very well captured in this paper, a seminal work by Klawonn and Castro almost 27 years back wherein they proposed this entire idea of extensionality and how and showed what happens to inferences especially CRI when you have different inputs related to each other in terms of extensionality how they give out identical outputs. And, this is what we will be looking into in the next lecture when we discuss the robustness of CRI.

Glad you could join us for this lecture and hope to see you soon in the next lecture.

Thank you again.