Approximate Reasoning using Fuzzy Set Theory
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Lecture - 45
Continuous Models of FRI

Hello and welcome to the next of the lectures in this week 9 of the course titled Approximate Reasoning using Fuzzy set theory. A course offered over the NPTEL platform. In this lecture, we will look at Continuous Models of Fuzzy Relational Inference. Schemes which will prepare the base to discuss continuous models that are obtainable from either the compositional rule of inference or the BKS inference scheme.

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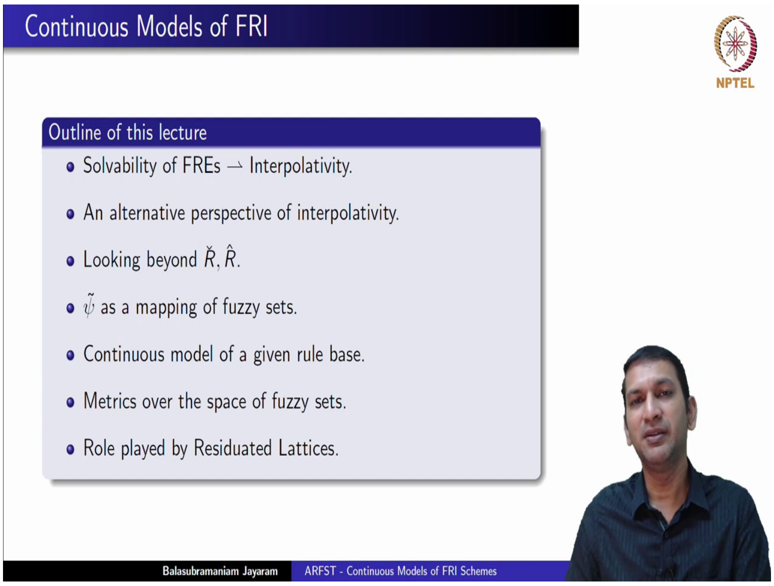

So, far we have studied interpolativity through the lens of solvability of the corresponding fuzzy relational equations. In this lecture, we will look at an alternative perspective of interpolativity which will enable us to look beyond these two relations R checked and R cap. We know a fuzzy inference system can be thought of as a mapping between fuzzy sets. And this will enable us to discuss what are proposed as continuous model of a given rule base in the literature.

And we will also justify why that particular definition can be seen as relating the continuity of a fuzzy inference scheme of course, this will also highlight the role played by residuated lattices.

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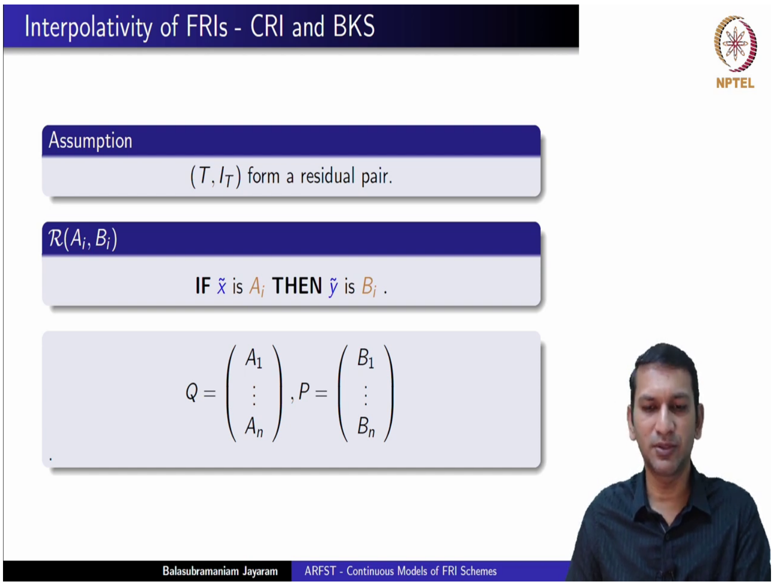

Now, when we discuss interpolativity of FRIs whether it is a compositional rule of inference or the (Refer Time: 01:42) coherence of product. Our basic assumption was we are in the realms of a residuated lattice; that means, T is a left continuous T norm and it is its corresponding residual implication. We have a set of if then rules, we obtain the matrix Q as from the antecedent of these rules; and the matrix P from the corresponding consequence of the rules.

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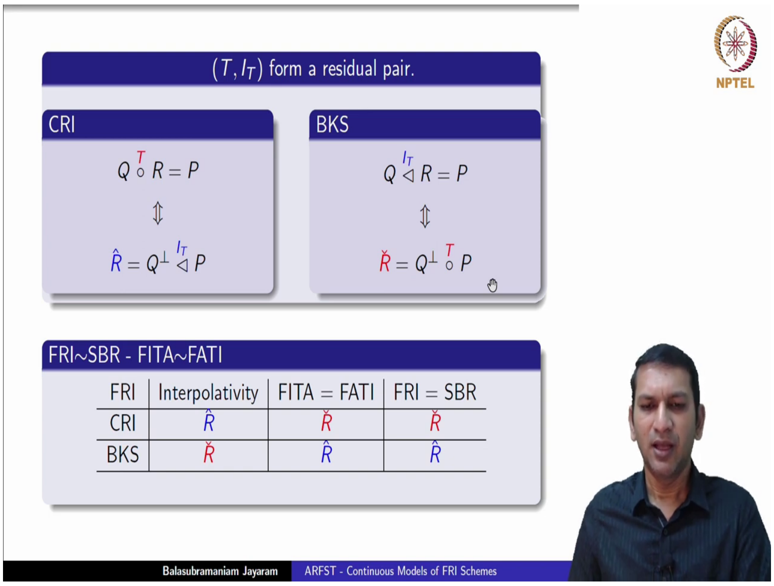

We knew that if we were discussing the solvability of this equation Q sup T R is equal to P where Q and P are fixed, and R is to be found out then we are essentially discussing the interpolativity of CRI. And we have seen that this has a solution that an R exists if and only if this R cap which is given as Q transpose in phi T P composition is a solution was can be shown that it is a maximal solution also.

Similarly, in the case of BKS if you are looking at the interpolativity of a BKS inference mechanism then essentially it boils down to solving this fuzzy relation equation. And we know that it has a solution only if R check as given like this is a solution. Now when we discussed the relationship between FRI and SBR or the equivalence between FITA and FATI we have working seen the role played by these two relations these are R cap and R check.

For instance if we consider CRI why interpolativity is ensured by R cap that for the FITA to be equivalent to FATI the inference from FITA to be equal to that of FATI and to be able to look at CRI itself as a similarity based reasoning scheme. What helped us was the R checked relation and in the case of BKS again the duality persists; that means, R check helps in interpolativity while it is R cap that actually helps us to show that the inferences from both FITA and FATI will be equal.

And also that you could look at BKS as a similarity based reasoning scheme for an appropriate matching function in the other operations.

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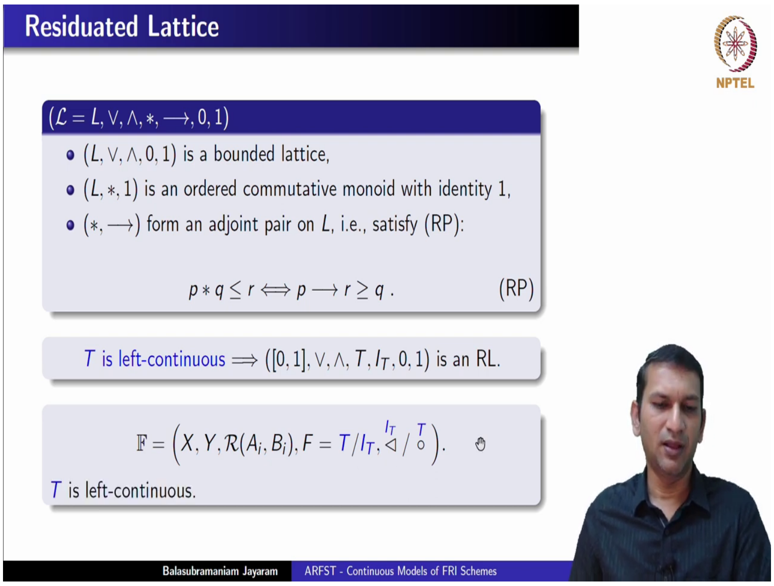

Now, needless to state we are in the lens of residuated lattices which is a bounded lattice it is an ordered commutative monoid with identity 1 and the star and the arrow operation are related by this adjoint property or the restoration property. In the case that we are considering we are picking up values from this unit interval [0,1]. So, essentially if you pick a left continuous T norm then immediately we get a supply of the residuated lattice.

Now, for an FRI that we are considering within this ambit this context all the operations whether it is for the composition or for aggregating the rules or even making the relation from the rule we are picking all the operations from the corresponding residuated lattice.

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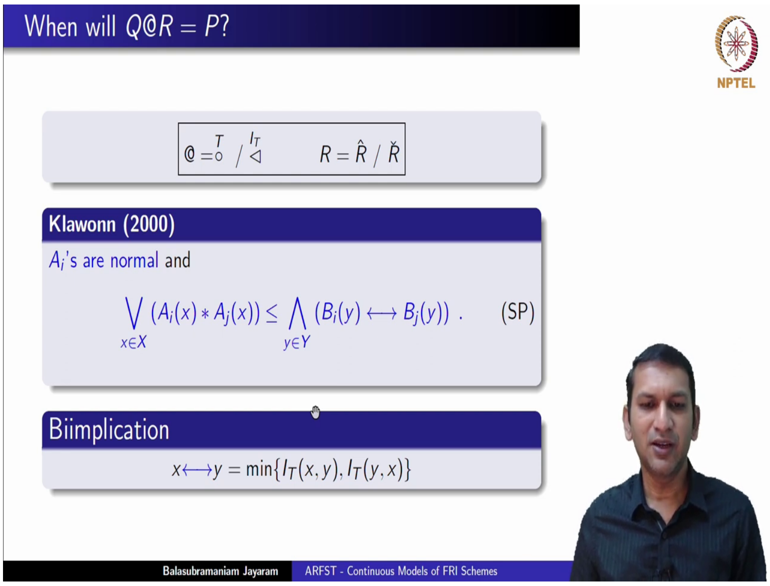

Now, discussing interpolativity of an inference scheme was essentially boiled down to discussing the solvability a of such an equation Q at the rate of R is equal to P where at the rate symbol stands for the composition and what we have done is we have either taken the sup T composition or in phi T composition.

And seen that to ensure that this system is interpolated we need to show that R cap or R check is a solution for this system; however, it was not known when R cap or R check itself will be a solution that is where the work with Klawonn stands out he proposed this sufficiency condition which showed that if this is valid then in the appropriate compositions that we consider with the CRI or BKS the corresponding relations of R cap or R check will indeed satisfy the relational equation.

Now, what is interesting about this condition is it decoupled the relation from its interpolativity for instance all it asked was that the antecedent should be normal and it related the antecedence of the rules with a consequence through some operations which are again taken from the residuated lattice note that this by implication is given like this.

So, we see here that the interpolativity of an FRI scheme can be thought of even beyond the relation R cap or R check purely in terms of the antecedence and the consequence. So, this gives us an avenue to think differently.

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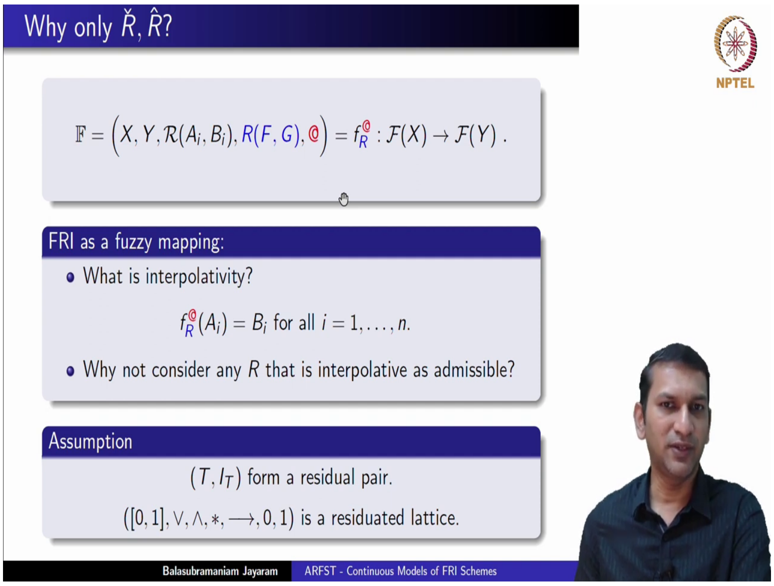

And let us see how to look at interpolativity of an FRI from a different perspective. Now if you look at the form of the FRI we know there are only two things that are important the composition and the final relation itself of course, they may be obtained from some operations performed on the rules which are operation performed on the antecedence and consequence of the rules.

However, if you look at it typically this is what is called the adjoint function of an FRI which is essentially characterized by the relation R and the composition which is denoted by the symbol at the rate of. Now this is the final fuzzy inference mechanism that we have. So, it can be looked at as a mapping from F(X) to F(Y) when we look at it that way as a fuzzy mapping if you ask the question what is interpolativity here essentially it is this function this adjoint function taking the value B i at A i for every i.

So, the question naturally arose why not consider any relation R that is interpolative as an admissible relation R. Of course, we are not getting out of the residuated lattice structure being within the residuated lattice structure we would like to explore if there are other solutions other relations are. Note that R cap or R check is just one solution which will ensure interpolativity and it will also be a solution, but there exist other solutions to ensure interpolativity.

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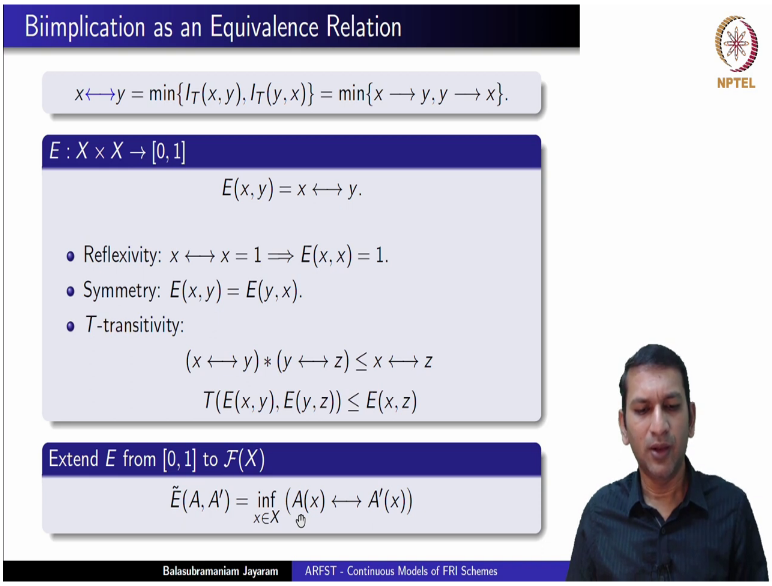

Let us look at the biimplication a little deeper before going any further we can look at biimplication itself as an equivalence relation. So, now, biimplication is given like this in terms of the infixed notation where the arrow stands for the residual implication coming from the residuated lattice this is how it is defined mean of x implies y comma y implies x.

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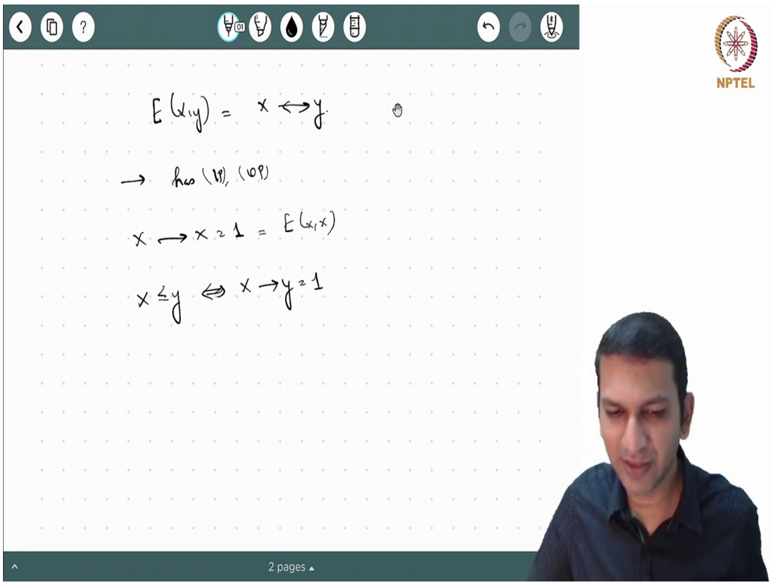

Now, if it define E of x y as x by implies y clearly this E looking at it as a reflect relation on x cross x we see that it is reflexive note that this is an R implication which means we know that this implication has IP and OP. IP means x implies x is 1 and OP says that if x is less than or equal to y if and only if x implies y is equal to 1.

So, x is x implies x is one is essentially E of x x. So, it is reflexive clearly it is symmetric is it transitive is it T transitive yes it is T transitive because by residual as it is all as it is also called in the literature and residuated lattices this by residual or biimplication also satisfies this property which is essentially written in the language of T norms where star is the corresponding T norm T here we see that this essentially means it is T transitive.

So; that means, this by residual when you use x by residual y it is essentially an equivalence relation a star equivalence relation. Of course, once you have a relation on [0,1] you can extended to any to the space of fuzzy sets as follows you can take the membership values and use the by residual on this and then take the infimum over all of those such sessions.

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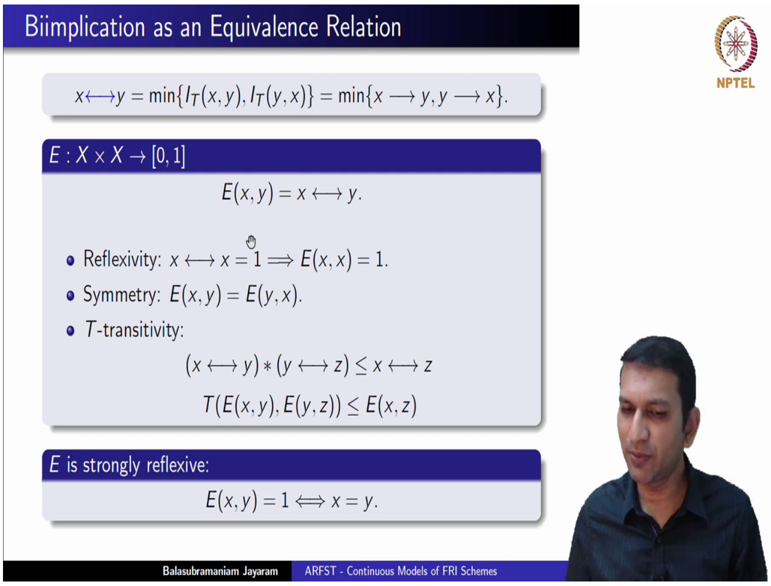

Well, there is one thing to be noted this E is. In fact, strongly reflexive what do you mean by this E of x y is 1 if and only if x is equal to y. Clearly if x is equal to y we know that E of x y is 1, but E of x y is 1 if and only if it also implies x is equal to y.

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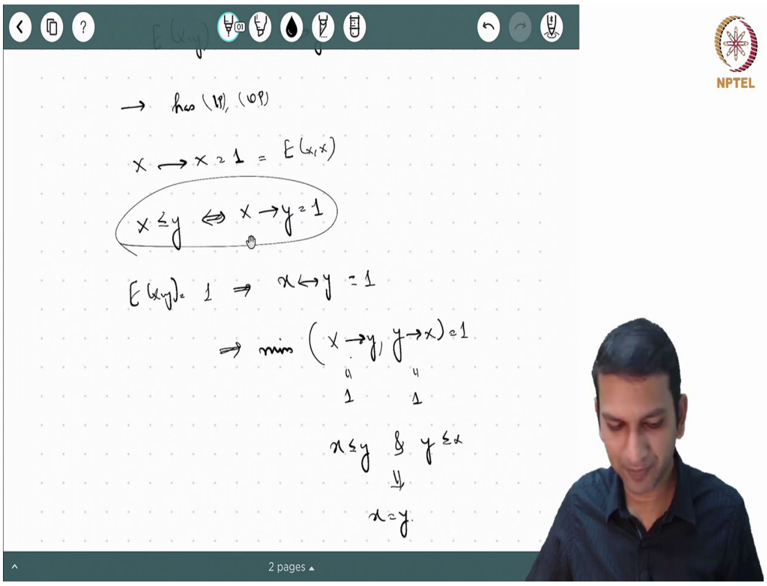

This is clear from the ordering property of the corresponding imply for instance if E of x y is equal to 1 this means x implies y is y implies y is equal to 1. Now, this implies minimum of x implies y comma y implies x this one which essentially means this is quantity is equal to 1 this quantity is also equal to 1.

But we know that from ordering property if this is equal to 1 then this implies x is less than or equal to y and this would imply y is less than or equal to x put together both imply that x is equal to y. So, this relation E is strongly reflexive a property that we will make use of again and again in our discussions later on.

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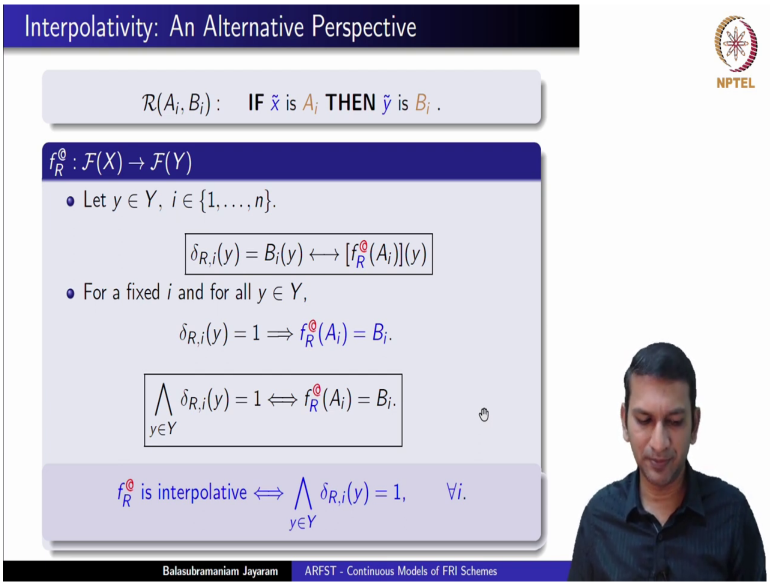

Now, given a SISO rule case we are looking at the adjoint function of an FRI as a mapping from F(X) to F(Y). Let us pick a y from the output domain y and i can be anything from one to n just a arbitrary index from this. Let us define this function delta of R note that this R is arbitrary any R that we are using in the FRI to represent the relationship rules this i is fixed y is fixed.

So, delta R i of y is given as this it is B i of y the membership value that y takes in the corresponding consequent of the ith rule with the by residual of the output obtained from this adjoint function f R at the rate of when A i is being given as the input. So, this will be a fuzzy set on y the membership value of that fuzzy set at that particular y that we have chosen.

So, this is delta R i this is the function that we would like to introduce. Now in terms of this note that for a fixed i if for every y element of Y if we know the delta R i of y is equal to 1, what does this mean?

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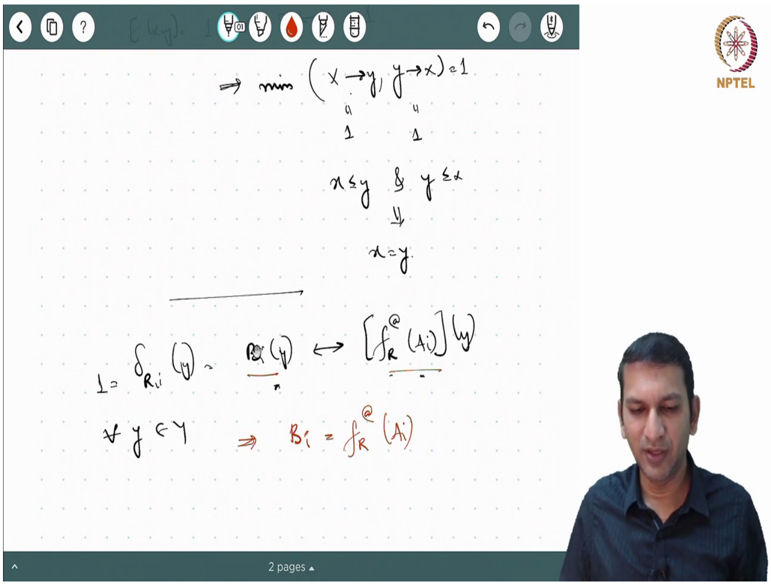

Note that delta R i of y is B i of y f R at the rate of A i y. Now, if you know that this is equal to 1; that means, both these quantities are same this quantity and this quantity are same these are membership values. Now if this happens for every y then; that means, for every y both these quantities are effectively same this implies B i is. In fact, equal to f R at the rate of A i. So, now, for every y del delta R i of y is equal to 1 implies f R at the rate of at A i is in fact, equal to B i.

Now, since this is equal to 1 for every y we could also represented like this if you take the infimum it will still be 1. So, now, we have obtained this from here. Now in terms of this delta R i function we could write or rewrite the interpolativity of this adjoint function as follows. You can say the adjoint function of an FRI f R at the rate of is interpolative if and only if the infimum over delta R i of y is equal to 1 for every A i. So, now, we have an alternate interpretation of interpolativity itself.

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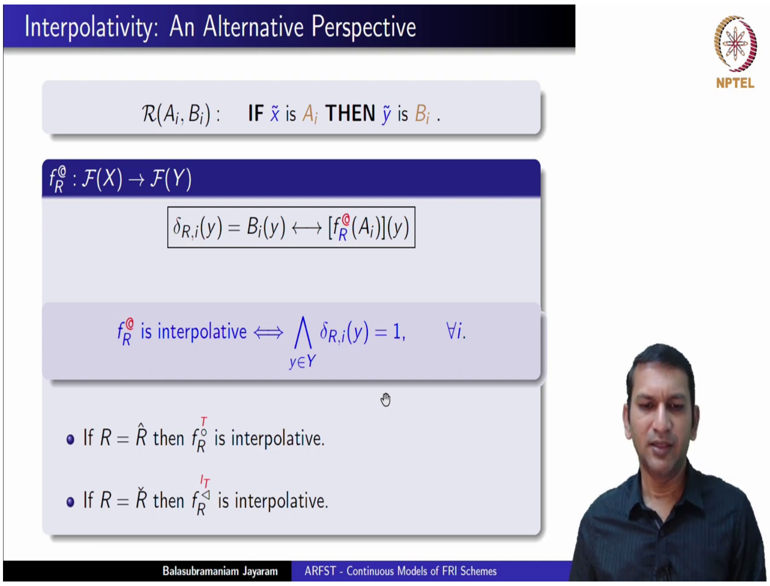

Well, now that we have an alternative perspective of interpolativity of an FRI. All we have seen earlier is this that if R is equal to R cap then f R circle T supremum T which is the adjoint function of an CRI is interpolative. Similarly if R is equal to R check the adjoint function of the BKS inference mechanism where the inference the implication is the residual implication it is interpolative.

So, now you see here these are some sufficient conditions and these specify an R to be of a particular type for it to be interpolative, but these are only sufficient conditions; that means, there exist other relations are under which CRI or BKS can well be interpolative.

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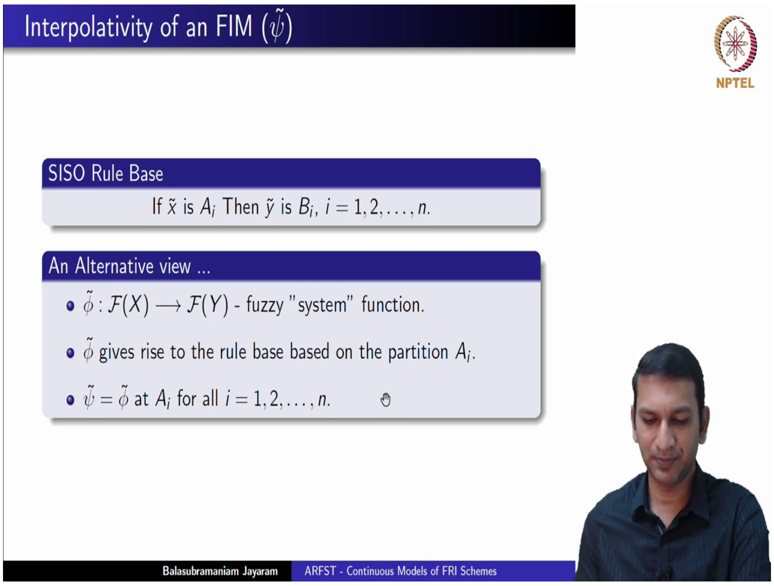

Now, let us look at what we understand by continuous model of an f R recall the given a SISO rule base an alternative view of fuzzy inference mechanism itself is as follows. You could think of the system function that we are trying to approximate using this fuzzy inference system itself is a fuzzy mapping.

That means you could think of it as phi tilde which is a mapping between F(X) to F(Y). Now, given this view and a partition A i we could think of the antecedence A i forming a partition on the input space x and phi tilde acting on A i to give the consequence B i. So, the antecedence and the consequence of the rule can be thought of as having been generated by the system function phi tilde which itself is a mapping from F(X) to F(Y) and where A i the antecedence are pieces in a fuzzy partition of the input domain and the consequences B i are the images of this these A i under the function phi tilde.

In that sense we saw that interpolativity is finding a fuzzy inference mechanism of system whose corresponding function psi tilde approximate this phi tilde at these A is or is equal to this phi tilde these A is that is how we looked at interpolativity of an FIM.

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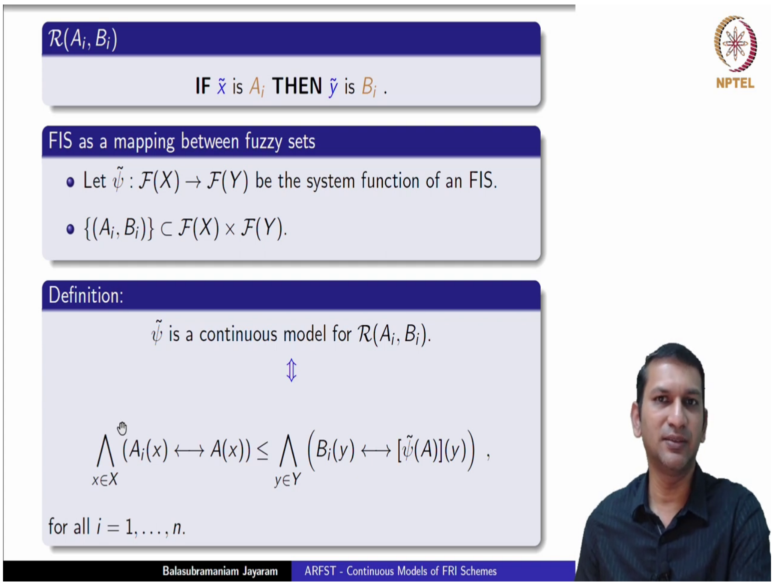

Now, let us extend this view. So, given this rule base we want to look at a fuzzy inference system as a mapping between F(X) to F(Y) now these antecedent consequent pairs can be thought of as coming from the Cartesian product of the input and output domains of fuzzy sets. Given this we could define psi tilde to be a continuous model for this rule base if and only if the following inequality is valid.

Notice this much like the condition proposed by Klawonn for interpolativity this condition is firstly, an inequality and it relates the given input A to the antecedence of the rules and the obtained output of A with respect to psi tilde; that means, the image of A under psi tilde to the corresponding consequence.

So, now we are looking at continuity in terms of the points that have been given these pairs of fuzzy points that we have been given. Now the question is why should we consider this as the continuous model why this term continuity coming into picture.

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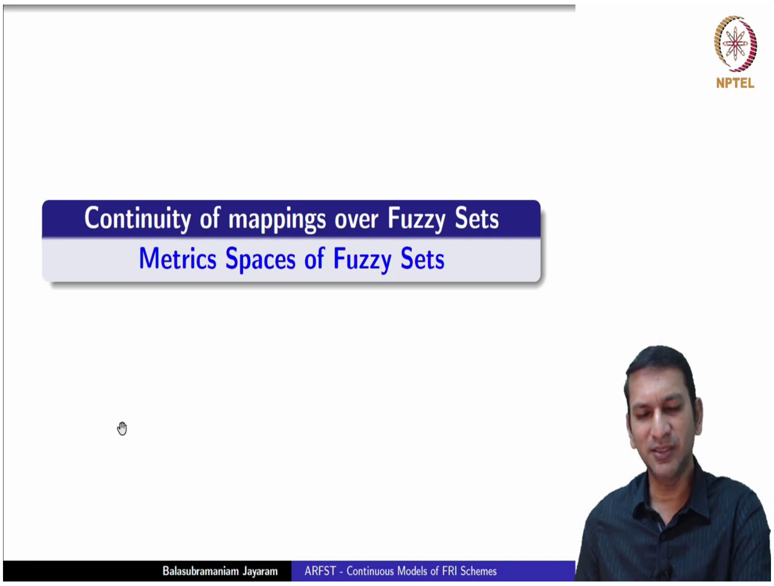

Now, let us take a short detail to understand this if you want to talk about continuity of mappings then we need a structure an analytical structure on the underlying set perhaps it could be a topological structure or we need the concept of sequences in their convergence or we could also have a metric. So, now let us look at having metrics on the spaces of fuzzy sets.

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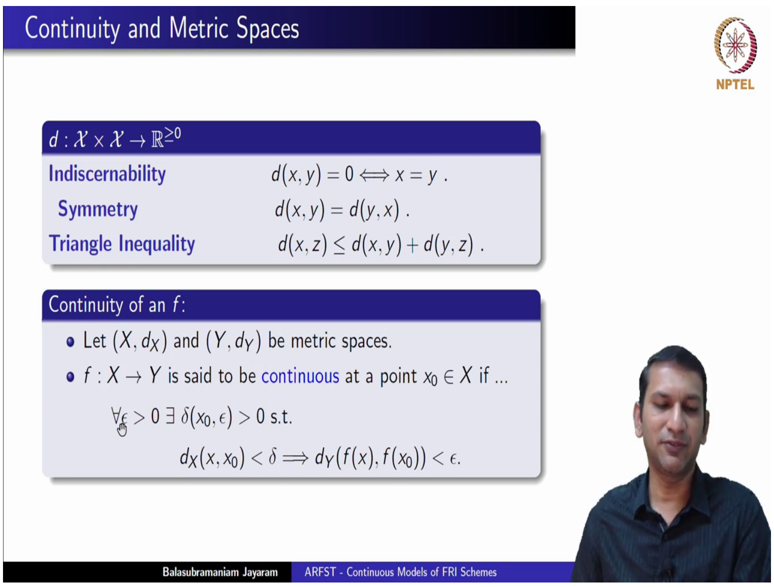

So, what is the metric the function d from x cross x to the non negative reals such that it satisfies three properties of indiscernability symmetry and the triangle inequality. Now with respect to metric space if you look at a function f if you want to talk about continuity of this function then we need metrics both on the domain and codomain. So, let us consider X and Y as the domain and codomain with metrics d x and d y respectively we say a function f from X to Y is continuous at a point x naught in X.

If for every epsilon greater than 0 there exists a delta which; obviously, depends on both x naught and epsilon such that whenever d x of x x naught is less than delta this implies d y of F(X) comma f of x naught less than epsilon. All we are saying is if x is close enough to x naught within the distance or measure of delta then the image of x under f should be close enough to the image of x naught under f.

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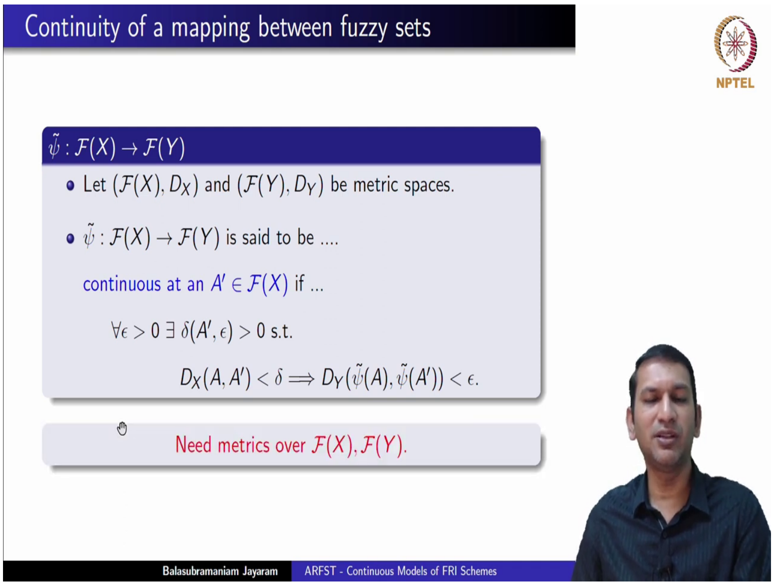

Now, we are looking at psi tilde as a function from F X to F Y, let us define some metrics D X and D Y on these spaces. So, now we can define what is continuity say psi tilde is continuous at an A dash if for every epsilon greater than 0 there exists a delta which once again depends on A dash and epsilon such that D X of A A dash is less than delta this implies D Y of psi tilde A and psi tilde A dash is less than epsilon.

Now this is a general definition of what we would mean as a continuous function psi tilde from F X to F Y given the metrics D X and D Y; however, we need metrics now on the spaces of fuzzy sets.

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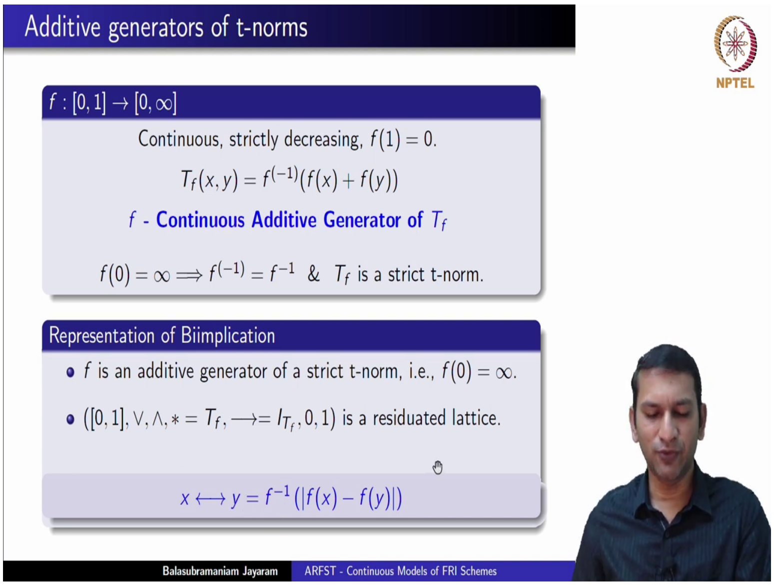

Let us turn towards the additive generators of T norms to see whether they can generate metric first. Recall an additive generator of a T norm continuous T norm is a function f from [0,1] to [0,infinity] which is continuous strictly decreasing and f of [0,1] using this with this formula we can get a T norm T f. This T f is called the generator T norm and f is called the continuous additive generator of T f.

Now, we have not specified what the value of f of 0 is; however, we know that if f of 0 is infinity then clearly this f pseudo inverse becomes actual inverse of f and the T f is in fact, a strict t norm. Now what is interesting is if we take such an additive generator of strict T norm that is f of 0 is infinity. And consider the residuated lattice obtained from this generator T norm note that this is continuous T f is continuous. So, it is left continuous.

So, definitely it will give rise to a residuated lattice structure on [0,1]. If you take this t norm and generate the residuated lattice structure then it is interesting to note the representation the by residual mass in this distributed lattice x biimplication y is given as f inverse of mod f x minus f y.

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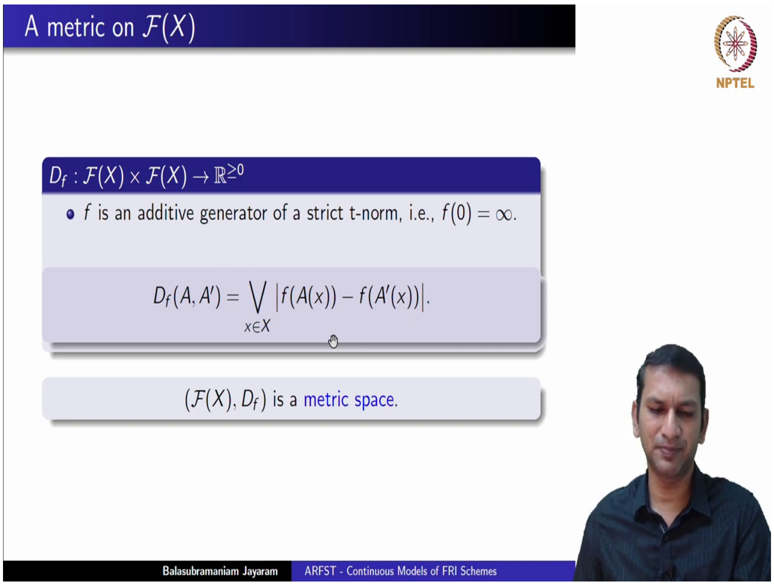

Now, why this representation? We will make use of this representation shortly. Let us define a function D f from F of X cross F(X) to the non negative reals where F is an additive generator of a strict t norm means f of 0 is infinity. Now let us define the function D f like this D f of A A dash is supremum over x element of x mod of f of A x minus f of A dash X note that A of X and A dash of X. They take values in [0,1] there membership values.

So, this is a valid definition what we can say is this D f in fact, is a metric on the set of fuzzy sets on x. Now, this can be easily seen the indesignability because it is supremum in supremum is 0 then we know that what we have a these values inside on which we are taking the supremum there are all non negative because its mod then every one of those values is going to be 0 and because of mod and strict decreasing or strict monotonicity and continuity of f we see that A of x has to be equal to A dash of x.

Similarly, symmetry is clear the triangle inequality follows from the triangle inequality of the mod function. So, on F X D f is a metric. Now why did we create this metric because we wanted to relate the equation which we call the continuous model the equation or inequality if it is satisfied by a psi tilde we said the psi tilde as continuous model of the given rule base we want to relate it how why do we use that term continuity or continuous model them.

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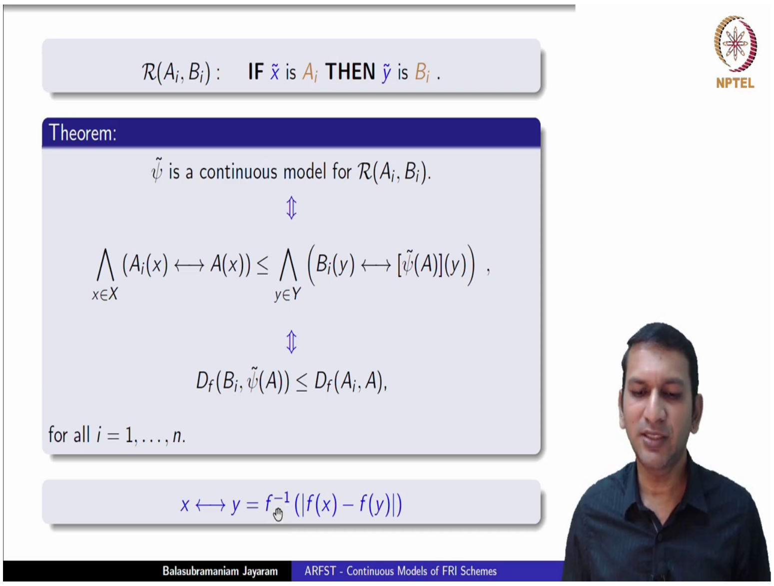

So, we say this is the rule base given to us we say psi tilde as a continuous model for this rule base by definition this is the inequality we expect which relates the given input to the antecedence and the output obtained from psi tilde to the consequence of the corresponding rules. This is what we have defined as a continuous model for a given rule base this we can show is actually equivalent to the following inequality.

So, what does it say, it says that this whenever this happens this equivalent to the psi tilde of A the output the of A the given input A and the psi tilde it says is much closer to the corresponding B i consequent B i compared to how close the given input A itself is to the corresponding antecedent. So, we insist the image of A is closer to B i much closer to B i than A itself to that of the antecedent A i as measured by this metric D f.

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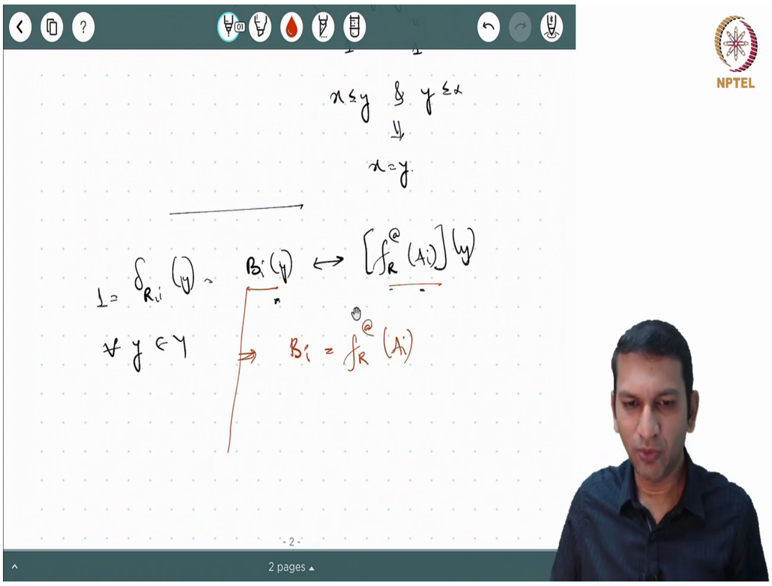

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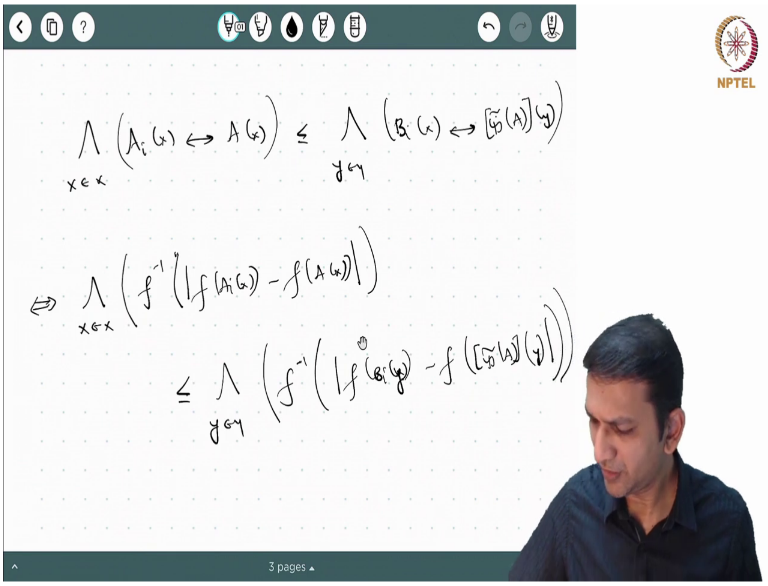

Now, the proof is not very difficult let us look at this not that what we need to prove is we are given this infimum over x A i of x mod A of x this is less than or equal to infimum over y B i of x residual psi tilde of A. So, now, this we need to show is. In fact, is equivalent to as you can see D f is a metric. So, this talks about some kind of continuity. In fact, you can look at it as one Lipschitz continuity at the points A i B i.

So, now let us start with this inequality and slowly go towards this. Remember we are looking at the operations coming from a residuated lattice which is obtained from this generated strict t norm T f in which case we know the biimplication has this kind of a representation. So, let us substitute that here; that means, given this substituting this is. In fact, equal to infimum over x f inverse of mod f of A i of x minus f of A i of x mod this is less than or equal to infimum over y f inverse of mod f of B i of x minus f of psi tilde of A of y.

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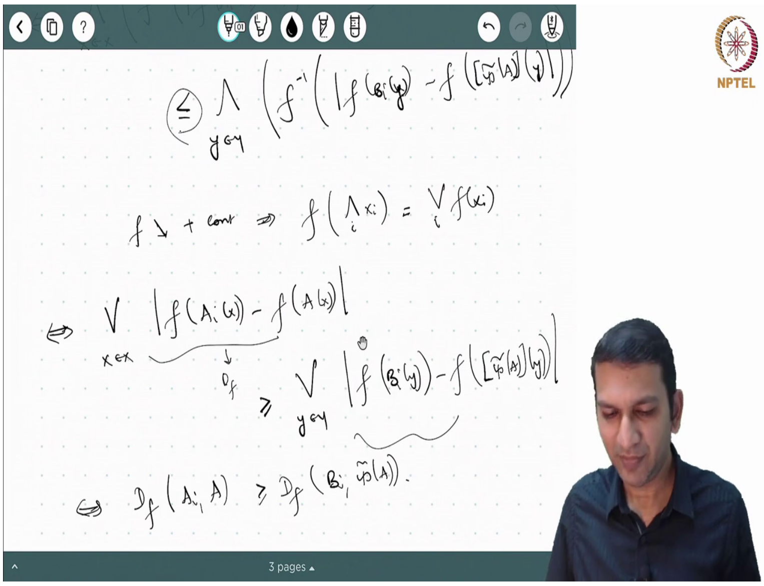

Now, note that f is strictly decreasing strictly monotony it is also continuous. So, this implies if we have infimum over some xi then this essentially becomes supremum over f of x i and now what we have here is inequality. So, if you apply f on both sides then the inequality also reverses. So, now, applying f on both sides what we get is the infimum comes out as the supremum.

So, we get supremum over x element of x f and f will get cancelled minus f of A of x is greater than or equal to because this inequality will change f is a decreasing function supremum over y mod f of B i of y minus f of psi tilde A of y. Now clearly if you look at it this is essentially how you have defined D f this on A ok. So, this is D f of A i comma A greater than or equal to D f of B i comma psi tilde of A this is exactly what we wanted to prove and now you see and everywhere it is an equivalent statement that we have written.

So, it is a biimplication. So, you could go back from this step to the inequality here which means we have shown that this inequality is. In fact, equivalent to this inequality which tells us that this psi tilde is continuous at these points A i B i. Now it kind of justifies why this inequality can be considered as the inequality capturing the continuity of a model at the given points essentially given the rule base considering A i comma B i pairs as the points at which we want the fuzzy inference mechanism itself the corresponding function system function to be continuous.

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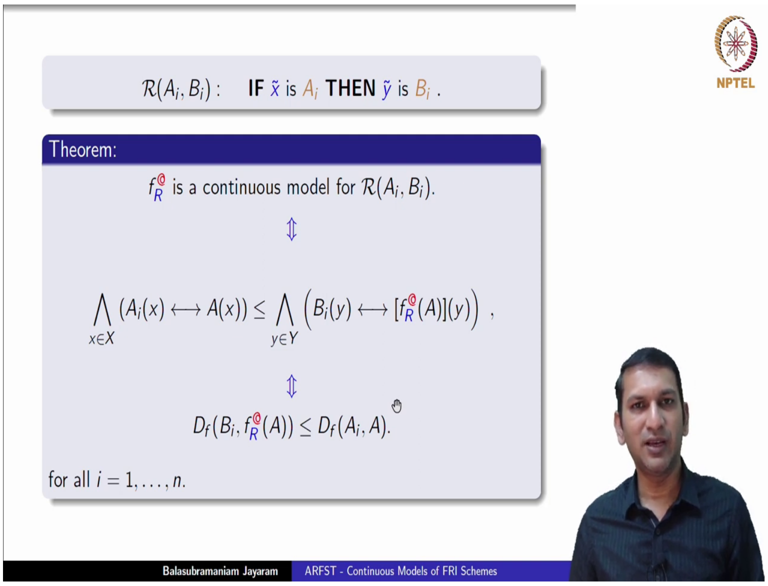

Well, now this was an arbitrary fuzzy inference mechanism, but we could also consider the adjoint function of an FRI so; that means, we can say f R at the rate of is a continuous model for a given rule base if this inequality is valid with f R at the rate of in the place of psi tilde and that is because of this.

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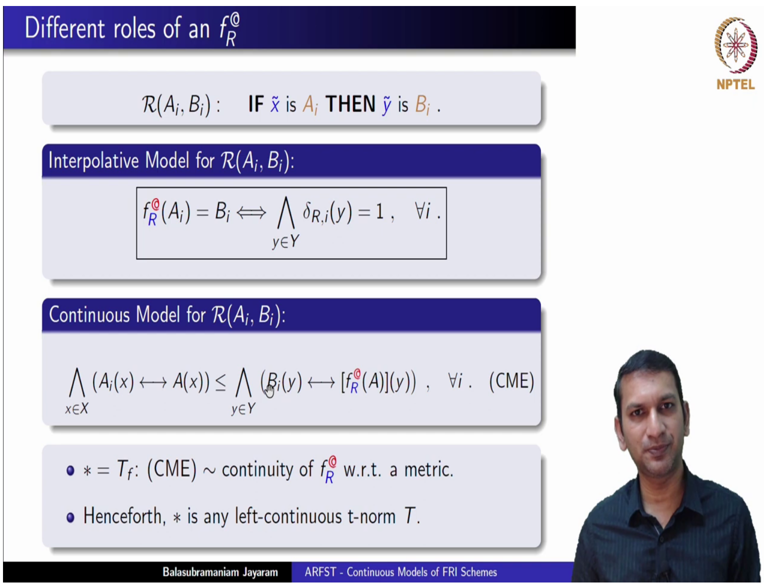

So, now what we have are these given a rule base we see we have an alternative interpretation of an interpolative model. So, we say that the corresponding FRI because this is the adjoint function of an FRI is interpolated if and only if this is valid for every A i. And similarly we can say the adjoint function of an FRI f R at the rate of is a continuous model for a given rule base if this inequality is well.

Now note that we have used the residuated lattice obtained from the generated strict T norm only to show that this inequalities can be looked at as can be related to the continuity of the adjoint function with respect to some metric it is only for that purpose that we have taken recourse to developing a metric generating a metric from their generators of additive generators of t norms strict t norms.

Henceforth, star will only be any left continuous t norm we will not fix it as the generated t norm. So, we are generalizing it is only contextually to show the relationship between this inequality and the continuity of the adjoint function on the fuzzy inference mechanism that we are considering. We took the help of the metric derived from additive generators of strict t norms. But henceforth the residuated lattice that we considered will be general in the sense that we could consider any left continuous t norm.

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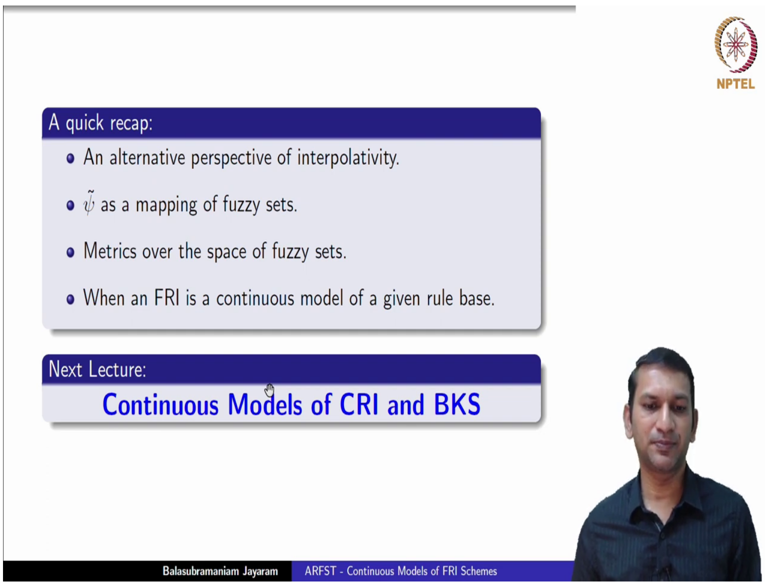

Now, a quick recap of what we have discussed in this lecture we looked at an alternative perspective of interpolativity and looking at psi tilde as mapping of fuzzy sets using cleverly designed metrics over the space of fuzzy sets, we have seen when and a fuzzy inference mechanism can be thought of as a continuous model of a given rule base and essentially; that means, the adjoint function of an FRI when it can be thought of as a continuous model of given rule base.

This is what we have seen in this lecture. In the next lecture, we will look at how to obtain continuous models of adjoint functions corresponding to the compositional rule of inference and then (Refer Time: 32:04) cohorts of product.

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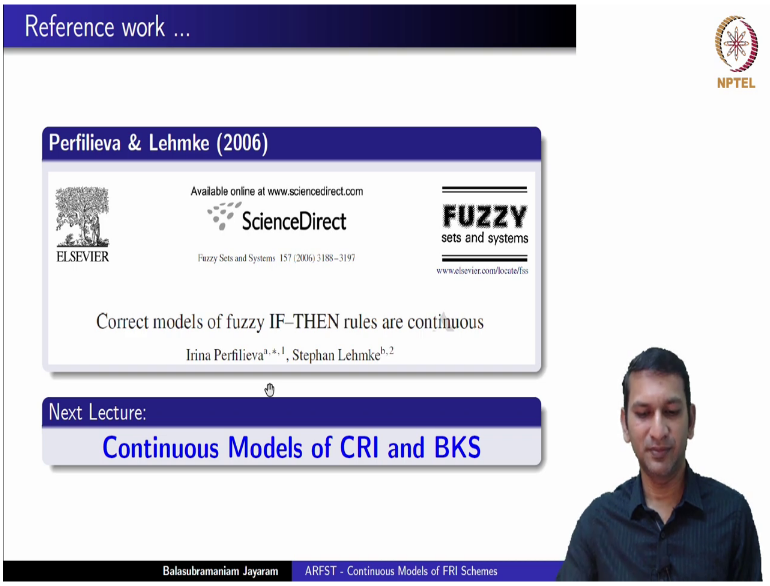

The topics covered in this lecture are predominantly picked up from this work of Perfilieva on Lehmke. We meet in the next lecture where we will discuss continuous models of CRI and BKS. Glad you could join us for this lecture and hope to see you soon in the next lecture.

Thank you again.