Approximate Reasoning using Fuzzy Set Theory

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Lecture - 47
Continuous Models of SBR

Hello and welcome to the last of the lectures, in this week 9 of this course titled Approximate Reasoning using Fuzzy Set Theory. A course offered over the NPTEL platform.

In this lecture, we will look at Continuous Models that can be obtained from Similarity Based Reasoning.

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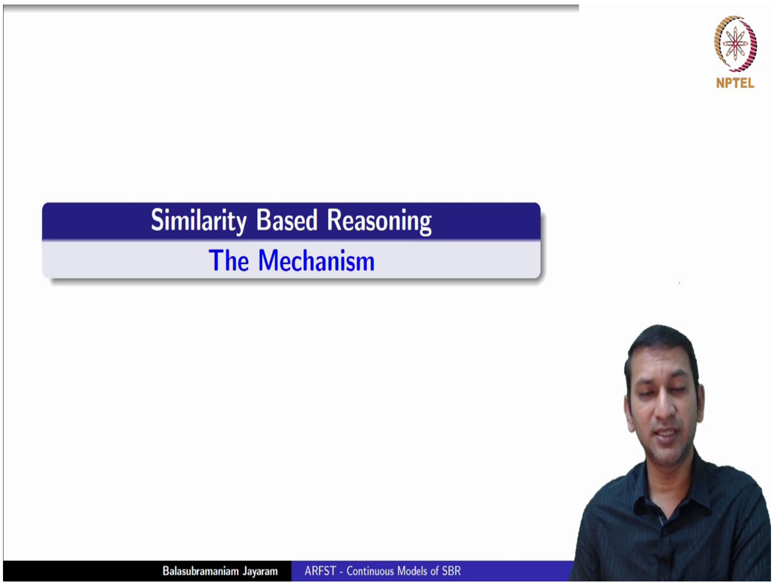

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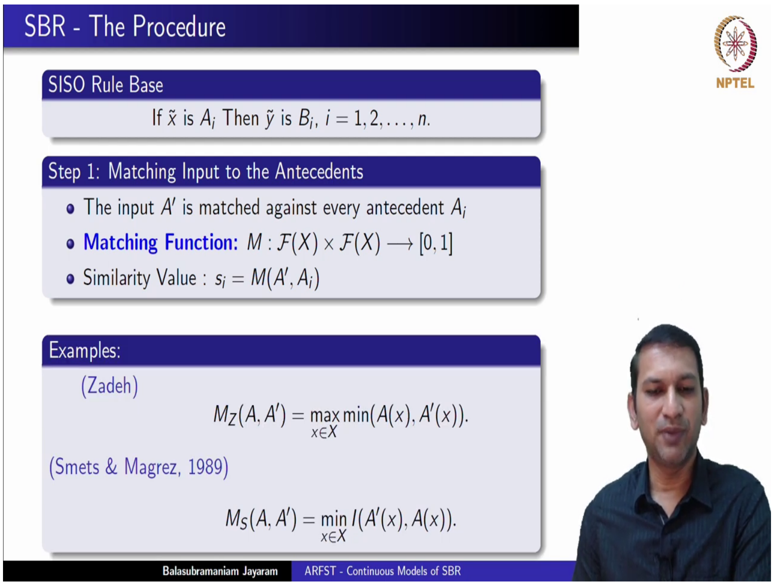

A quick recap of the mechanism of the similarity based reasoning inference schemes. We are given a set of SISO if-then rules. Given an input A dash, first thing we do is match the input with each of the antecedents A i, we use a matching function for that and obtain n similarity values s i.

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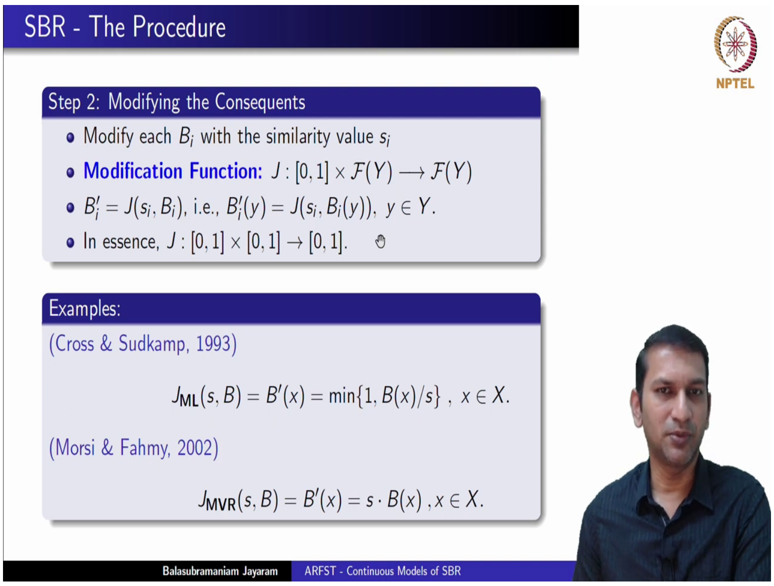

In step 2, these are some examples of matching functions we have seen earlier too. In step 2, we take these similarity values s i and modify the corresponding consequent of the rule ith rule. For this we use a modification function. So, the modification function J is essentially from [0,1] cross F(Y) to F(Y). However, we have seen that it can be seen as a binary function on the unit interval.

These are some examples of modification functions that by now we are quite accustomed with.

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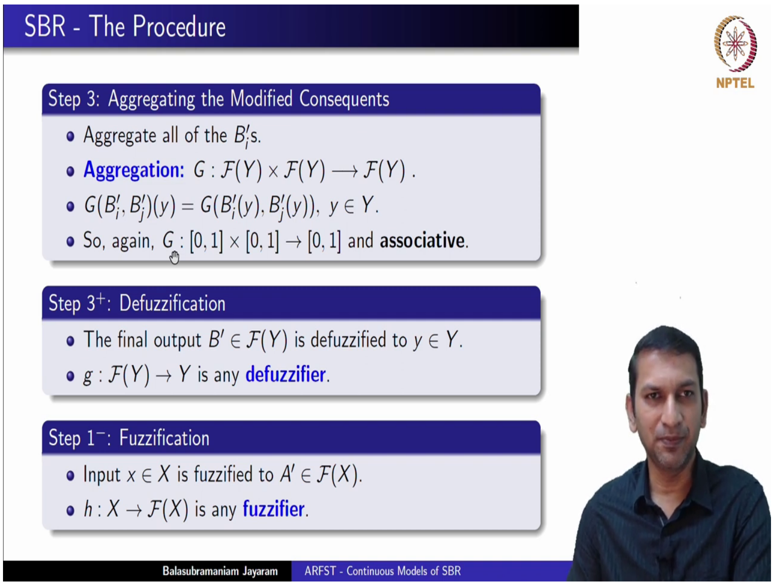

In the third step, all of the modified consequent fuzzy sets B i dash s, they are all aggregated using a function G. For this we use a function G which takes two or more fuzzy sets on Y and gives us out a fuzzy set of Y. We have seen that this G can be considered once again as a binary function on [0,1] along with the associativity property.

Now, if you stay within the realms of mapping from fuzzy sets to fuzzy sets, this is sufficient. However, often there is a need to defuzzify the output fuzzy set or the input may be in the form of a real number or real vector which needs to be fuzzified. So, we need a defuzzifier and a fuzzifier, as some as pre and post-processing steps.

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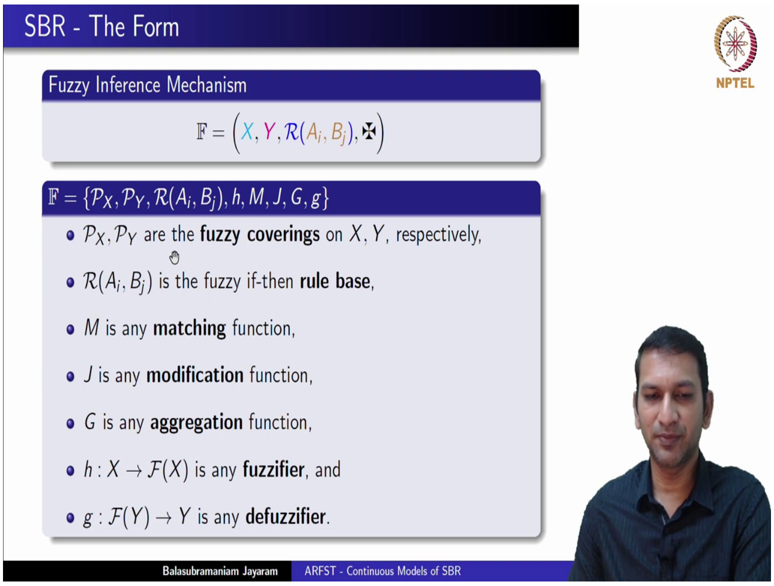

Now, this is a general form of a fuzzy inference mechanism. In the case of an SBR, we know these are the corresponding components. P X and P Y are the fuzzy coverings on X and Y, respectively. R of A i, B j gives us the fuzzy if-then rule base where this A i's and B j's, the antecedents and consequence are actually picked from the fuzzy coverings P x and P Y.

M is any matching function, J is any modification function, G is any aggregation function, h is the fuzzifier and small g is the defuzzifier.

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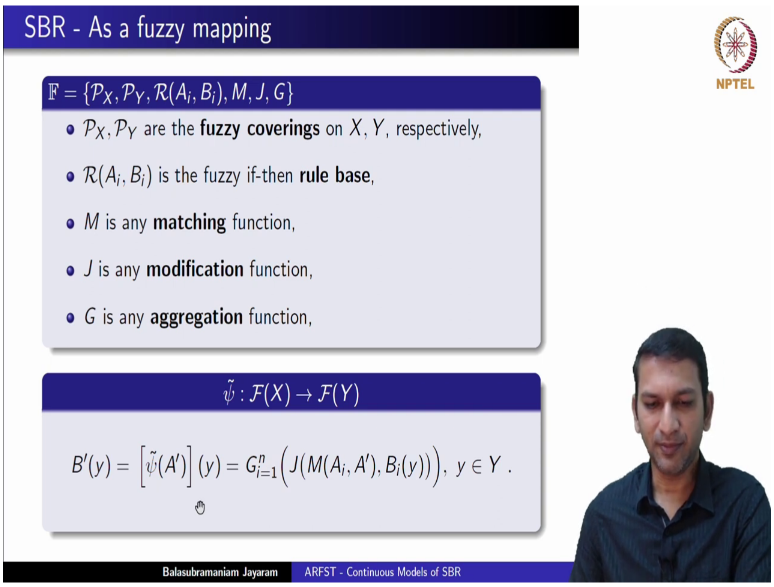

However, if you look at SBR as fuzzy mapping that is a mapping between the set of all fuzzy sets on X to set of all fuzzy sets on Y, then that fuzzifier h and the d fuzzifier small g, they do not play a role. So, these are the factors that will affect the inference.

And given this how will the overall function look like? We denote it by psi tilde. So, given A dash the corresponding B dash of B dash which is a fuzzy set on Y is specified like this, B dash of y is, we are aggregating over the rules given an A dash.

First we find the similarity value with each of the antecedents using the matching function. Use this s i to modify the consequence, the corresponding consequence with the modification function G J and then aggregate it using the function G.

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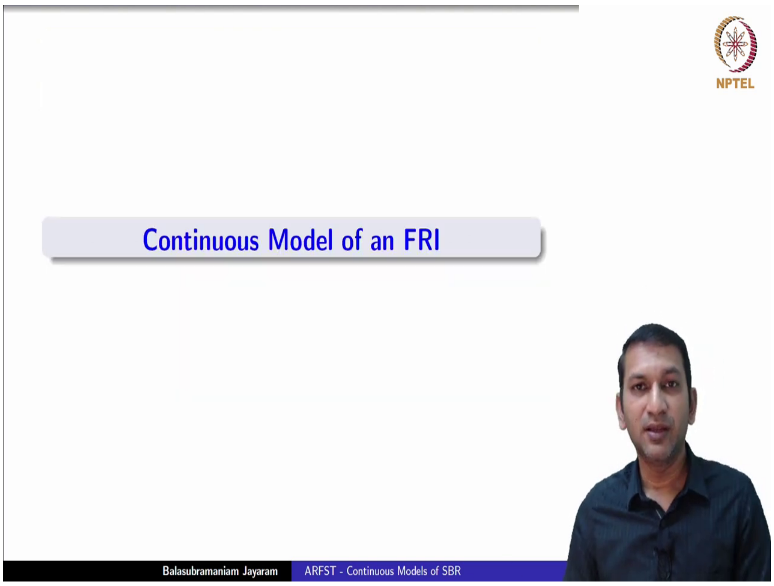

Well, let us revisit the continuous model of an FRI and define the continuous model of an SBR accordingly.

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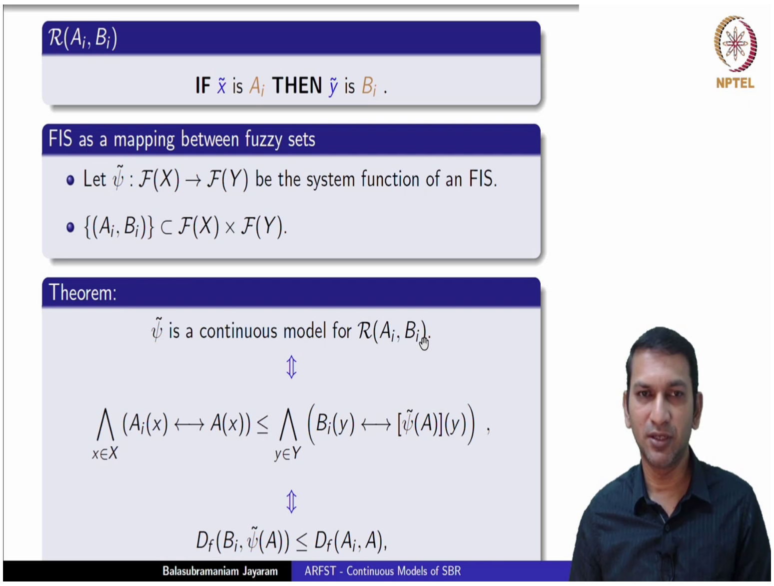

So, we have this system of if-then rules, now we are looking at a fuzzy inference system as a mapping between fuzzy sets. And the antecedent consequent pairs as coming from the corresponding Cartesian product of the input and output fuzzy spaces. Based on this, we said a psi tilde is a continuous model for the given rule base, if and only if, this particular inequality is valid.

Of course, we decoded or interpreted this as the continuity equation or inequality because we could show that in some sense it is giving us Lipchitz continuity with at the points A i, B i with respect to a particular metric that we have actually constructed from the additive generators of t nodes.

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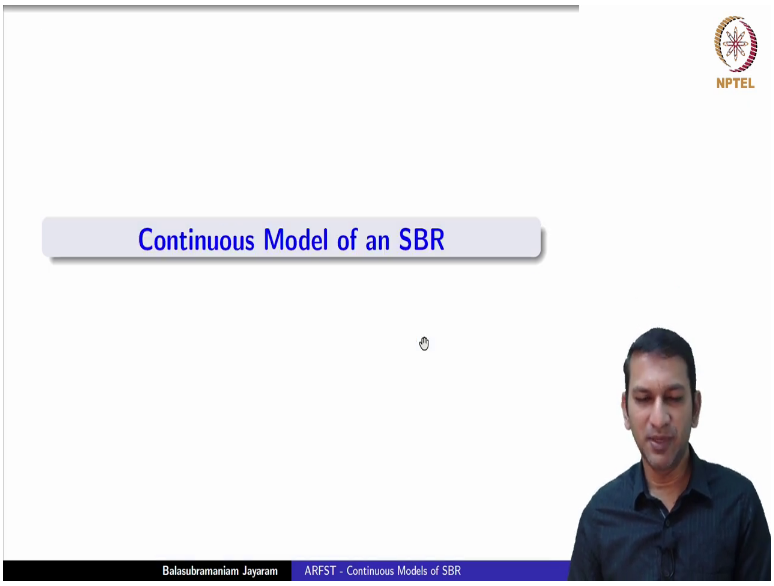

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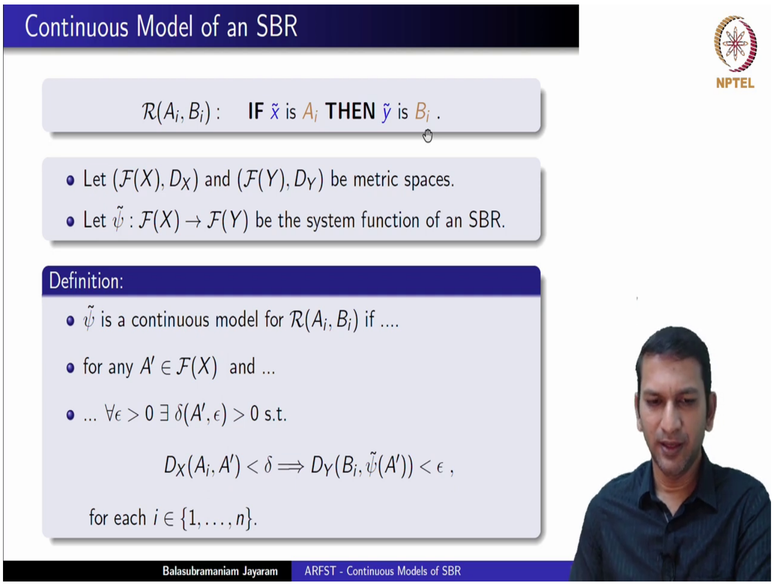

Now, we will try to do a similar thing now. Once again we have the rule base. Let us assume there are metrics defined on the input and output spaces. They are given as D X and D Y. And now we are looking at the system function of an SBR psi tilde as a mapping from F(X) to F(Y).

We define psi tilde to be a continuous model for this given rule base, if for any input A dash coming from F(X) and for every epsilon any epsilon greater than 0, if we are able to find a delta of course, this delta will depend on both A dash and epsilon, such that whenever A dash is delta close to A i on the input space. Then, we want that the output from psi tilde at A dash is epsilon close to B i with respect to the metric that is defined on the output space of fuzzy sets Y F(Y). And this should happen for every i.

So, this is what we would like to define as the continuous model of an SBR and this is how it has been dealt with in general in FRI also.

We want to base it on the given ground truth of values which are these pairs of fuzzy sets is what we called fuzzy points A i, B i which are the antecedents and consequence in the fuzzy if-then rule base that is given to us.

With this definition let us try to proceed further.

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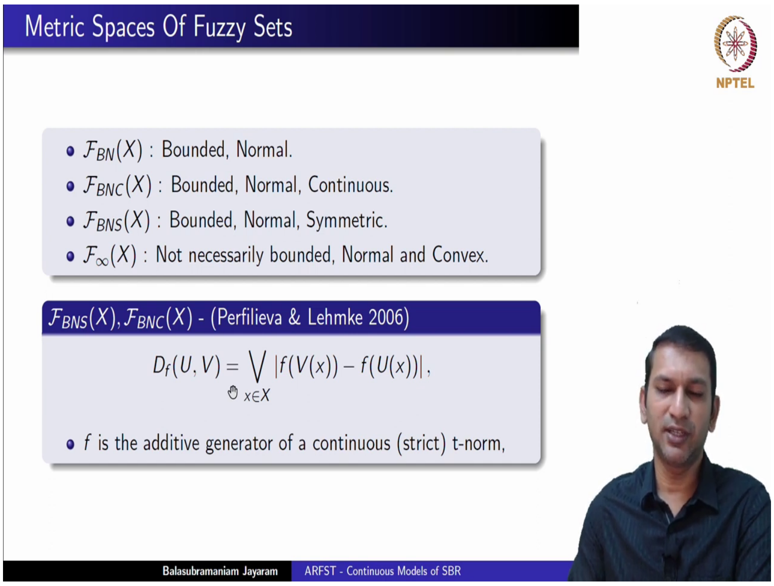

That means we need metrics on the spaces of fuzzy sets.

Let us look at a few examples of such metrics. We have seen one in the previous lecture. We will see a few more in this lecture. By F P and F X given an X, we denote the set of all bounded normal fuzzy sets on X. F BNC would mean bounded normal and continuous fuzzy sets on X. BNS would imply bounded normal and symmetric fuzzy sets. And by this F infinity of X, we consider normal and convex fuzzy sets, the set of all normal and convex fuzzy sets defined on X, but they need not necessarily be bounded.

So, now we have seen this particular metric in the previous lecture that was proposed or defined by Perfilieva and Lehmke in the earlier work 2006. We know that this is a metric on the space of all fuzzy sets and in fact, specifically on the space of all bounded normal, symmetric or continuous fuzzy sets, it has some interesting properties too. Of course, note that this f is the additive generator of a continuous t-norm, not necessarily strict, it could also be generalized to the case of non-strict or nill potent t-norms.

But yesterday for convenience sake, we have seen coming from the strict t-norm generators. Of course, in that context the purpose was to interpret the inequality which we called as the continuous model inequality because that is what was specified as to be satisfied for a psi tilde to be called a continuous model of the given rule base. So, in that limited context, we considered only the additive generators of strict t-norms.

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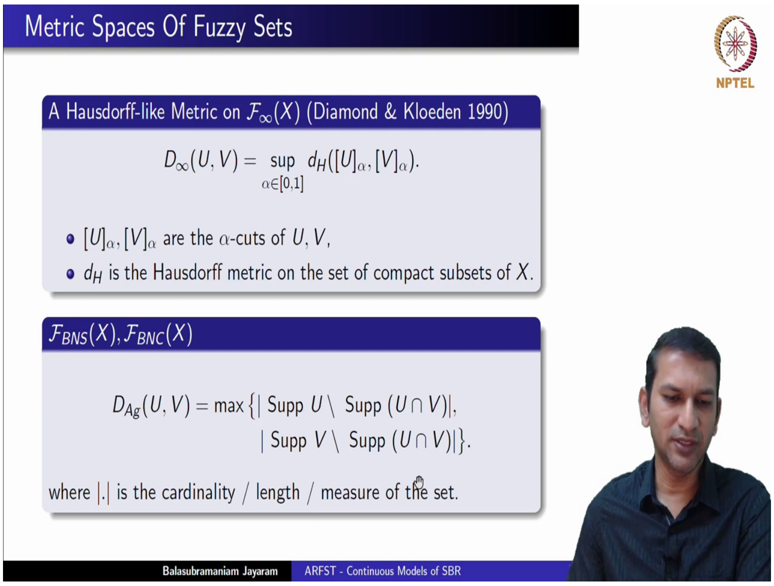

Now, on the space of unbounded fuzzy sets, we could define such a metric where this U alpha and V alpha the alpha cuts of U and V, and d H is the Hausdorff metric, the usual Hausdorff metric and the set of compact subsets of X.

If we consider bounded normal, symmetric or continuous fuzzy sets on X, we could also define such a metric where by this mod operation, it could denote either the cardinality in case it is finite or the length if it is an interval or the measure of the set.

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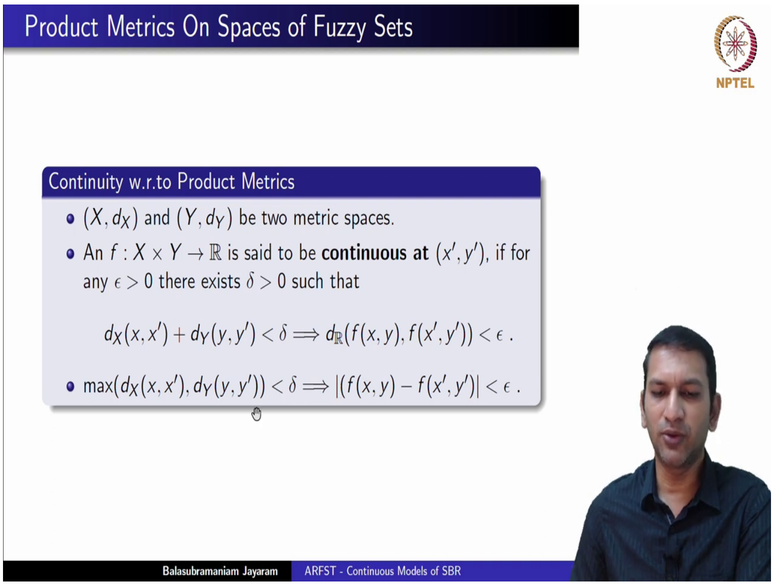

Now, we will need to deal with continuity of binary functions. So, let us look at how to define continuity with respect to product metrics. So, if you have two metric spaces X and Y with the corresponding matrix D X and D Y, and F from X cross Y to R is said to be continuous at a point X dash Y dash.

If for any epsilon greater than 0, there is a delta once again depends on the points and also the epsilon such that D X of x dash plus D Y of y y dash less than delta implies. With respect to the corresponding metric defined on R the function values at x, y and x dash, y dash, the distance between them d R of f x, y, F of x dash, y dash should be less than epsilon.

Of course, there is one way to define continuity. Instead of taking the addition the sum of these two values, we could also take the max of these two values and insist that when this is less than delta, then we want this to happen with respect to the usual metric on R.

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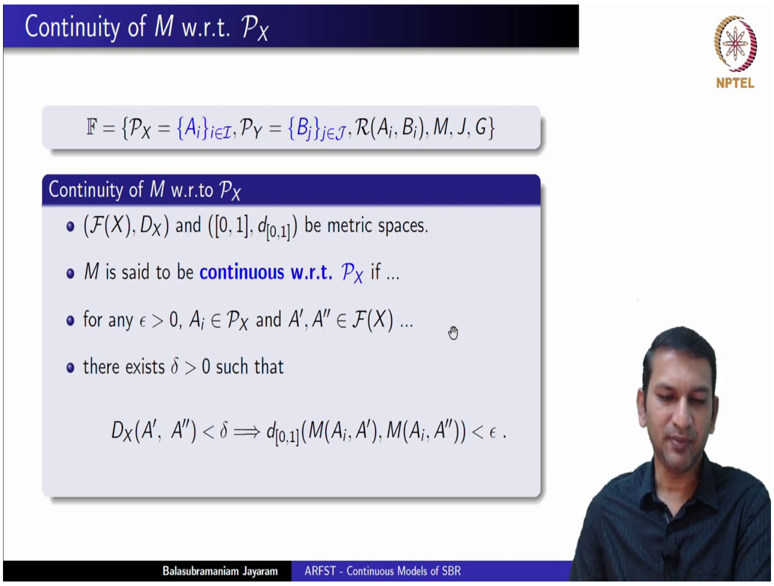

Well, to discuss continuous models of SBR, we know that there are many factors that go into making the SBR inference mechanism, the modification function, the matching function, also the aggregation function. So, these components also need to be continuous in a certain way. Let us look at those specifications.

Now, in the case of SBR as a mapping from F(X) to F(Y) these are the components and P x and P Y are the coverings, A i's are the pieces in the partition, similarly B j are the pieces in the fuzzy covering over Y.

Given F X and D X is the metric on it, and the unit interval [0,1] and a metric d [0,1] on it, we say M is continuous with respect to this covering P X, if for any epsilon greater than 0 and any A that you pick from P X and given any two fuzzy sets A dash and A double dash coming from F(X). There should exist a delta depending on all these parameters.

Such that D X of A dash, A double dash, whenever it is less than delta it should imply that d of [0,1] of the with respect to the metric defined on [0,1], the similarity values of A dash to A i and A double dash to A i should be smaller than epsilon.

Now, remember this A i is also picked arbitrarily, so we do not specify i separately again. So, we pick an epsilon A i and two fuzzy sets A dash A double dash, for that we should have a delta such that whenever A dash and A double dash are delta close, then we know that similarity values to a the a that we have picked is also epsilon close.

So, this is when we say the matching function is continuous with respect to the fuzzy covering that we have considered.

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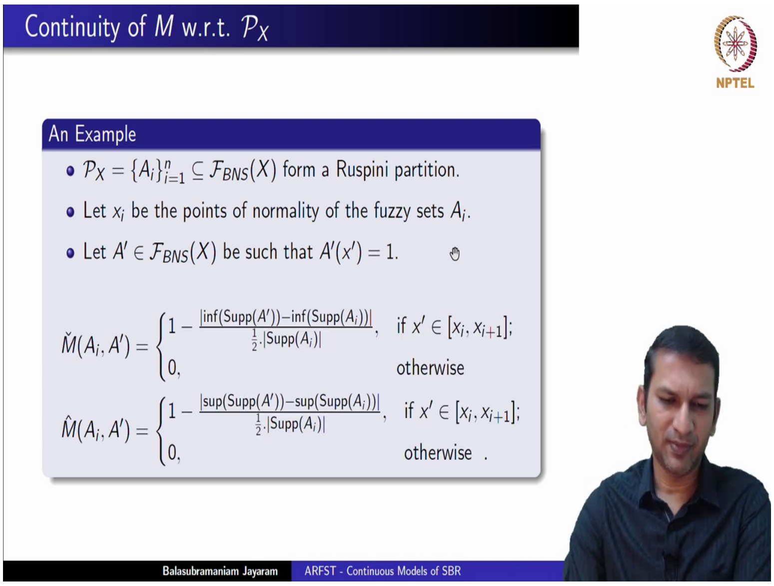

Are there examples of such matching functions? Well, yes look at a covering of a set based on fuzzy sets which are bounded normal and symmetric.

Let us assume that these form a Ruspini partition of X. We could also assume X itself to be bounded. And let x i be the points of normality of this fuzzy sets A i. And given an A dash which is a once again coming from this F BNS of X so that means, it is bounded normal and symmetric, let it attain normality at X dash and consider these two matching functions.

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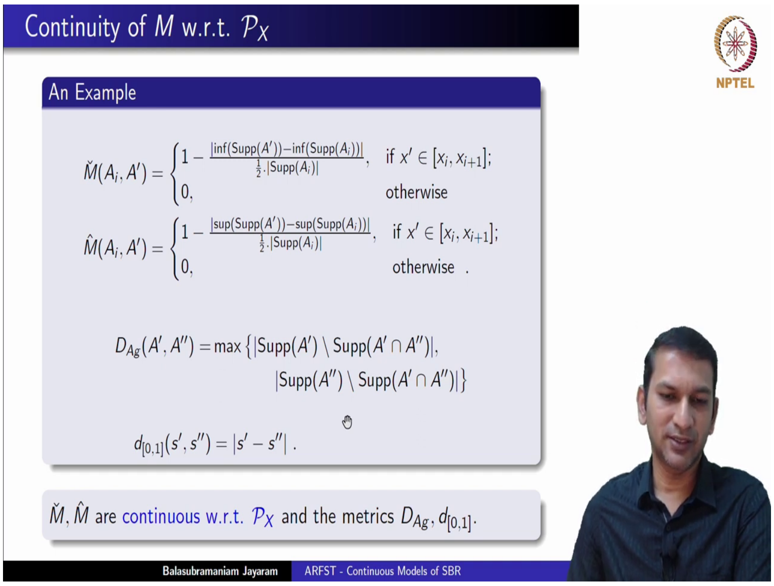

Now, it can be shown not without much difficulty that if we consider these two metrics the metric D Ag on F(X) and the usual metric mod on [0,1] then both these matching functions are in fact, continuous with respect to the P X that we have considered and the matrix that we are we have listed out here. So, there are examples of matching functions which are continuous with respect to a given covering of X.

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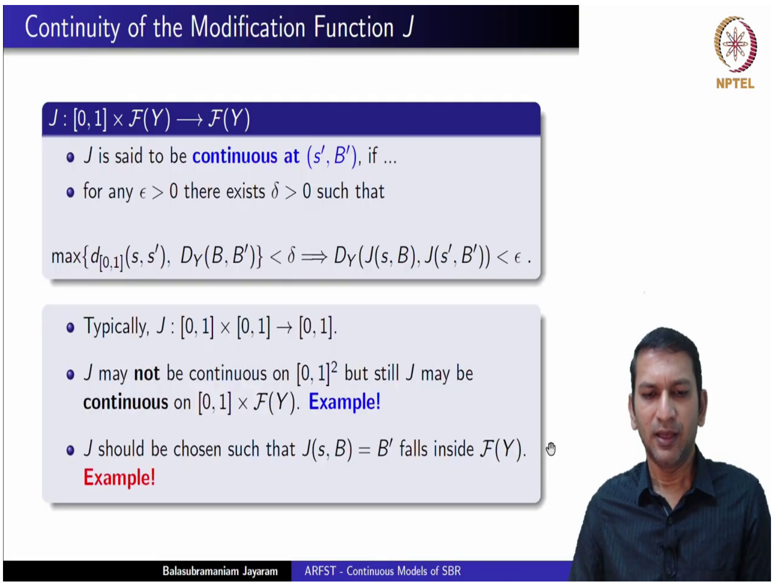

Now, let us look at the modification function. We know that it takes a similarity value which comes from [0,1], takes the consequent B i which is a fuzzy set on Y, modifies this and gives us a B i dash which is again a fuzzy set on Y. So, it is a function from [0,1] plus F(Y) to F(Y).

We say this J is continuous at a point s dash B dash, s dash is from [0,1], B dash is a fuzzy set on Y. If for any epsilon greater than 0, there is delta greater than 0, such that max of this quantity whenever it is less than delta, then we want that the corresponding modified fuzzy sets with respect to the metric on Y is less than epsilon.

Now, there is the usual definition of continuity with respect to the product matrix that we have considered. Note that we have seen that j typically can be considered to be a binary function on the unit interval. What needs to be seen is we can have a J which perhaps is not continuous on [0,1] square, but still J may be continuous on this set. That means, J may be continuous as a function on [0,1] cross F(Y) to F(Y).

Now, let us look at an example.

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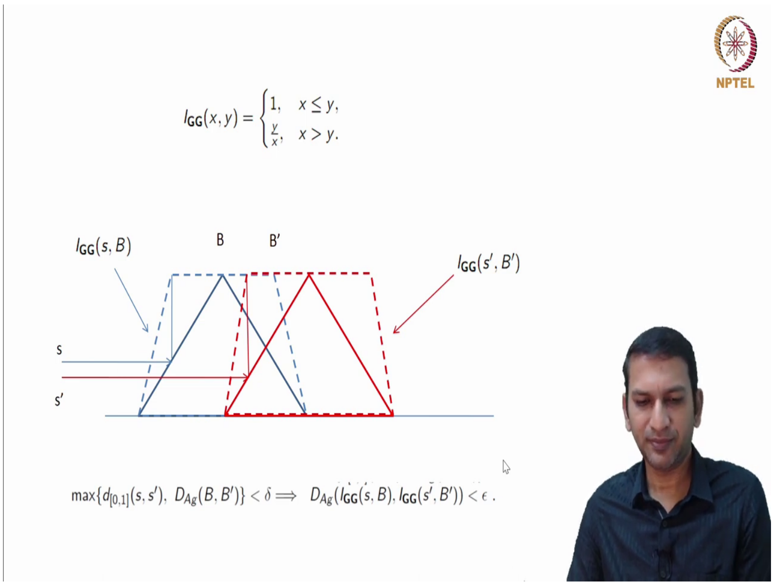

Let us take the Goguen implication because for modification often we use the implication function. Clearly the Gogusen implication is not a continuous implication.

Let us consider two fuzzy sets B and B dash. These are the fuzzy sets that we want to modify. Let us assume that s is the similarity value coming from [0,1] with which we want to modify this B. This is the value s. Now, using the Goguen implication, if you modify this then this is the modified fuzzy set that we get.

Similarly, you can also note that at this point where s is it attains normality because y by x, so it goes above 1. It its it touches 1. So, similarly, if this is the similarity value s dash, using this if you are modifying B dash, this is how it would look like using the Goguen implication.

Now, it can be clearly seen it can be proven based on the formula that whenever s, s dash and B B dash are such that this inequality is held, for any given epsilon, we can find a delta such that if this is smaller than delta, then using the same metric on y the corresponding fuzzy sets B B dash are in fact, less than epsilon. So, it can be shown that continuity can be had even when the function J that we use is not a continuous function on [0,1] square.

However, we should be careful in choosing the J, because the modified fuzzy set should again fall back within the space of fuzzy sets that we are considering for Y.

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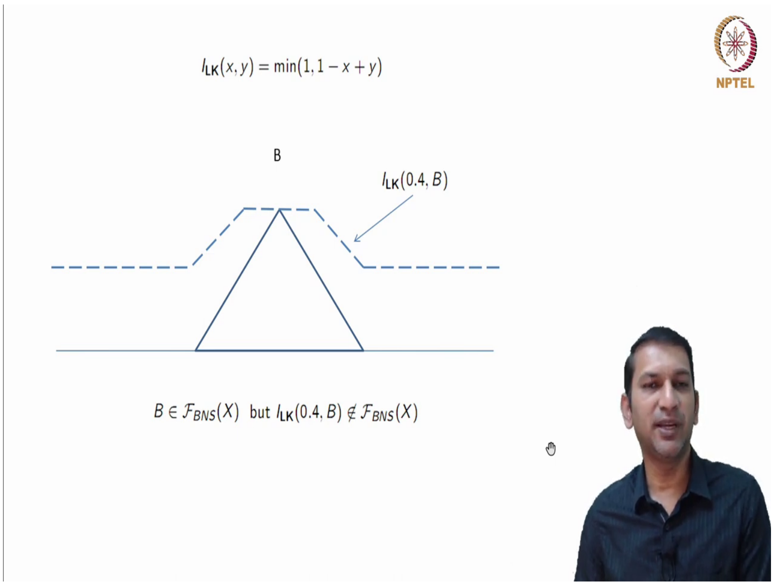

Consider for instance the Lukasiewicz implication and this particular B. Clearly, this is a bounded normal symmetric fuzzy set. So, if you are assuming the to come from F BNS of Y, look at what happens if you take the similarity value 0.4 and use the Lukasiewicz implication as the modification function.

Then, the modified fuzzy set is something like this. While it is normal and seems symmetric, it definitely is not bounded. So, we should be careful in the choice of the modification function J, even if you are considering them to be binary functions on [0,1].

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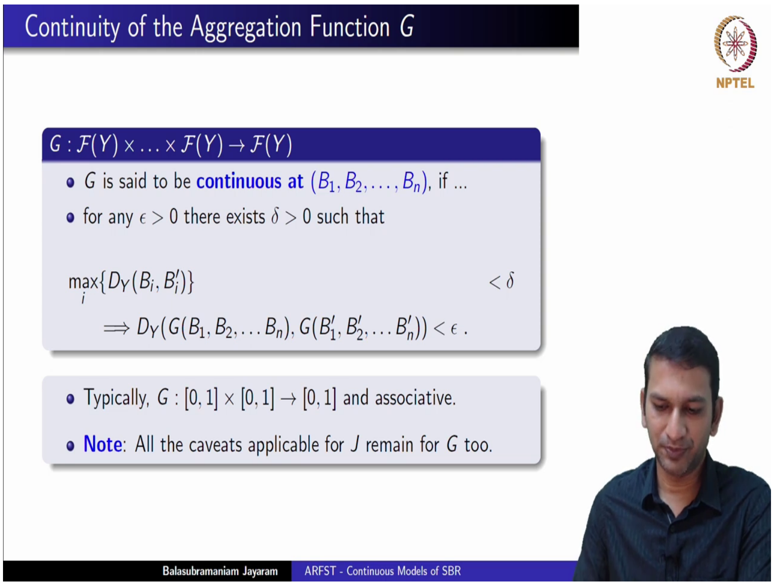

Well, we can again define the continuity of the aggregation function. So, this takes a few fuzzy sets on Y and gives out a fuzzy set on Y. We say that for a fixed n it is continuous at B 1, B 2 so on till B n. If for any epsilon greater than 0, there exist delta greater than 0, such that when the maximum of B i B i, B i dash, if this is less than delta, then this implies that the corresponding aggregated values should be smaller than epsilon with respect to the metric Y that we have on F Y.

Once again, we have seen that typically G can be any binary operation on [0,1], but needs to be associative. And note that all the caveats that are applicable for J remain for G too.

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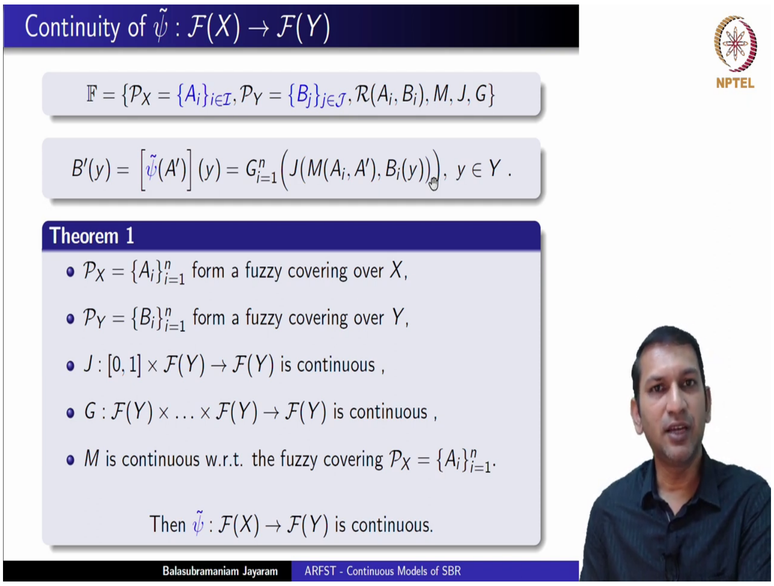

Now, with this we are in a position to talk about continuity of the system function obtained from the similarity based reasoning. Note that this is the form of the similarity based reasoning that we are considering. And now given an A dash, the output B dash is obtained as follows B dash of y is psi tilde A dash of y which is nothing, but this.

Now, the main result on this topic of this lecture is this. Let P X form a fuzzy covering over X that is these are the ones that we pick from the antecedents. P Y form a fuzzy covering over Y. J as a function on [0,1] cross F(Y) to F(Y) is continuous. G as a function on F(Y) cross F(Y) so on n times n copies or some arbitrary copies of F(Y) to F(Y) is continuous.

And M is continuous with respect to the fuzzy covering P X that we have considered. If you have all these things then what we can claim is the system function of SBR psi tilde is in fact, continuous as a function from F(X) to F(Y).

Note that we are, this result is stronger. It claims psi tilde to be continuous, not just a continuous model of the given rule base. So, we will take a look at the proof of this result. As was mentioned, it is a stronger result we need to prove psi tilde is in fact, continuous as a function from F(X) to F(Y).

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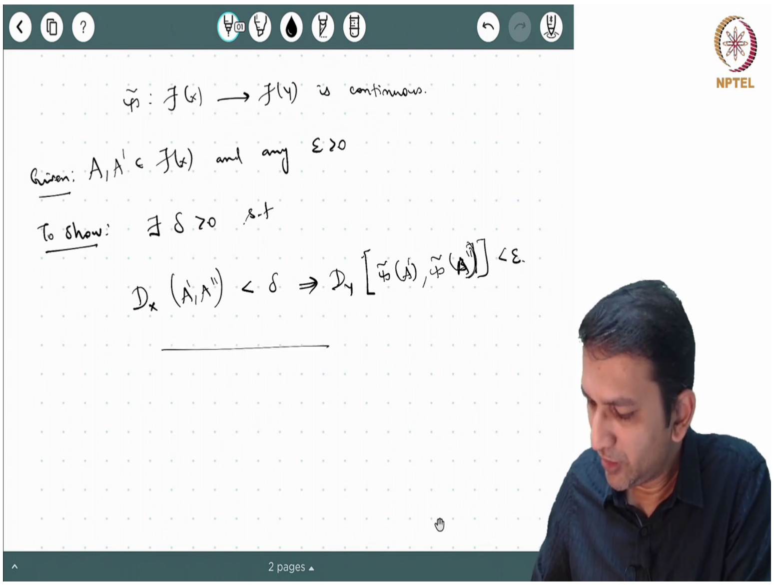

Well, what does it mean to show that psi tilde as a function from F x to F y is continuous? Given any A A dash element of F(X), we need to show and any epsilon greater than 0, we need to show there exists delta greater than 0, such that whenever D X of a A dash is less than delta this should imply D Y of psi tilde of A dash become a psi tilde of a double dash is less than epsilon. So, this is what we need to show.

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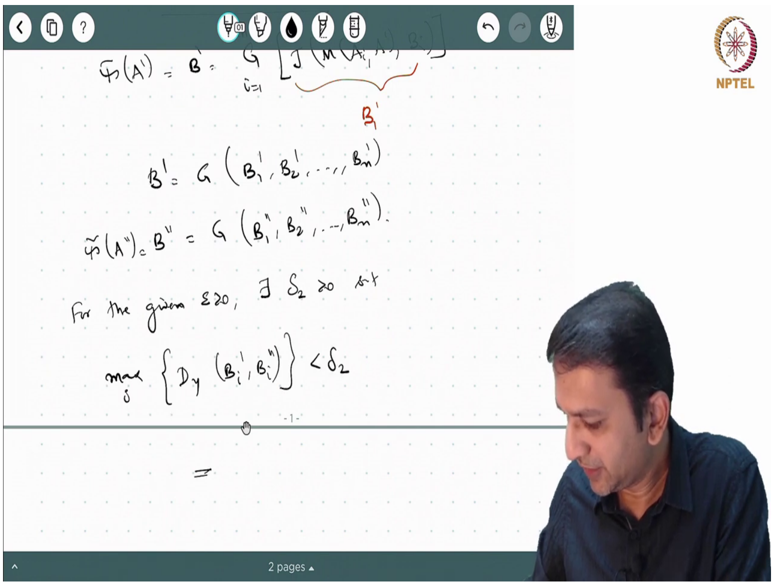

Let us look at what is the form of B dash itself. Here given A dash psi tilde of A dash is B dash. Now, this is given as G over i is equal to 1 to n. So, this is the formula that we have finally, for psi tilde. J of M of A i comma A dash comma B i.

So, now, writing this quantity as B i dash, we could write B dash as G of B 1 dash B 2 dash B n dash. Similarly, psi tilde of a double dash is B double dash can be written as B 1 double dash B 2 double dash so on till B n double dash.

Now, let us make use of the conditions that are given to us. G we know is continuous. So, now, for the given epsilon greater than 0, they will always exist a delta 2 greater than 0. Such that whenever max over i D Y of B i dash B i double dash if this is less than delta 2, then this implies D Y of G of B 1, B 1 dash, B n dash comma G of B 1 double dash B n double dash essentially that is D Y of B dash B double dash is less than epsilon. So, for a given epsilon we know such a delta 2 exists.

So, let us map this. Because of the continuity property of G.

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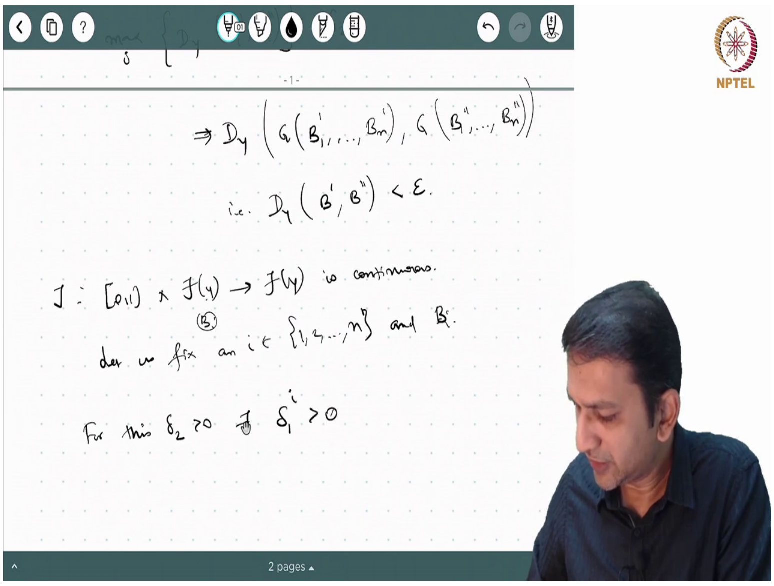

So, now we also know J as a function of [0,1] cross F(Y) to F(Y) is in fact, continuous. Let us fix an, i from 1 to n and hence fix a B i.

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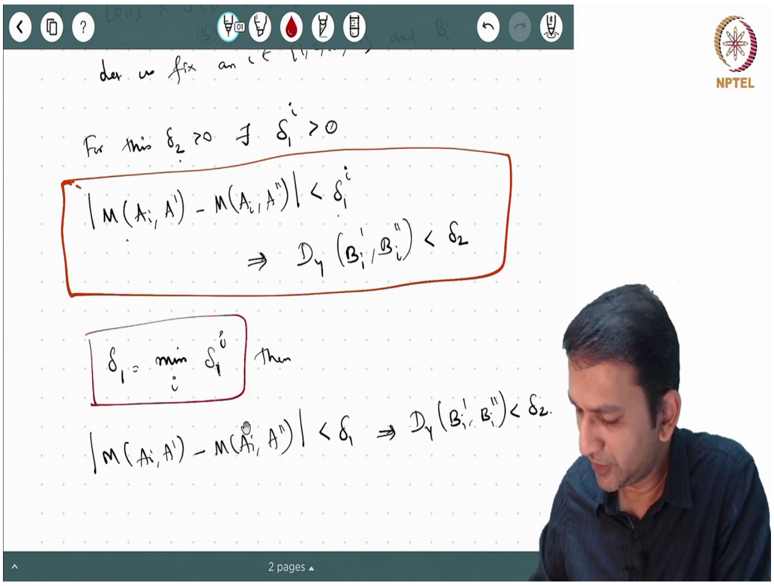

Now, if you fix this B i from this space, then what we can see is since it is continuous for this delta 2 that we have, considering it as an epsilon, locally there will exist a delta 1 i greater than 0 this i is related to this i that we have fixed. Such that mod M of A i comma A dash minus M of A i comma A double dash this, whenever this is less than delta 1 i this implies D Y of B i dash B i double dash is in fact, less than delta 2.

Note that we are getting this because of continuity of G. We have fixed. What have we done? We have fixed a particular i, and once we fix a particular i, we are looking fixing the B i we are looking at what happens to J as a function of single variable because one of the components we have fixed.

We see that when delta 2 is fixed looking at it as an epsilon because of continuity of J, we see that for this delta 2 and a fixed i there will always exist a delta 1 i. Such that mod M of A i comma A dash minus M of A i comma A double dash whenever it is less than delta 1 i this implies D i D Y of B i dash B i double dash will be less than delta 2.

Now, note that this is for one i that we have done. So, if we consider delta 1 to be minimum over all the i's delta 1 i, then what we can say is M of A i comma A dash minus M of A i comma A double dash whenever this is less than delta 1, this implies the corresponding B i B i dash, B i double dash is less than delta 2. So, delta 1 is essentially the minimum of all these delta 1 values.

Well, we are almost there, in the final step.

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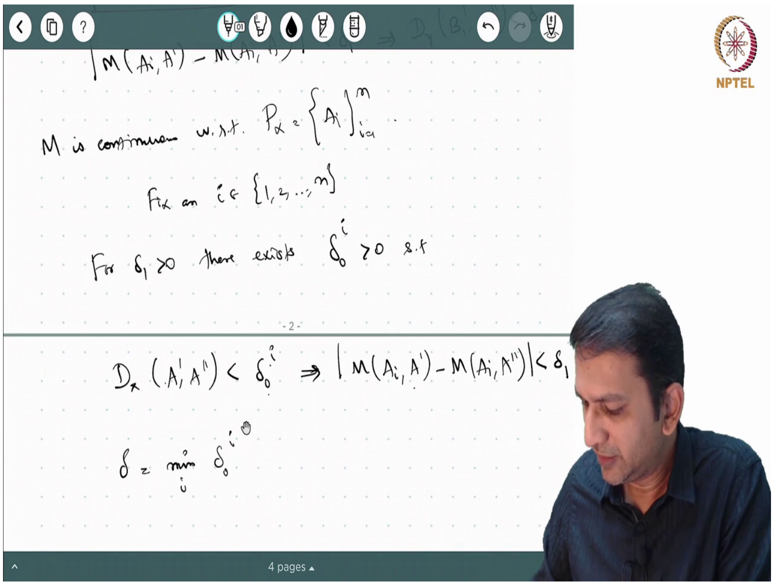

Notice once again that M is continuous with respect to P X which is A i, i is equal to 1 to n. What it means is if you once again fix an i, note that we fixed an i here, but we generalized it in the sense that we took delta 1 to be the minimum of all of them. So, now, this delta 1 will work across any i.

So, now, in the next step we are fixing an i to again discuss as above. Let us fix an i. Now, by the continuity of M, we know that for the delta 1 greater than 0, there exists let us say a delta naught i greater than 0, such that whenever D X of A dash A double dash is less than delta naught i this implies the corresponding matching values M A i A dash minus M A i A double dash, this will be less than delta 1.

From that this i is related to what we have taken. Now, this is an arbitrary i because it is continuous with respect to every A i the whole partition or whole covering P X. Now, let us define delta to be once again the minimum over i delta naught i.

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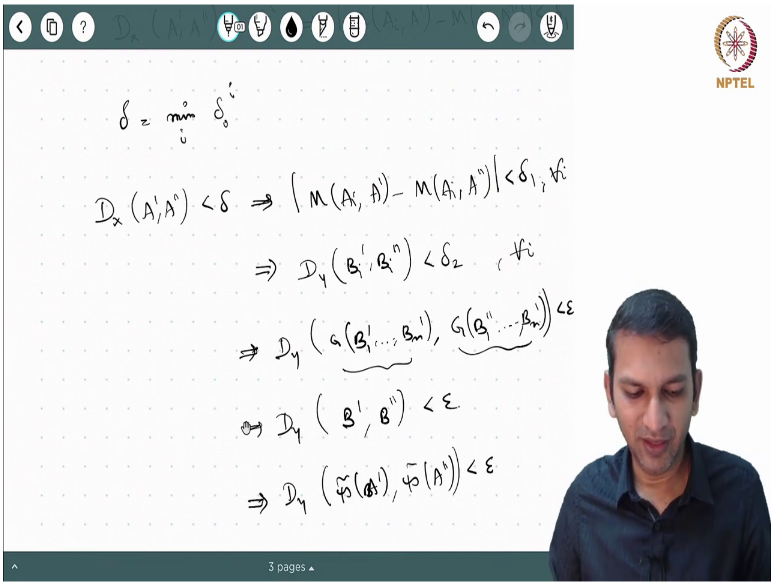

And clearly what we have is whenever D X if A and A dash are such that they are less than delta, this implies mod of M A i A dash minus M A i comma A double dash is less than delta 1.

Now, this is the missing piece. Now, we know that if this happens then retracing of steps back, you can see that here itself that whenever this happens for this will happen for every i. Now, this would imply that the corresponding D Y B i dash B B i double dash they are less than delta 2. Once again this is for all i.

Now, retracing the step back. When this happens we know that from here the max of them will for also for every i it is true that means, that implies D Y of G of B 1 dash B n dash comma G of B 1 double dash B n dash is less than epsilon. But what is this? This is nothing, but this quantity is B dash and this quantity is B double dash. So, we know that this is less than epsilon. So, this is exactly what we wanted to prove.

We have taken A dash and A double dash to be arbitrary. And what we have shown is for any given epsilon there will exist a delta, such that if A dash and A double dash are close enough in the X space with respect to smaller than delta, closer than delta. Then D Y of psi tilde of A dash and psi tilde of A double dash, they are closer than epsilon with respect to the metric that we have on Y.

Now, interestingly what we have shown is continuity of the psi tilde, the mapping obtained from SBR as a mapping from F(X) to F(Y). Now, what we are interested in is, is it also a continuous model for the given rule test.

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Now, let us look at some of the sufficiency and equivalence condition that we have obtained for SBR inference scheme to have interpolativity. You might recall this from the last lecture of the previous week.

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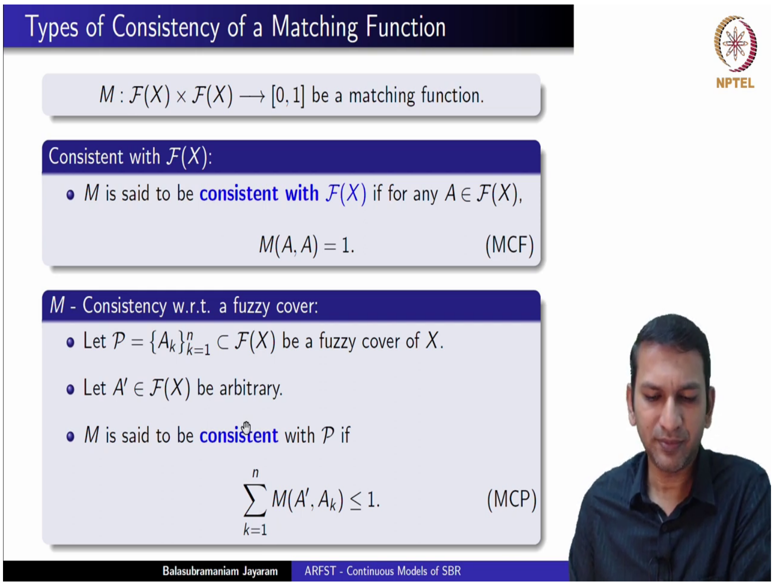

If you look at it towards discussing this, we had insisted on some properties on M specifically. Given a matching function M, we say it is consistent with the space from which the fuzzy sets are being taken, its domain essentially.

We say it is consistent with its domain if for any A, M of A is 1 and we say it is consistent with respect to a fuzzy cover much like how we said it is continuous with respect to a fuzzy cover. If we are given a fuzzy cover on X and an arbitrary A dash from F(X), we say that it is consistent with respect to this P.

If summation of the matching values of A dash to each of the pieces in the covering that their sum, does not add up into more than 1. So, sum is less than or equal to 1.

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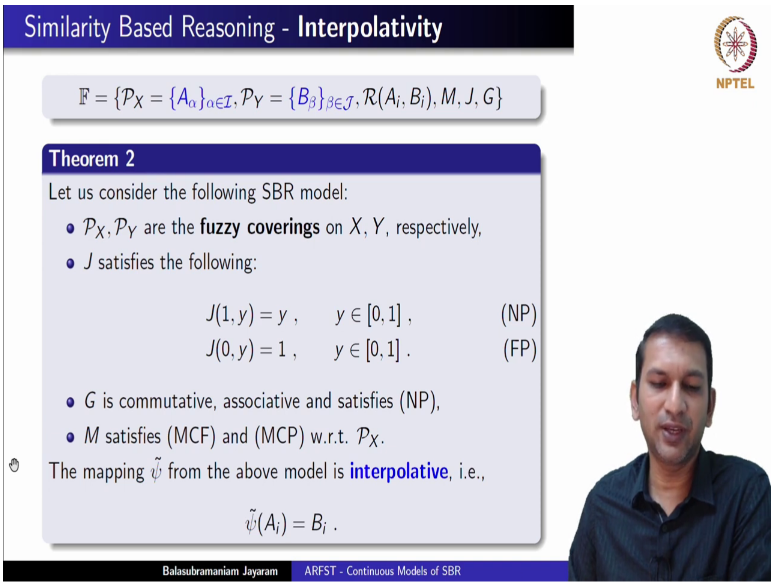

Now, under this condition, we have seen that an SBR model is interpolative. What are the conditions P X and P Y are fuzzy coverings. J satisfies these two properties, that is NP and also what we call the falsity principle. G is commutative associative and satisfies NP. And M satisfies both MCF, it is consistent with respect to its domain and also consistent with respect to the fuzzy covering P X.

Then, we have seen that the mod, the psi tilde obtained from here is in fact, interpolative that means, psi tilde of A i is B i.

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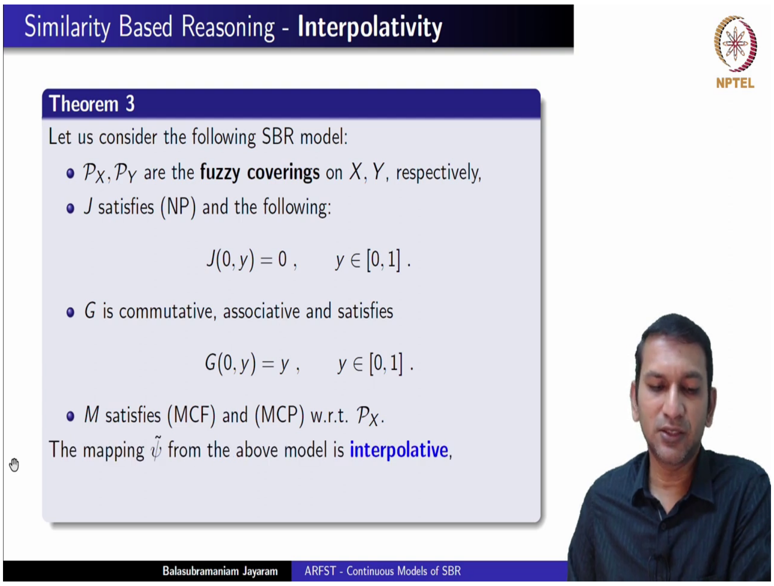

We also had couple of other results. One other sufficiency condition ensuring interpolativity is this, that P X and P Y has fuzzy coverings on X and Y. J satisfying NP and J of 0 Y being 0. G is commutative associative, and now 0 is the neutral element of y. And M still satisfies MCF and MCP. We can then show the corresponding psi tilde is in fact, interpolative.

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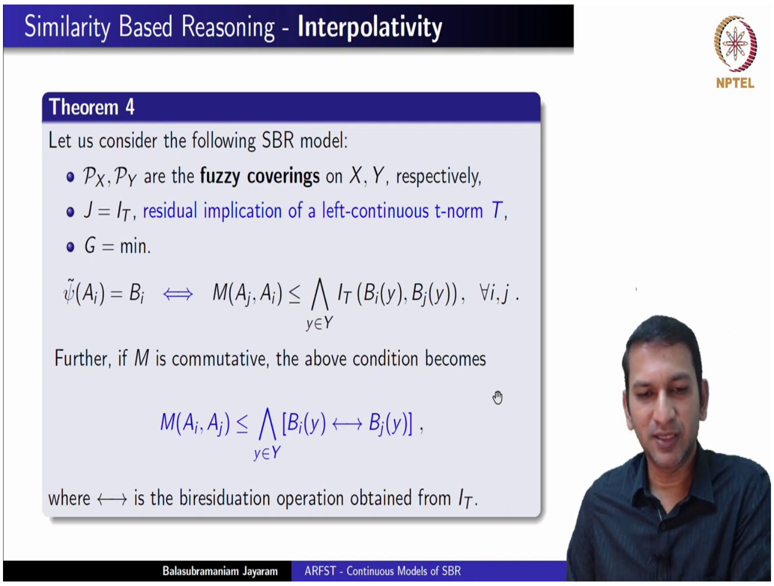

And finally, we have also seen some necessary and sufficiency conditions. Once again P X and P Y are fuzzy coverings. Now, the operation J is not an operation with NP and positive principle. We specifically pick a residual implication that means, an implication R implication from a left continuous t-norm. And G to be min. Then, we say this psi tilde that we obtain from SBR is interpolative, if and only if, it satisfies this inequality.

We have also shown that if M is commutative, then that inequality turns out to be this where this is the bi-residue.

Well, (Refer Time: 34:16) discuss this interpolativity results.

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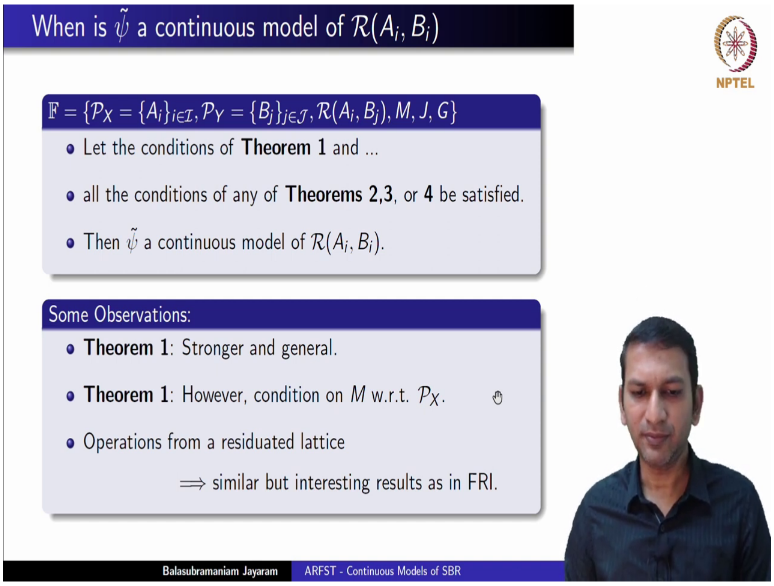

The question is, when is psi tilde a continuous model of the given rule base? Now, putting all of them together, it is clear that this is the form of the SBR we that we are considering. If P X, P Y, M, J and G satisfy all the conditions of theorem 1, and all the conditions of any of theorems 2, 3, or 4, which are required for interpolativity. If they satisfy this, then it is clear that psi tilde is in fact, a continuous model of the given rule base.

This is immediate to see because what we have shown is for any A dash A double dash D of D X of A A double dash less than delta implies this.

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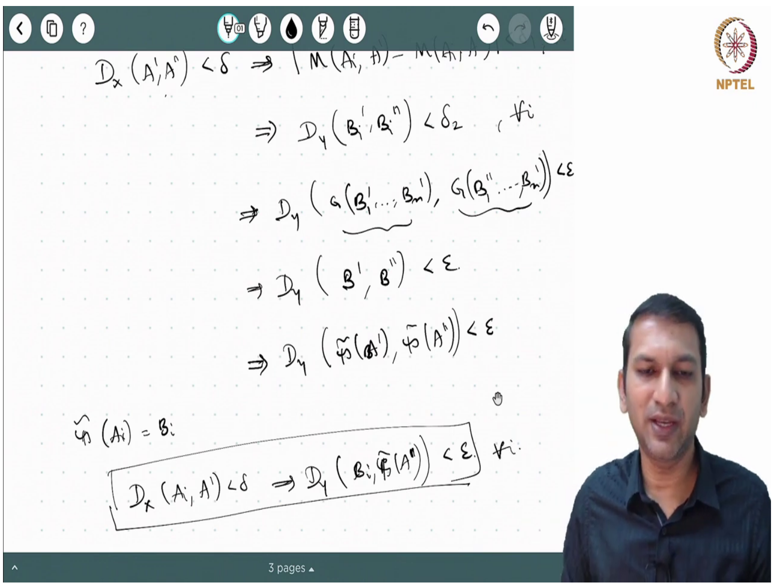

Now, if psi is interpolated, psi tilde is interpolated and if you give A i for any i, then we are going to get B i, so that means, essentially what we will get here is D X of A i comma A dash less than delta implies D Y of B i comma psi delta of A double dash, A dash is less than epsilon. This is essentially the condition that we need to satisfy for psi tilde to be a continuous model of the given rule base, of course, for every i that we have there.

So, putting all of them together, we see that we can talk about when an SBR is a continuous model of a given rule base. A few observations. Note that, theorem 1 is in general stronger and it is also general because we have not put any conditions on J except for G except on the continuity part.

We are not asking for some ah boundary conditions on J, we are not insisting that they should come from resituated lattice or so now. And it is also stronger, in the sense that it is showing continuity between spaces, that psi tilde is actually a continuous function, between spaces not just continuous model of the given rule base.

However, we are asking for a condition on M which ensures continuity with respect to the fuzzy cover P X. In this sense, we are also tying it up to the given rule base. Finally, if you allow the operations to come from a residuated lattice as we have seen for interpolativity, we can get similar, but very interesting results as in the case of an FRI because in the context of continuous models of an SBR.

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Well, the topics covered in this lecture are predominantly picked up from this work of Mandal and Jayaram. With this, we will wind up with discussing continuity or continuous models of fuzzy inference mechanisms of FRI and SBR.

In the next week of lectures, we will begin by discussing robustness of fuzzy inference mechanisms. We will clearly highlight how it is different from continuity, and discuss their robustness.

Glad, you could join for this lecture. Hope to see you soon in the next lecture.

Thank you again.