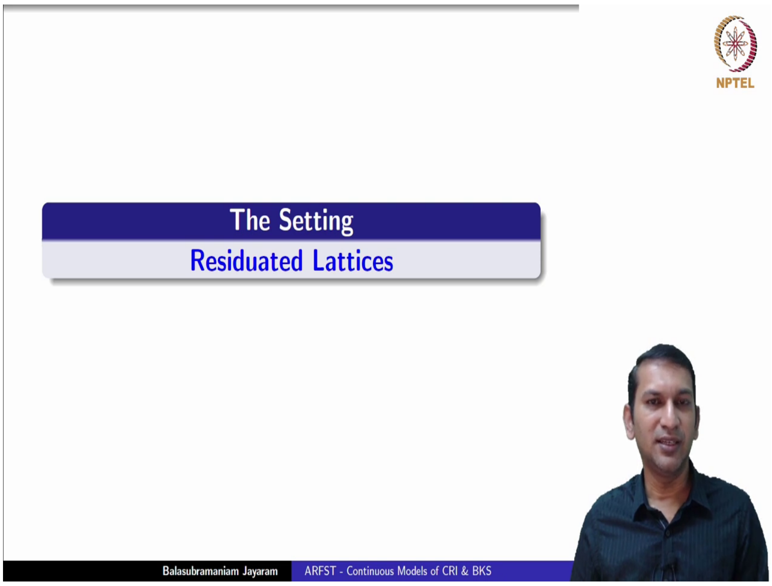
Approximate Reasoning using Fuzzy Set Theory
Prof. Balasubramaniam Jayaram
Department of Mathematics
Indian Institute of Technology, Hyderabad

Lecture - 46
Continuous Models of CRI and BSK

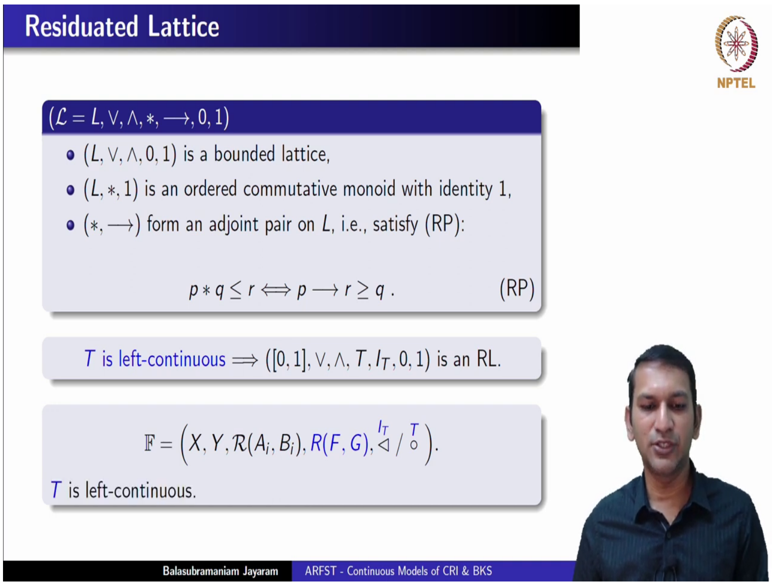
Hello and welcome to the next of the lectures in this week 9 of the course titled Approximate Reasoning using Fuzzy Set Theory. A course offered over the NPTEL platform. In this lecture we will discuss the Continuous Models Obtainable from Compositional Role of Inference and the Bandler Kohout Subproduct inference schemes.

(Refer Slide Time: 00:43)



A quick recap of what we have seen so far related to the continuous models of FRI once again the setting is that of residual lattice.

(Refer Slide Time: 00:53)

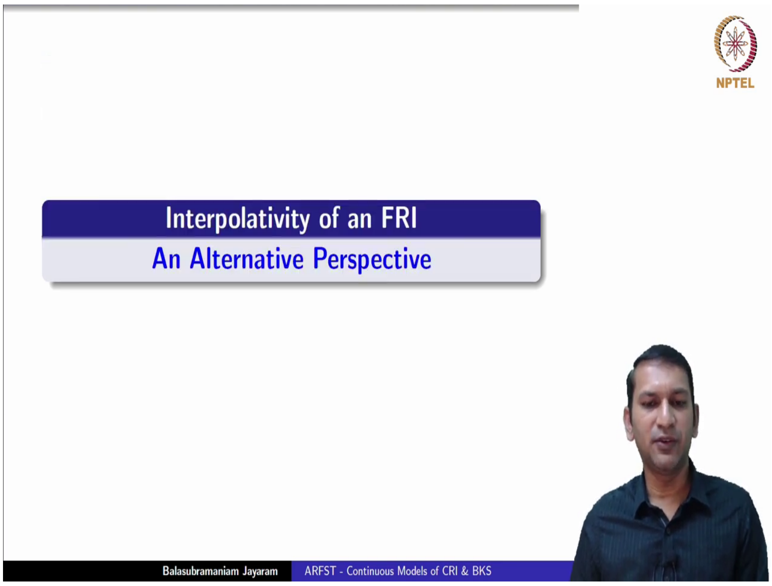


We know what a residual lattice is, it is a boundary lattice ordered commutative monoid with identity one. And the star and the arrow operation they are related by the adjoint equation or which is also known as the residuation property.

So, in the case we consider the set L to be the unit interval [0,1] by taking a left continuous T norm and its corresponding residual implication we were able to impose a residual lattice structure on 0 and now we are discussing FRIs where, both the operation of composition whether it is sup T or inf I both T and I are taken from this distributed lattice.

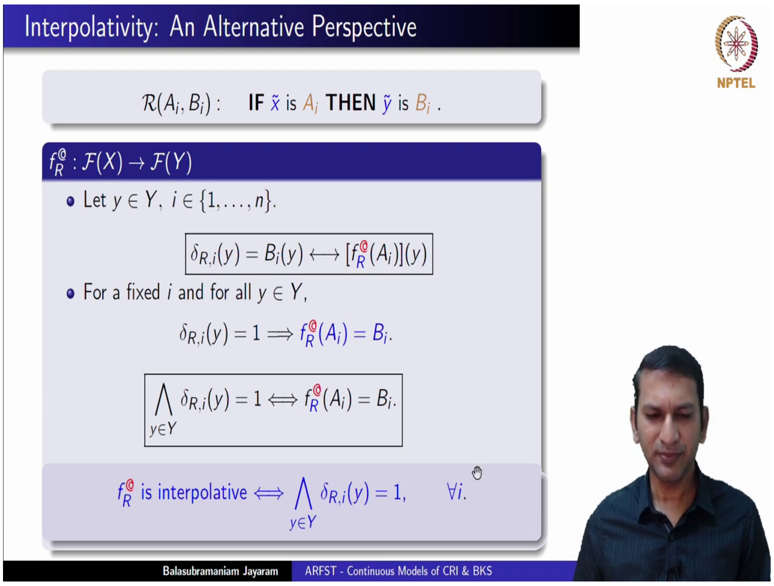
So, also all the other operations like that of f or G aggregation operation or the operation used to relate rules into a relation once again T we assume is to be left continuous.

(Refer Slide Time: 01:53)



We have seen an alternative perspective of interpolativity of an FRI.

(Refer Slide Time: 01:58)



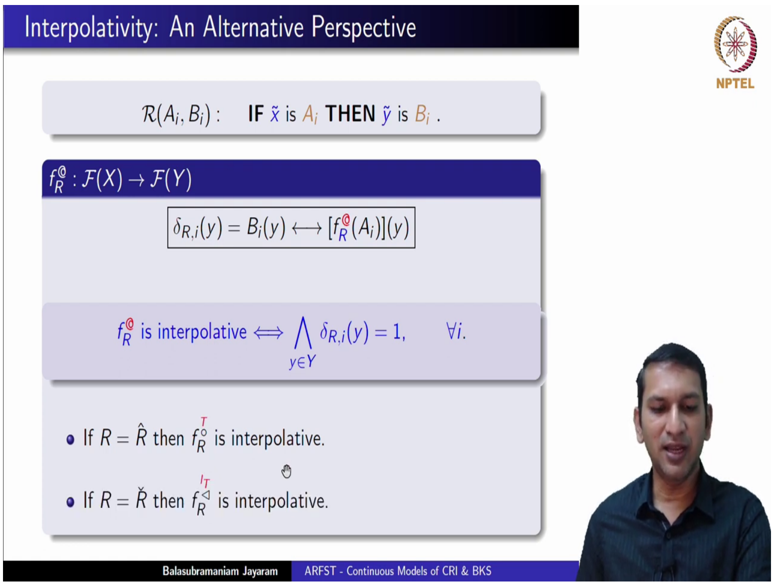
Now, given a set of it then rules we could consider the FRI to be given as its adjoint function F R hat which is a mapping from F(X) to F(Y) this is how we wanted to look at fuzzy inference mechanism as a mapping from F(X) to F(Y). Now, if you see here if it is this mapping is coming from an FRI, the two important things that we need to specify are the relation R and the composition.

So, the relation R is capturing all the rules into a single relation. Now let us pick a y from the output Y and pick any one of the i s from the n rules we define delta R i for this fixed R and i and y we define delta R i at y as follows this is the bi implication operation which we know is an equivalence relation and T equivalence relation with respect to the star if you are considering the residuated lattice structure.

Now we have seen that if you fix an i and if for all y if delta R i of y is 1 then clearly it means that f R at this adjoint function of the FRI is in fact, giving us the output as B i when the input is A i. So, for this i if the antecedent A is given as output input then f R at A i is essentially giving us B at the corresponding consequence.

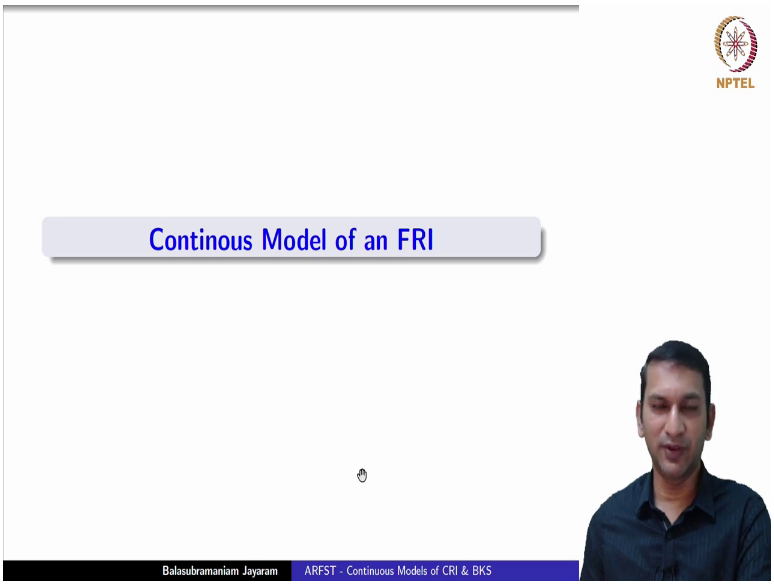
Now this could also be written like this because this is true for every y, which means this adjoint function f R at is interpolative with respect to this rule base that we have considered if and only if this happens delta R i of y is infimum over all y this is equal to 1 for every i so; that means, f R at when given an input a always gives us A i B i for every. So, now, you look at it interpolativity has been looked at from a completely different perspective by defining a function delta R i.

(Refer Slide Time: 04:14)



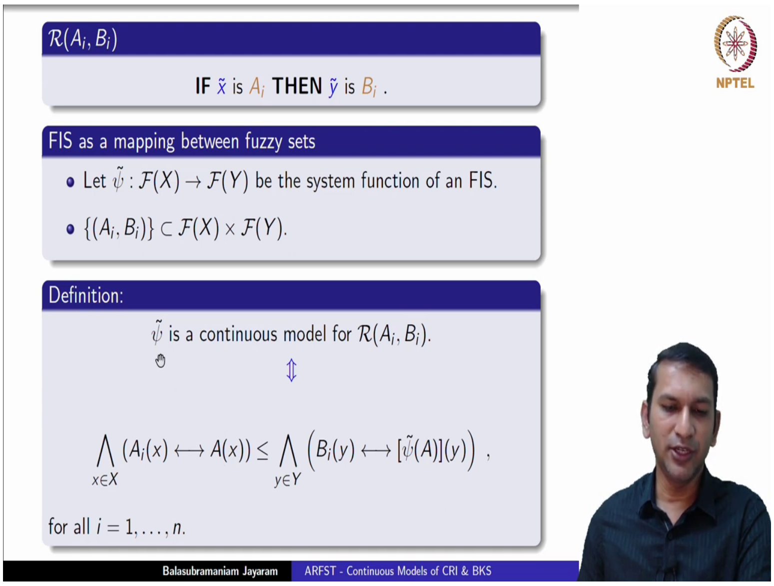
Now, seen from this lens what we have seen earlier is that if R is R cap then f R sup t is interpolated and if R is R check then f R delta I T is interpolated. So, these are just two relations which make them interpolative, but there can also be other relations which make it interpolative.

(Refer Slide Time: 04:39)



Now, we are interested in discussing the continuous models of an FRI.

(Refer Slide Time: 04:43)

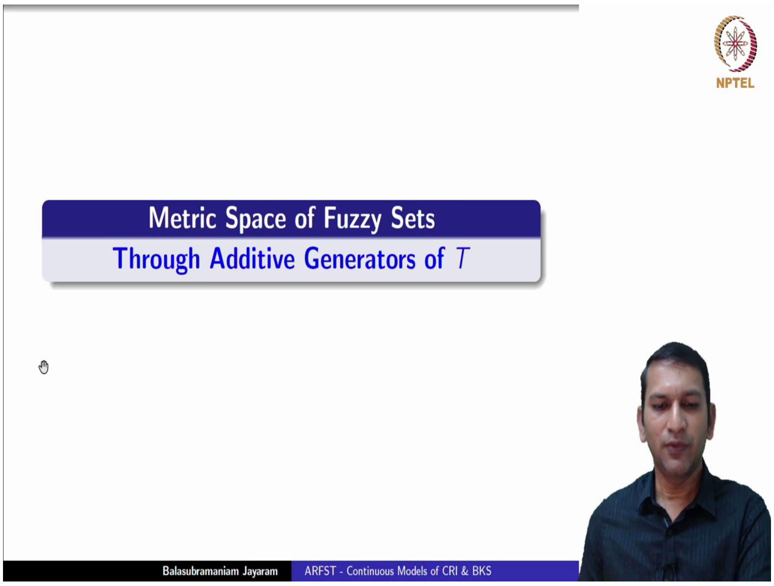


Towards this end once again we look at the corresponding adjoint function as the system function as a mapping from F(X) to F(Y). Given this rule base we look at antecedent consequent pairs as being picked up as the ground truth coming from the Cartesian product of F(X) plus F(Y), then we can define the psi tilde to be a continuous model for this given group base at these points A i B i.

If and only if this inequality, which essentially involves the antecedents and the consequence the given input and the output as obtained by psi tilde. If this inequality is valid that is when we call this psi tilde a continuous model of the given rule base this.

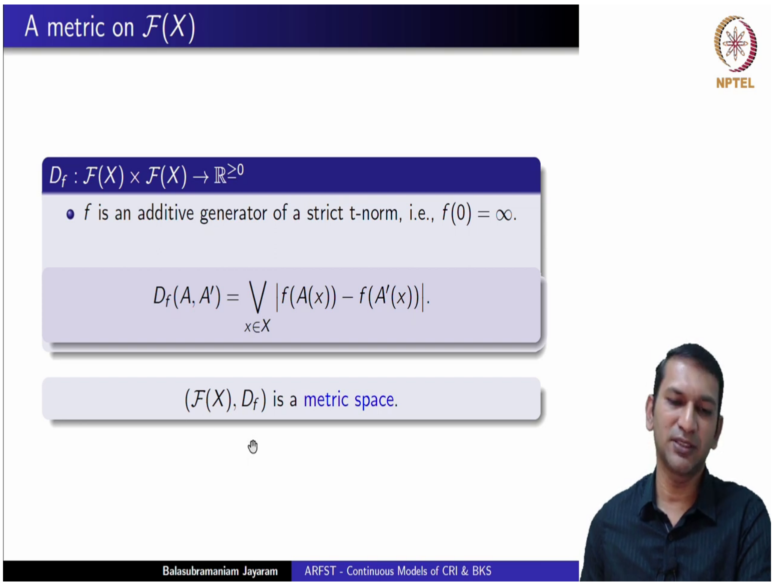
Note that we are not talking about continuity of a mapping from F(X) to F(Y) in general we want to talk about the continuity of adjoint function of an FRI seen as a mapping from F(X) to F(Y) given the rule based A i B i. So, in that sense this is the inequality that it should satisfy to be called a continuous model for the given rule base.

(Refer Slide Time: 05:58)



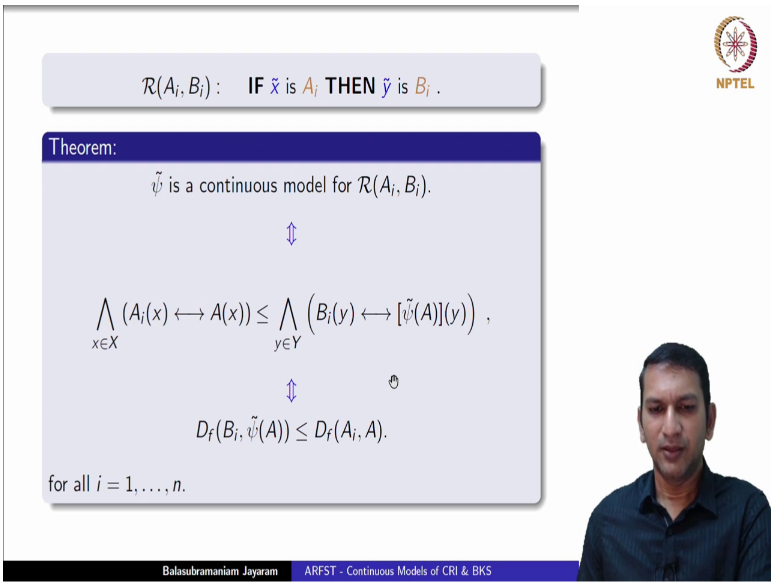
Now, we have seen also why this is called the continuity. Towards this end we came up with a metric on the space of fuzzy sets through the additive generators of strict t norms.

(Refer Slide Time: 06:11)



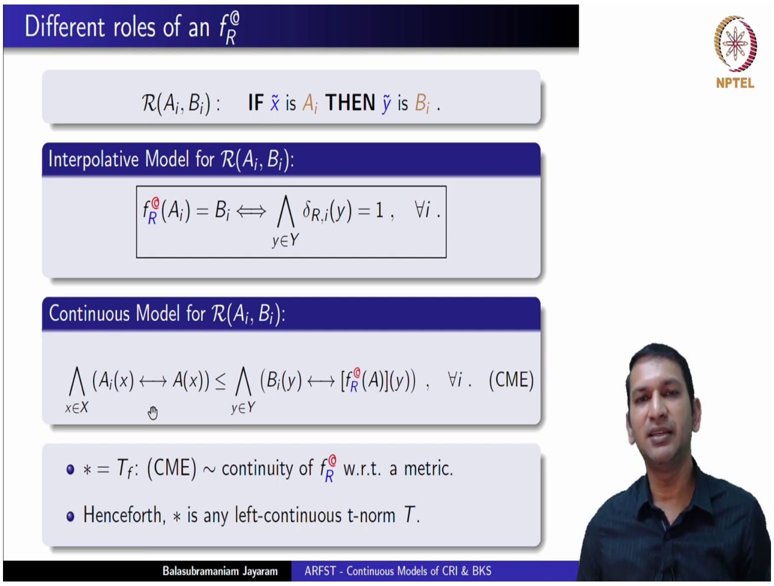
So, now let us take an additive generator of a strict t norm f we know that f is a strictly decreasing continuous function from [0,1] to [0,infinity]. So, that f of 1 is 0 and since it is an additive generator of a strict t norm f of 0 is infinity, we define such a function D f which acts on A, A dash these are two fuzzy sets over x it is defined like this we have seen that this D f is in fact, a metric and makes F(X) a metric space.

(Refer Slide Time: 06:42)



And what we have seen is, if psi tilde is a continuous model for this rule based; that means, it satisfies this inequality for every i then it can be shown that it is in fact, satisfying this inequality with respect to the metric D f what does it say? It says given an input A psi tilde of A is the output the output psi tilde of A should be closer to B i than A itself is closer to the corresponding A i in that sense you can look at this as some kind of a one Lipchitz continuity at the points A i B i.

(Refer Slide Time: 07:22)

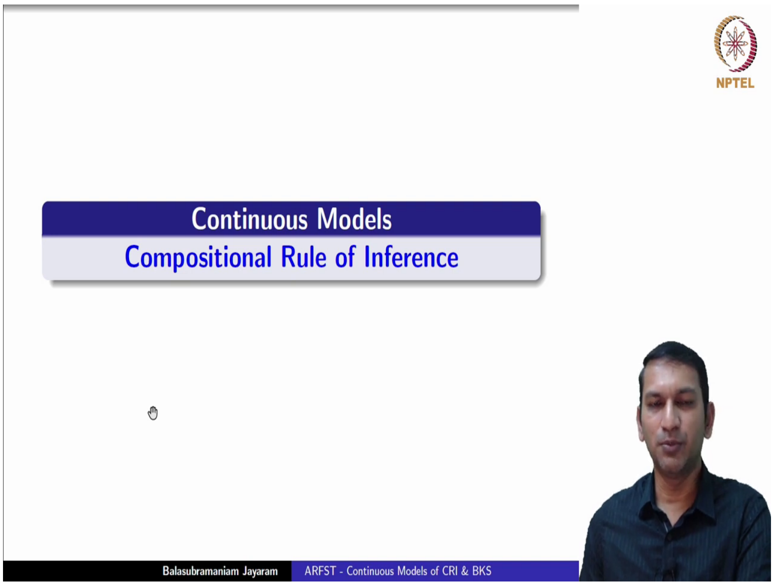


Well, if you look at the adjoint function of FRI where we need only the R and the composition operator at we know that it is an interpolative model for this rule base if this is satisfied and it is a continuous model for the rule base if this functional inequality is satisfied which we probably will call as the continuous model equation see I mean.

Note that in the previous lecture we picked up start to be T f a strict t norm obtained from an additive generator f only towards showing that this continuity equation continuity modelling equation CME.

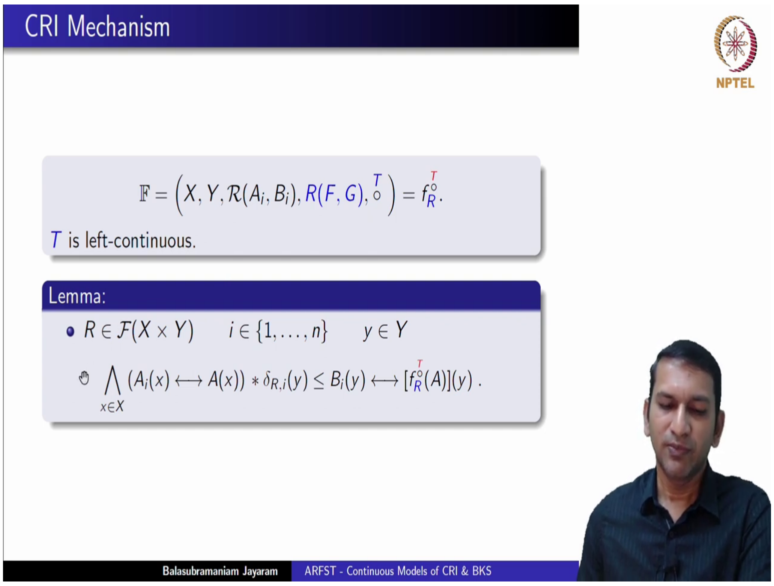
This inequality is related to the continuity of the adjoint function of the FRI which is FRI with respect to some metric this is all that we wanted to show essentially interpreting this as continuity of the underlying adjoint function; however, henceforth we will go back to being in a general residuated lattice; that means, the star can be any left continuous t norm.

(Refer Slide Time: 08:34)



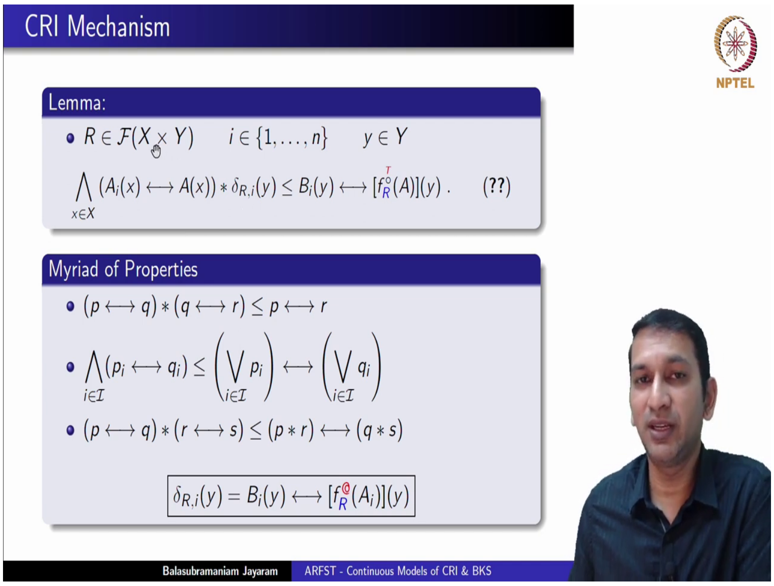
Well, now let us look at what are the continuous models of compositional rule of inference.

(Refer Slide Time: 08:40)



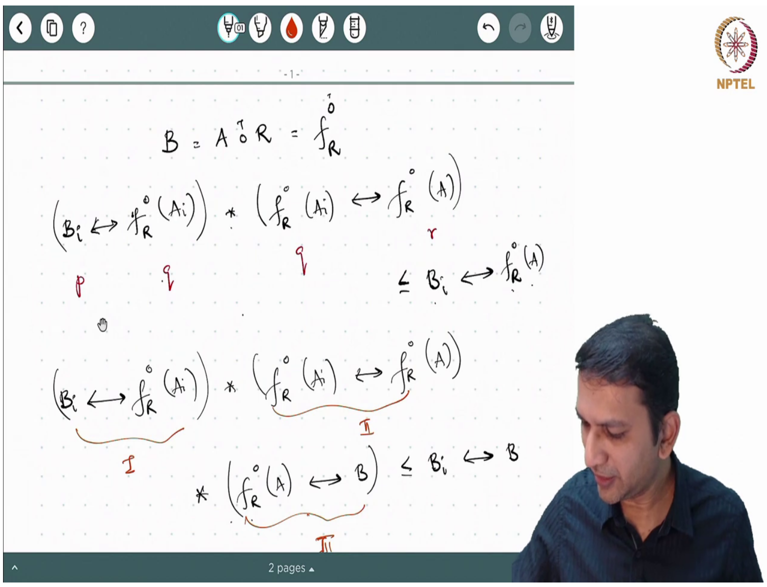
So, all that we are doing is fixing the composition that it to be sup T of course, T is left continuous t norm and all the other operation that we need are coming from the corresponding residuated lattice. The first result says take any R which is a fuzzy relation on X cross Y take any i from 1 to n and any y from the output domain Y then this inequality is valid. So, let us prove this.

(Refer Slide Time: 09:18)



First let us prove this then we will see how to interpret this lemma with respect to the theorem where it is being used where it is going to be employed. Now note that the star this bi implication the T everything is coming from a residuated lattice structure. And we know in the residuated lattice it is an extremely rich structure and it has myriad of properties we will make use of a few of those properties towards proving this. The first property that we are going to make use of is what we have seen as the transitivity of this bi implication.

(Refer Slide Time: 10:04)



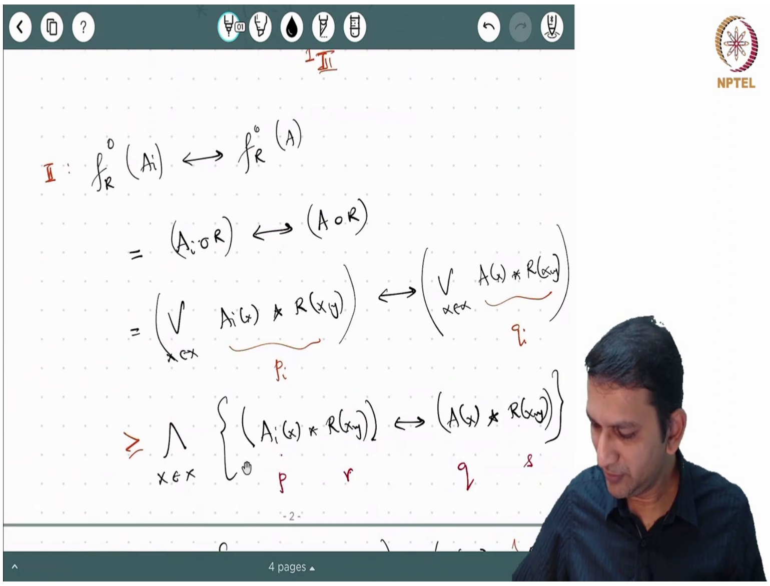
Now, with a slight abuse of notation we will write this given a now we are in the realm of CRI. So, given an input a let and R let B be equal to A. So, T composed with R now we can write this as f R circle to make it easier to write to avoid the cumbersome notation we will skip this T on top and just use only circle since there will not be any confusion in this context. Now what does this say? p bi implication q star q bi implication r is less than or equal to p bi implication r.

So, let us start with B and for some i now look at this. As I said its once again for the moment please bear with me with respect to the notations we are using note that this bi implication will actually act on the membership values on the values from [0,1]; however, for the moment we will write it like this and then finally, apply it on a particular y. So, now, if you take this star f R circle A i bi implication f R circle a. We know this is in fact, less than or equal to by the transitivity property look at this as p and this as q this as q and this as r.

So, then what we get is, it should be less than or equal to p bi implies r now the p is B i bi implies f R 0. Now we could also go a little further using the same logic, but perhaps write it like this also star f R circle of A B this will be less than or equal to note that because we are applying star again here by the monotonicity we will have b i if and only y implies f R of circle at A star f R circle at A bi implies B once again by the transitivity this entire thing can be written as.

Well, now this is what we can start with now look at this fellow look at each one of these terms. Now, this term essentially B by definition is in fact, f R circle at A. So, now, by the bi implication property of bi implication we know this is 1.

(Refer Slide Time: 13:35)

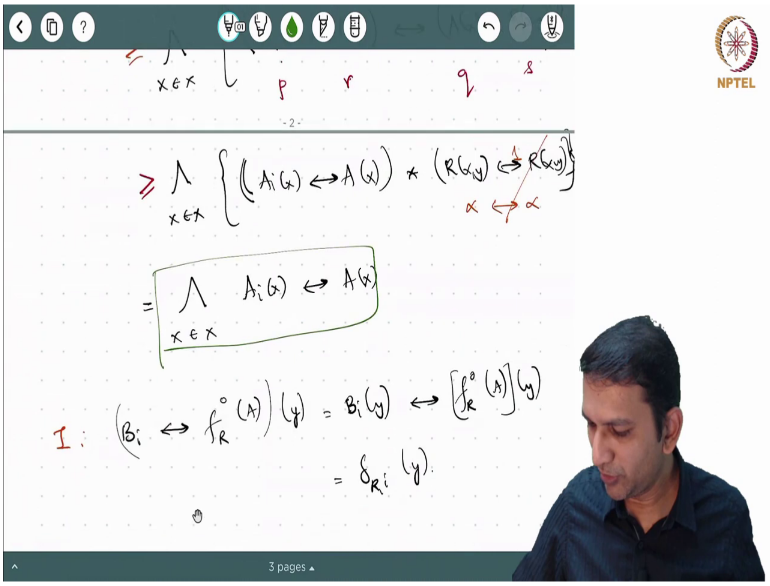


Let us consider the 2nd term this is f R circle A i bi implies f R circle of A and this is equal to A i circle R. Now this is sup t composition supremum over x A of x which is star R of x i x y supremum of x element of x A of x star R x y this is what we have so far. Now look at this property we can easily match this with this.

So, what we have is supremum over x A of x star R x y which is can be looked at a supremum over p i. Similarly the supreme over x A of x star R of x y which can be looked at as supremum over qi now, we know that this should be greater than or equal to.

So, looking at this quantity as p i and this quantity of q i what we knows this is greater than or equal to infimum over x element of x p i bi implies qi which is A of x star R of xy b i implies a of x star r of x y right what is the next step now? We also have this property p bi implies q star r bi implies s is less than or equal to p star r bi implies q star s. Once again look at this what we want is looking at this as p, and this as r, and this as q this as s.

(Refer Slide Time: 16:04)

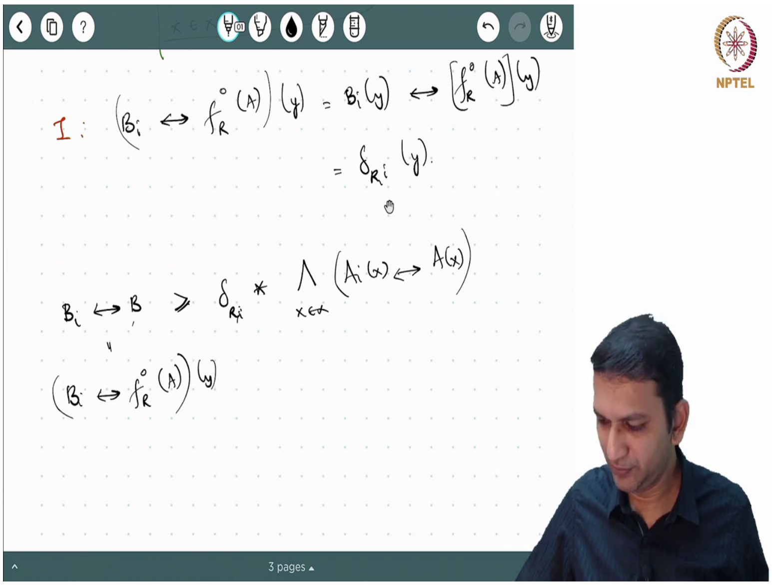


So, we know that p star r bi implies q star s is greater than or equal to infimum over x, x is p bi implies q A i of x is A of x star r bi implies s which is nothing but R of x y bi implies R of x y. Now what is this? Equal to look at this is R of x y bi implies this. So, it is essentially alpha bi implies alpha we know that that will be equal to 1.

So, now this entire thing is in fact, equal to and x infimum over x A i of x bi implies A of x. Now this is essentially the second term now what is this term here? If you look at the first term you have taken that to be B i A y this is going to be equal to exactly this because this is B i of y bi implies f R circle at a of y. Now that this is essentially this and this we know is nothing but delta R i at y.

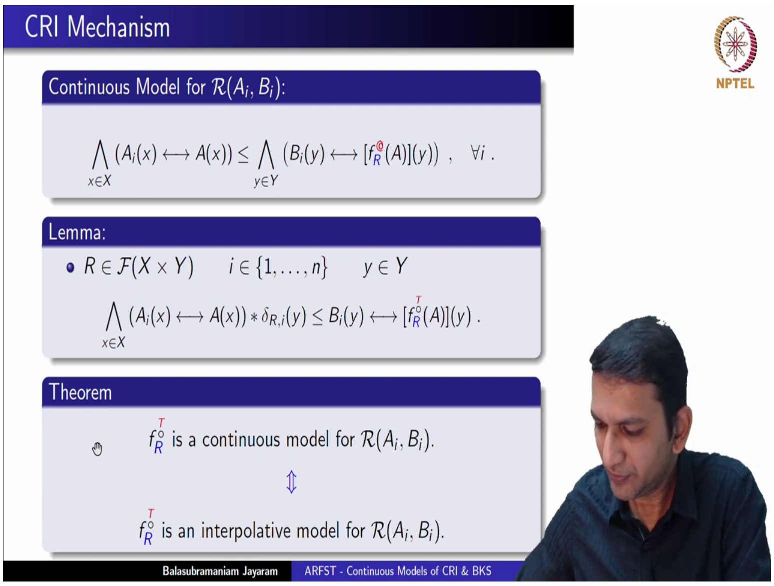
So, now, substituting them here so, this quantity is essentially delta R i this quantity is greater than or equal to this quantity here and the last quantity is 1 (Refer Time: 18:21) this quantity is 1. So, this is less than or equal to B i bi implies b. So, now, putting them all together what we get is less than or equal to delta R i star is exactly what we wanted to prove.

(Refer Slide Time: 18:34)



Note that this B is essentially f R circle of A greater than or equal to this point. So, just by making use of all these properties, we have been able to show that for any R we are not specifying what the relation is any relation R fuzzy relation on X cross Y and pick an i and y we know that this inequality is valid.

(Refer Slide Time: 19:32)

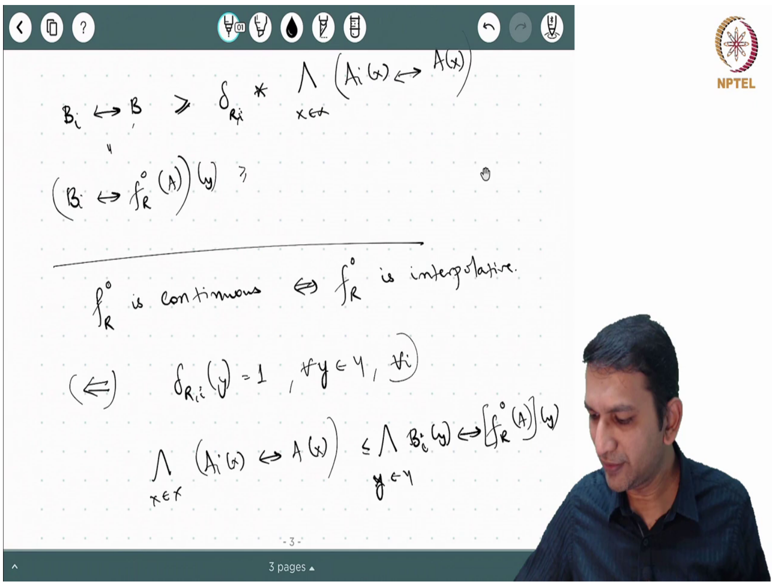


Now, how will this small research help us? The main research says, if we consider any R and the sup T composition and the corresponding adjoint function that is the fuzzy inference mapping that we get out of this, it is a continuous model for the given rule base if and only if it is also an interpolative model for the given rule base.

So, continuous model we explain or define continuous model as that one which satisfies this equation CM equation which is there on the top this is inequality and we saw why it is called continuous? Because we could show in some special cases that it can be seen as continuity with respect to a kind of a metric obtained from the additive generators of strict t norms.

Now what was essentially seen as continuous model is also now equivalent to an interpolative model for the given rule base is now how do we prove this? The proof is quite simple and straightforward let us look at it like this.

(Refer Slide Time: 20:45)

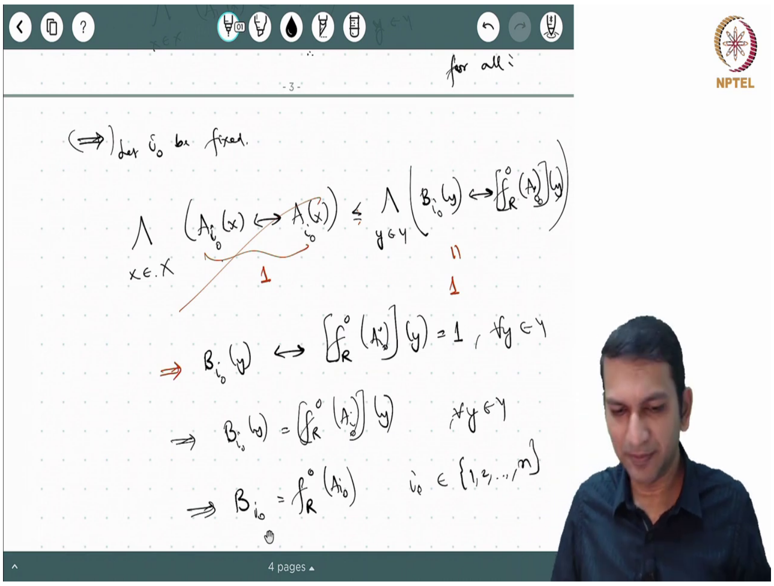


So, we need to prove f R circle is a continuous model if and only if f R circle is an interpolative model.

The reverse implication is straightforward if it is interpolative what we do now is delta R i of y is actually equal to 1 because that is what interpolativity is this is 1 for every y and every i which means this becomes 1 and what we have immediately is this inequality and since this happens for so; that means, delta R i of y is equal to 1, for every 1 and for all i which means substituting in this lemma what we see is and x element of x A of x bi implies C of x is less than or equal to B i of y f R circle of u at y now this is true because it is interpolative.

So, by substituting 1 here this is what we have got now since this is true for every y we could also put take the infimum over y and it will be true and this essentially is our continuity equation the inequality and this happens for every i. So, if it is interpolative that it is continuous that it is continuous is very clear. Now let us show that if it is continuous then it is also interpolative. Once again the proof is not very difficult we need to show the forward implication. So, what we have is that it is continuous.

(Refer Slide Time: 22:50)



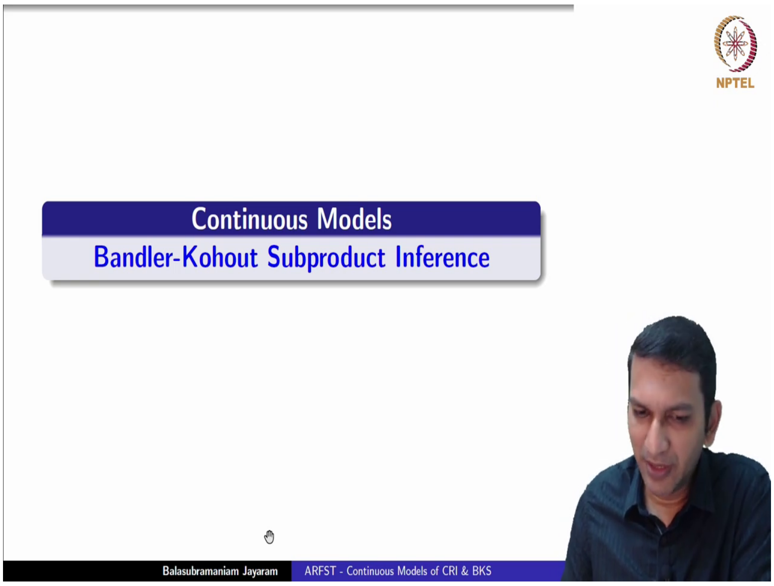
Now, continuous means, this equation is valid for all i. So, let us fix an i let i naught be fixed no it is valid for this i naught also now; that means, i is less not equal to u for y B i naught of y f R circle A at y. Now let us give to show that it is interpolative we need to show that if A is in fact, A i naught B should be this output from here should be B i naught.

So, now let us keep A i naught now if it is A i naught then we know this quantity in fact, is 1 for every x which means this is 1 totally because infimum of all of them will be 1 now this is greater than or equal to this; that means, this is also equal to 1. Now infimum of all of them is 1 implies each one of the terms is 1 means B i naught of y bi implies f R circle of A i the output of A i at y this is equal to 1.

But now this is a y implication, we know that if it is equal to 1 then both of them are same; that means, B i naught of y is equal to f R circle A i at y. Now this happens for every y in y, which means B i naught is in fact, equal to f R circle of A i to. Now this i 0 is arbitrary so; that means, it will happen for any one of those ns.

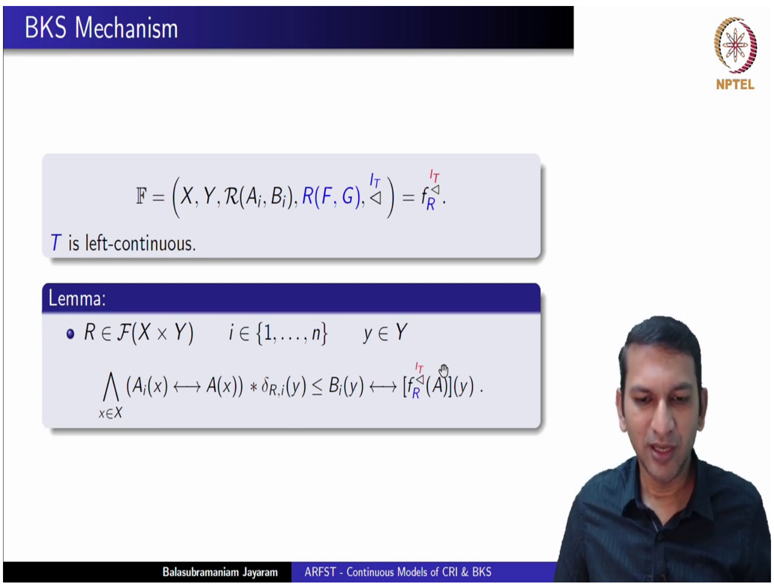
So, now, what we have seen is in a nutshell we looked at interpolativity of an f R A in a different way we introduced what is a continuous model of the corresponding adjoint function of an FRI. And we managed to show that the adjoint function is a continuous model of a given rule base if and only if it is also interpolative. This is what we have seen for the CRI mechanism.

(Refer Slide Time: 25:40)



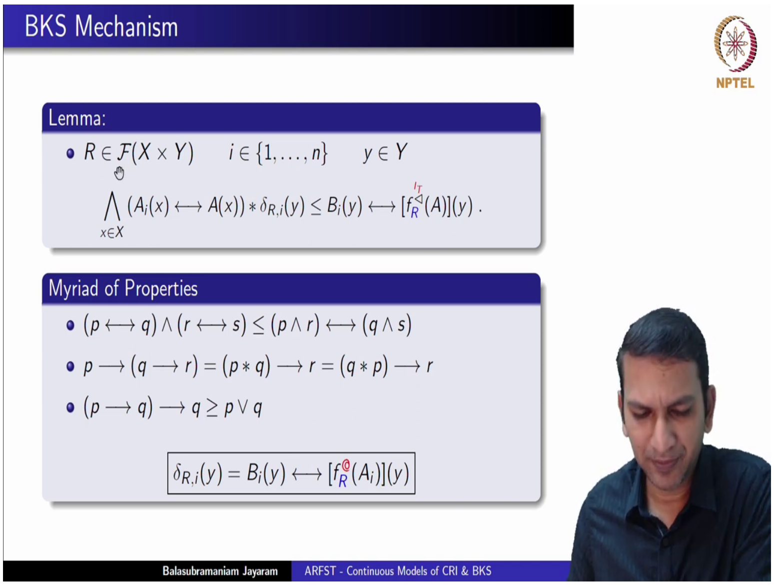
Now what happens when we look at the Bandler-Kohout Subproduct Inference. Well, similar results are available let us go through them.

(Refer Slide Time: 25:44)



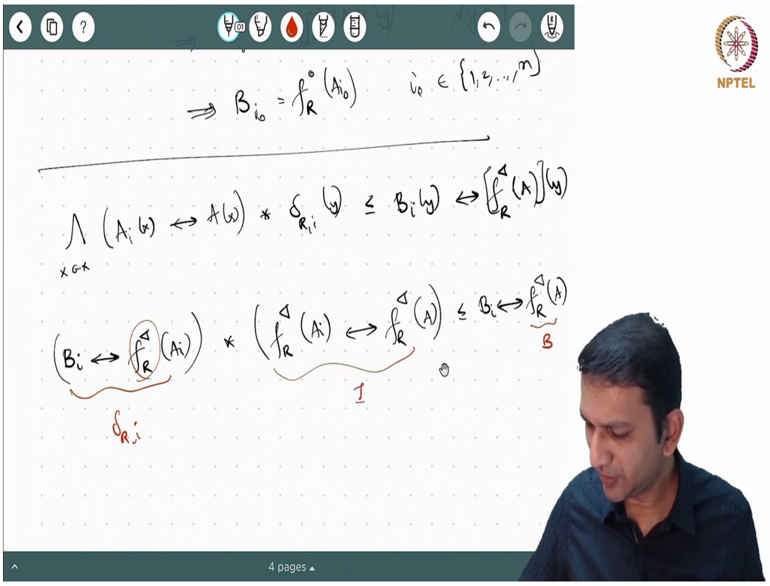
Once again we are fixing the composition operator of course, the implication is residual implication of the corresponding left continuously T norm that we have considered the residuated lattice. And the operations all the other operations are coming from the residuated lattice itself. Once again if you take any arbitrary relation R and fix an i and any y the same inequality is valid when we consider the BKS inference. Now once again the proof here follows it is straight forward.

(Refer Slide Time: 26:23)



However, we need to make use of different properties of the residuated lattice especially of the bi implication. So, let us go through with this proof now what we need to show is this.

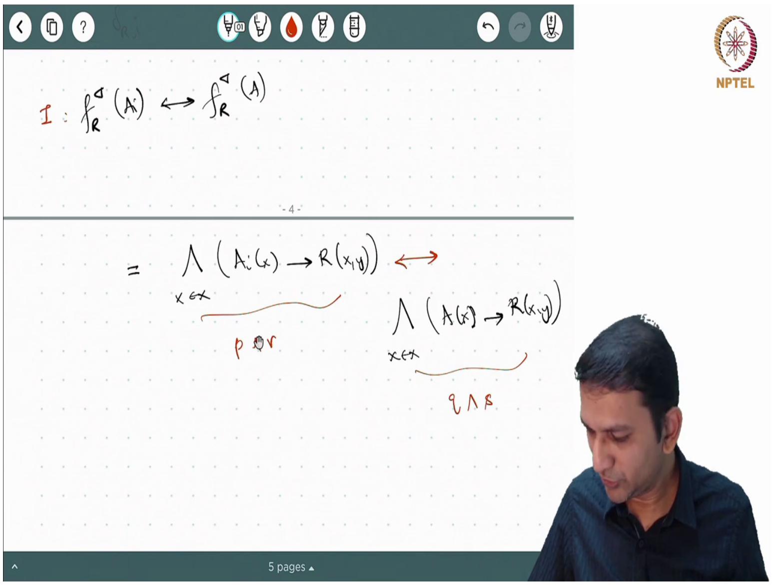
(Refer Slide Time: 26:36)



Once again we need to show an x infimum over x A of x bi implies A of x star delta R i y is less than or equal to d f of y bi implies f R. Once again we will just use the delta that it is the implication is an it is a residual implication of the corresponding left continuous t norm is obvious f. So, this is what we want to prove let us start with taking the same.

So, we know that B i f R delta at A i star f R delta here bi implies f R delta A is less than or equal to B i bi implies f R delta at A note that this is what we have called B. So, given an A this B that we are going to get. So, now, this comes from the corresponding transitivity equation which we have seen. So, now, it is immediately clear that this is nothing but delta R i for this particular composition. So, it essentially boils down to looking at this term.

(Refer Slide Time: 28:16)

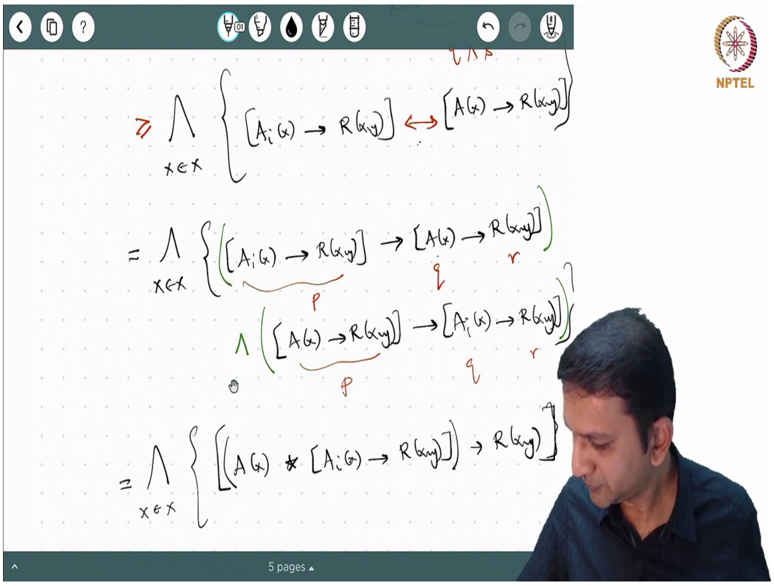


Once again let us look at the term here. So, this is f R delta of A i bi implies f R delta of A we have to write this yeah nothing but. Now, that it is the BKS sub product. So, this is A i of x implies R i r of x, y bi implies infimum over x element of x A of x bi implies perhaps.

Let us use a different colour here this is bi implication and this is the implication which is essentially the residual implication this is what we have now. Once again let us make use of some of the properties available to us from the residuated lattice structure. Now look at this what does it say?

p and p meet r bi implies q meet s is greater than or equal to p bi implies q meet r bi implies s. So, now let us compare it with this what we have here note that. So, you could look at this as this part as p meet r and this part as p meet s the associated you can extend. So, now, this would be we see that this is greater than or equal to corresponding terms that we should take.

(Refer Slide Time: 30:09)

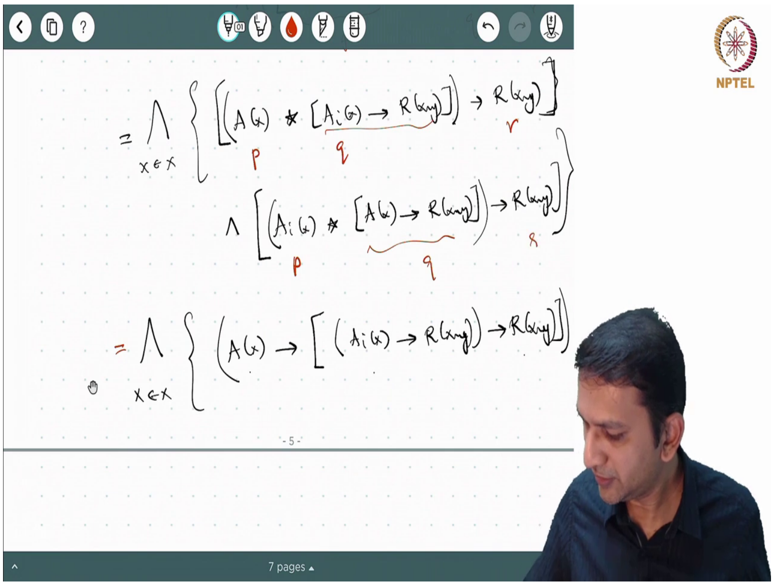


So, this is greater than or equal to we have essentially pulling out the ampersand the infimum need to be careful here. So, now, this entire thing A i of x implies R i of x y bi implies A of x implies R of x. So, the next step let us use the definition of the bi implication and split this. So, what we have is bi implication is A i of x implies R of x, y.

Now just implies A of x implies R of x y and meet their and meet opposite A of x implies R xy implies A i of x implies R of xy. So, this is essentially from the definition of the corresponding bi implication operation. Now we also know this law of importation which says that p implies q implies r is equal to p star q implies r or q star implies p implies r, because of the commutativity of star.

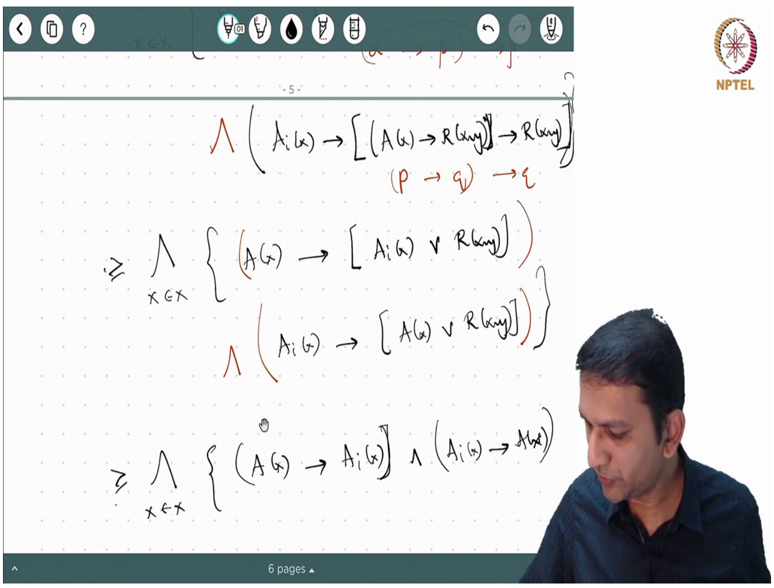
So, let us try to use that now for that what are we going to choose as p q and r? Let us choose this as p this as q and this as r similarly we will choose this as p this is as q and this as r. So, what you will have is equality here now this comes here A of x star A i of x implies R of xy implies R of xy this ampersand remains.

(Refer Slide Time: 32:42)



And the same thing we will do here A i of x star A of x implies R of xy implies q star q implies R of xy. Now, once again we can apply the same law of importation p star q implies r and then getting a different form for that we will take this p and this as q and r. This entire thing is q and this is r. So, now, what will you do that with this infimum over x now p implies q implies r. So, this is p implies this is q A i of x implies R of x y implies R. Note that this is p this is q and this is r similarly we can do the same here this is p this entire thing is q and this r.

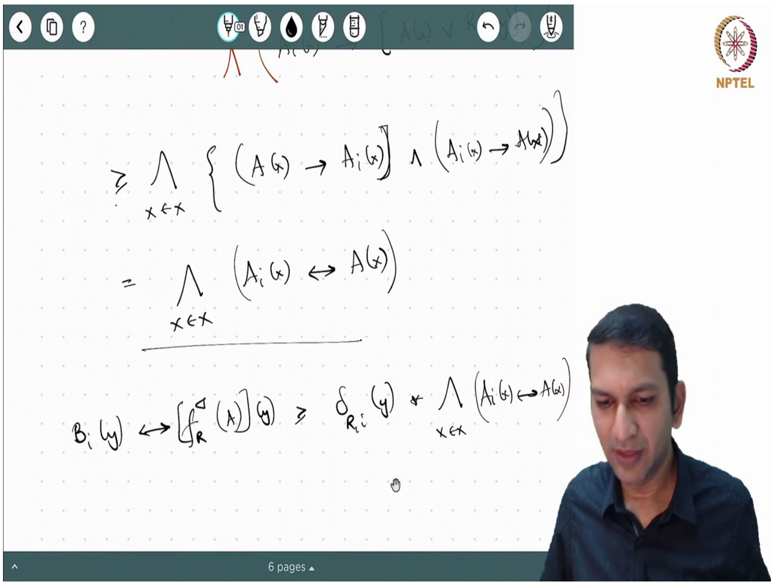
(Refer Slide Time: 34:09)



So, we have this in the meant inside the bracket A i of x implies and to do that (Refer Time: 34:42) from the (Refer Time: 34:43) now what is the next step? Look at this quantity here, this is essentially alpha implies beta implies beta or p implies q implies q this alpha implies, beta implies, beta or p implies q implies q which we know is greater than or equal to p r q. So, now, we could write that as this is greater than or equal to A of x implies A i of x join R of x y and we do the same here. So, now, this is p implies q implies q.

So, now, this would be this A i will remain A i of x implies A of x joint r of x y. Now, once again we know that a implies a or b is greater than or equal to a implies b or c is greater than or equal to a implies b or a implies c because implication is increasing the second variable and it is distributed over max here. So, now, this entire thing can be written as greater than or equal to A of x implies A i of x and A i of x implies A of x; note that they have accounted for the increase in inequality here, but what is this?

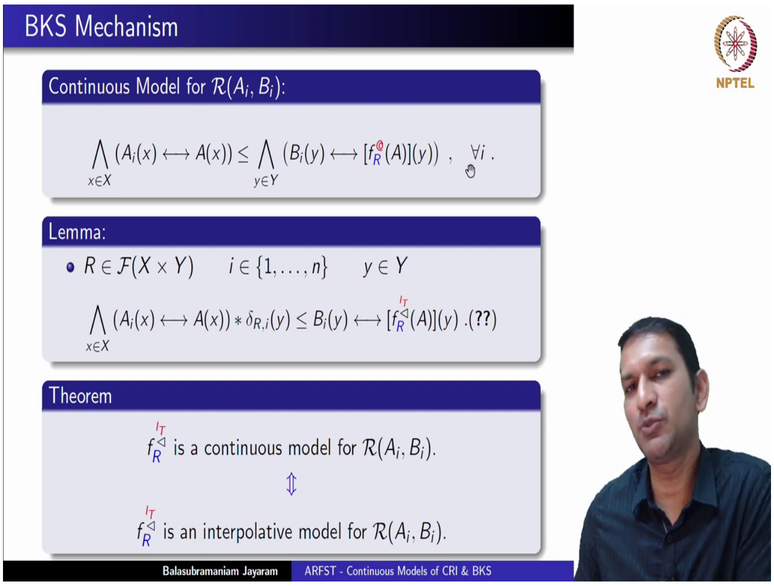
(Refer Slide Time: 36:42)



This is essentially equal to infer x A i of x bi implication A of x. So, what we have shown is that this quantity here is in fact, greater than or equal to infimum over x A i this is delta R i. Now, that is smaller than this. So, overall what we get is this once again as in the previous case we get exactly what we are having here that B i y f R delta of A at y is greater than or equal to delta R i y star.

Note that what this quantity is essentially this is what we have considered B i of y by implies f R at any of the compositions in general this is the general definition of delta A i of y is A i of x. This is exactly the inequality we wanted and we have seen that the same inequality that is this lemma is also valid when we use BKS inference or the inf i t composition. Once again this is valid for any r as in the case of CRI.

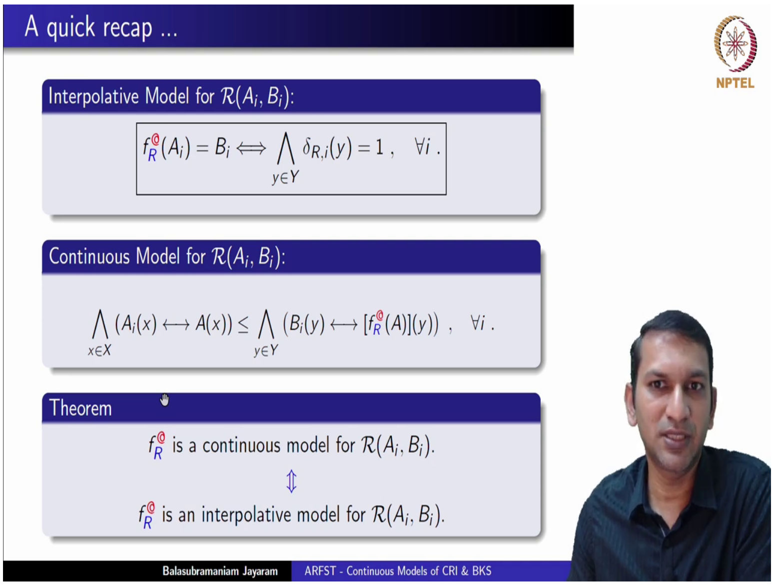
(Refer Slide Time: 38:30)



Now, it is now straightforward to show that with respect to an r f R delta I T is a continuous model for given rule base if and only if f R delta I T is in fact, an interpolative model for the corresponding rule base. The proof goes similarly as in the case of CRI if we know that it is interpolative this is 1; that means, this quantity is smaller than this quantity for everyone. And so, since it is true for every y we could take infimum over this and then we will get this inequality which means an interpolative model is also a continuous model.

If it is a continuous model we know that this is available for us and now if you fix A i here for some i because this is true for every i then this becomes 1; that means, infimum is greater than or equal to 1; that means, every element here is 1 and if we know that bi implication of a means alpha and beta is 1 if and only if alpha is equal to beta, which means B i is in fact, equal to f R delta I T at A i and this happens for A i and arbitrary I So, it happens for every i which means it is interpolative.

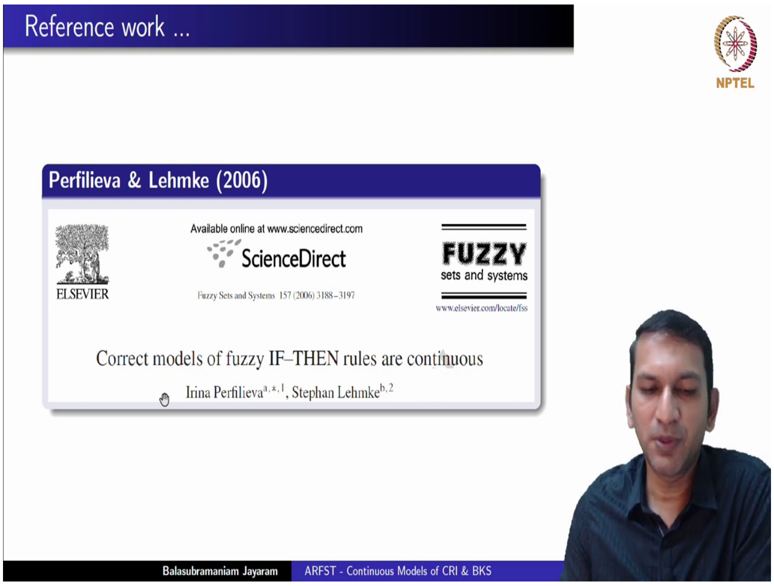
(Refer Slide Time: 39:45)



Now, a quick recap what have we done over the last two lectures? We have looked at interpolativity of an FRA in a different perspective, this alternative perspective allowed us to write or disentangle ourselves from either R cap or R check we said that use any composition any relation if it has to be interpolative, then this is the equality that should be valid for every i over every y also that is what we have taken here as infimum.

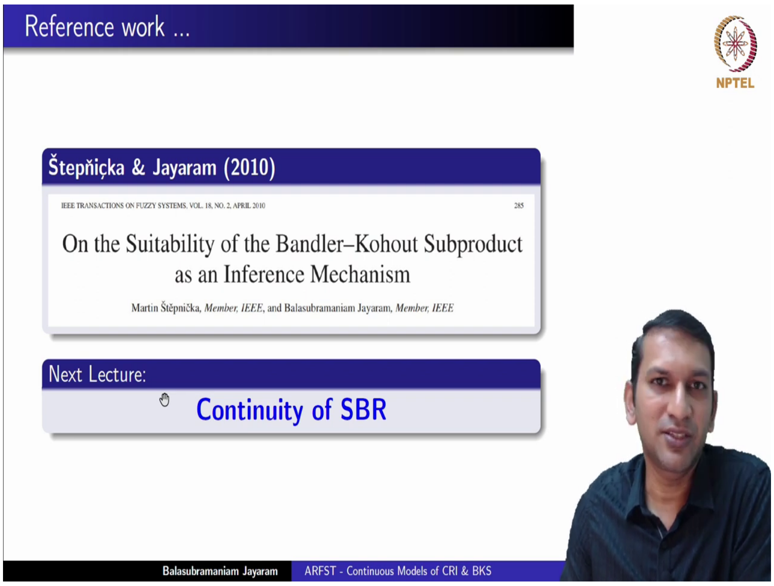
We looked at the continuous model of the corresponding adjoint function in terms of an inequality which related the input to the antecedence and the output to the corresponding consequence through this inequality and what we have finally, found is at least in the case of CRI and BKS inferences any R we will use it there if the corresponding adjoint function is a continuous model for the given rule base, then it is also an interpolative model the given rule base and vice versa. So, these are two equivalent statements.

(Refer Slide Time: 40:58)



Well, for these two lectures this and the previous one predominantly we have sourced it from this particular work of Perfilieva and Lehmke.

(Refer Slide Time: 41:10)



And also the topics related to the Bandler-Kohouts Subproduct from that of Stepnicka and Jayaram. In the next lecture we will look at Continuity of SBR the similarity based reasoning scheme glad that you could join us for this lecture hope to meet you soon in the next lecture.

Thank you again.