**Approximate Reasoning using Fuzzy Set Theory**
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**Lecture - 43**
**Similarity Based Reasoning - Interpolativity**

Hello and welcome to the last of the lectures in this week 8 of the course titled Approximate Reasoning using Fuzzy Set Theory, a course offered over the NPTEL platform. In this week we have been discussing the interpolativity of fuzzy inference mechanisms, predominantly in the last four lectures we have discussed the interpolativity of an FRI scheme a fuzzy relation inference scheme.

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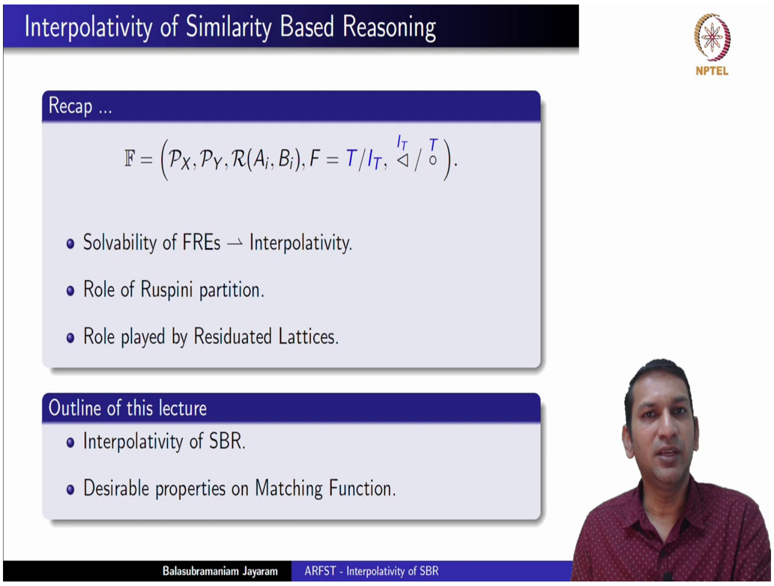


We have seen that to specify a fuzzy relation inference scheme we need to specify the coverings the fuzzy covers over the input and output space. The rules the way they are formed where the antecedent A i and the consequence B i are actually picked up from these fuzzy covers.

The operation used to relate the antecedent to the consequent to obtain the rules the relations from the rules and finally the composition itself. And what we have seen is if you want to discuss the interpolativity of an FRI then practically every degree of freedom imposes some conditions or we need to impose some conditions on all of these different parameters for instance.

When we looked at the composition then we had to look into the solvability of fuzzy relational equations using those compositions and we have seen in the last lecture how they directly impinge upon the interpolativity. Not only that we have also seen the underlying coverings they should become a partition and at least on the input domain we need a Ruspini partition to ensure interpolativity. Of course, when we look at the operations involved themselves.

Then we see the role played by residuated lattices in this lecture we will look at interpolativity of similarity based reasoning systems. And here we will see that we would want some interesting or desirable properties on the matching functions that we employ in the similarity based reasoning schemes.

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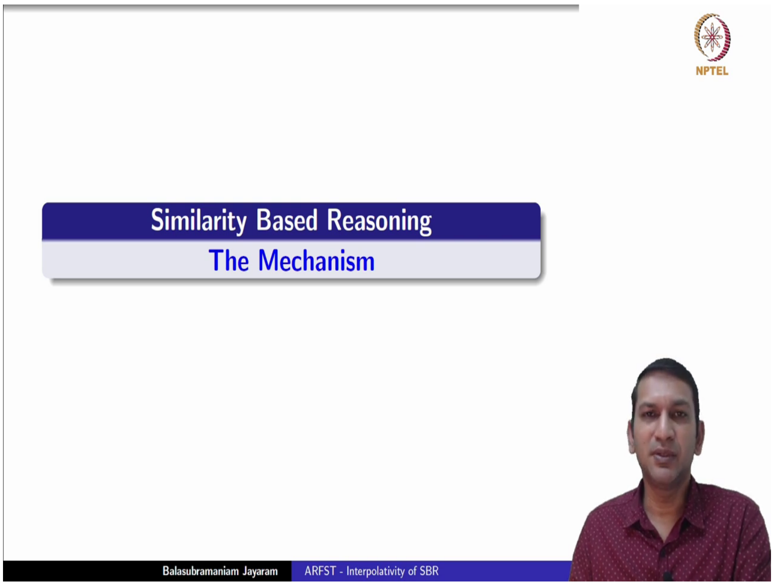


A quick recap of the mechanism underlying a similarity based reasoning scheme.

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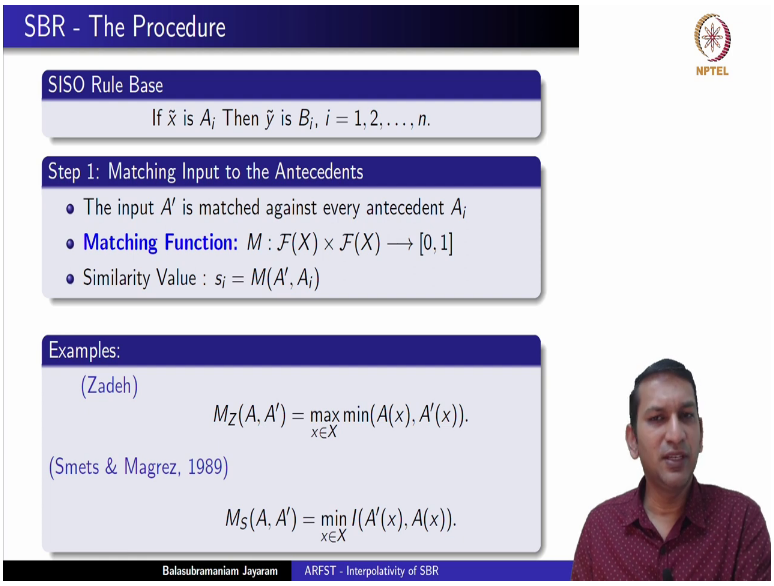


So, we are given a SISO rule base x is A i then y is B i given an input A dash we use some matching function to obtain a similarity value, which typically lies between 0, 1 and the 0,1 interval these are some examples that we have seen earlier too.

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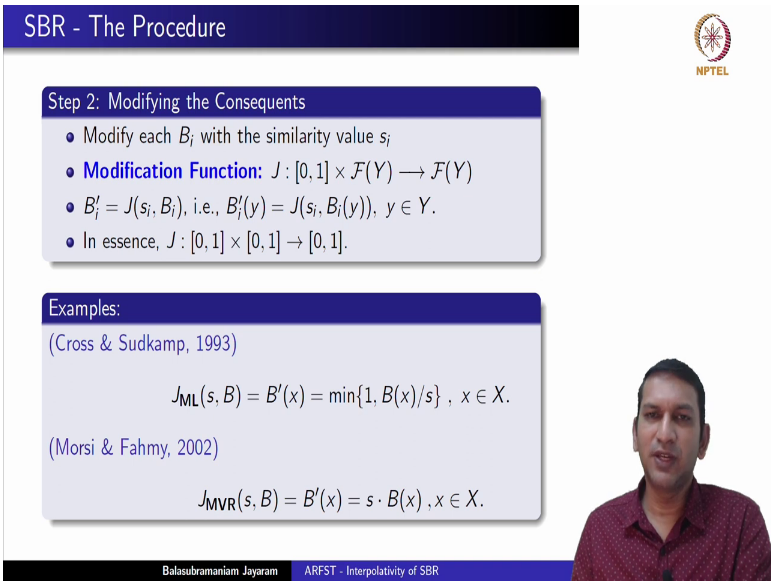


Using the similarity value we modify the consequence of each of the rules and for this we use a modification function J which we have anyway seen as can be thought of as a just a binary fuzzy logic operation. So, these are some modification functions we have seen earlier.

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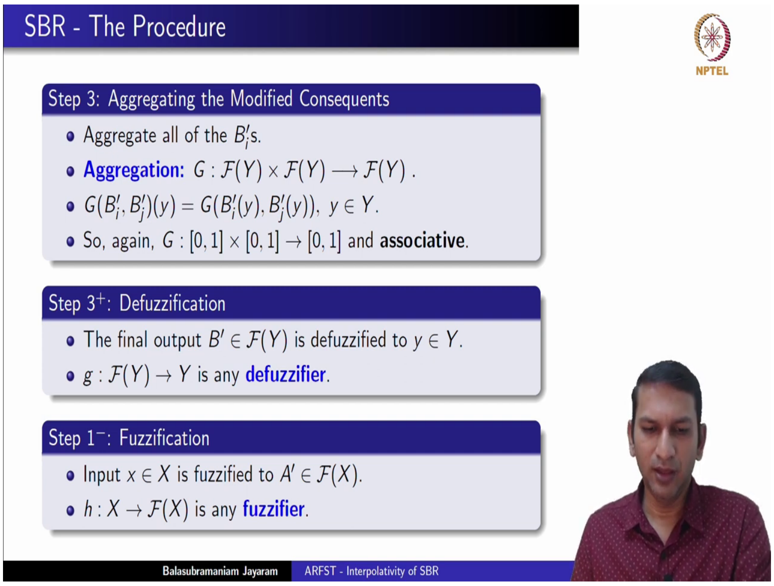


And once we obtain the modified consequence from each of the roots that is we get Bi dashes, then what we do is we use an aggregation function G to aggregate all of them into an overall output B dash. Now once again we have seen that this G could be a binary fuzzy logic operation just that it needs to be associative. Of course, if you want to defuzzify the overall output we use a defuzzifier and if the initial input if it were a real value or a real vector we will appropriately fuzzify it to obtain an input fuzzy set a dash.

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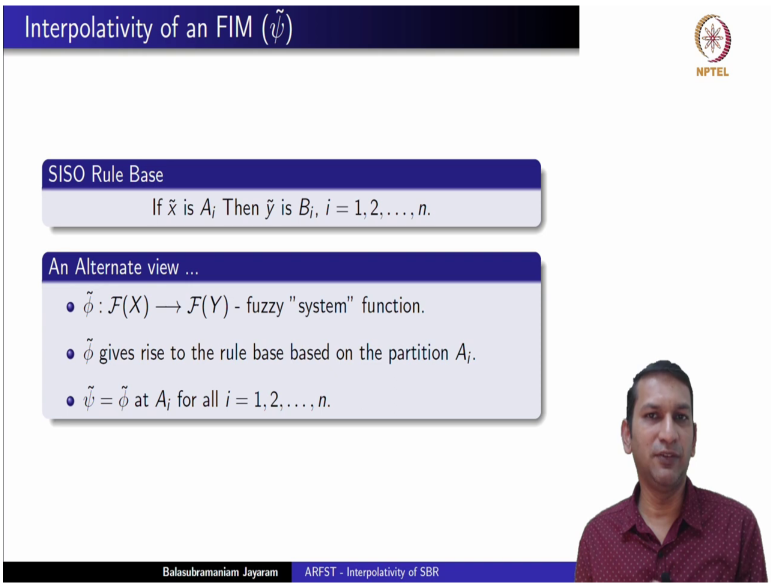

Let us look at interpolativity itself we have seen that we are given a set of if then rules SISO rules, we have discussed in the classical case what is interpolativity and we took cue from there. However, there is also an alternate view we said that fuzzy inference system tries to capture the system function.

However we could also look at the system function itself as a fuzzy mapping in which case the antecedent and consequence of a rule of rules can be thought of as actually coming from this system function, which gives rise to the partition A i the antecedents there, and hence when we are trying to approximate it using our fuzzy inference mechanism. Now you can think of it as a psi tilde, which is a function from F of X F of Y and we will insist that this psi tilde is as close to phi tilde as possible.

And as we have seen one measure of correctness to ensure this is that it is interpolative at these antecedent points. So, this is the same as what we have seen earlier, but now it could also be seen from a different view as the system function itself being a fuzzy matter well.

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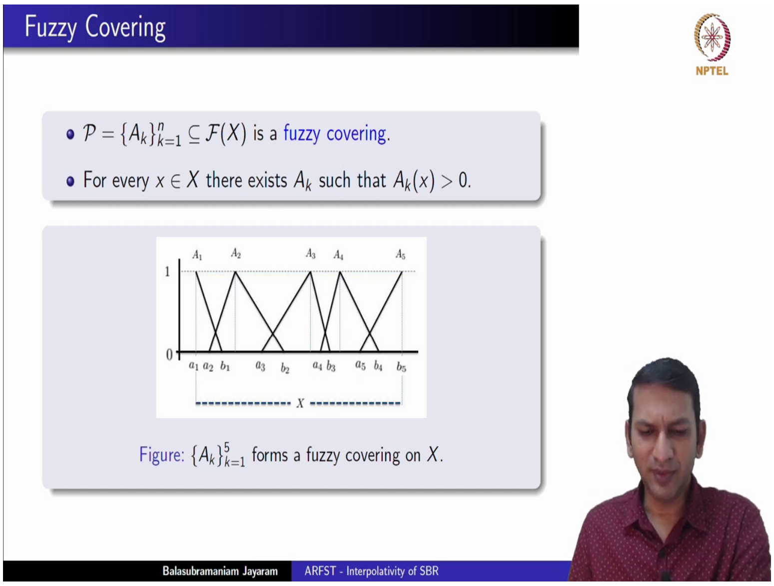


We know what a fuzzy covering is the collection of fuzzy sets on the space says that the union of its supports is actually equal to the space or contains that space.

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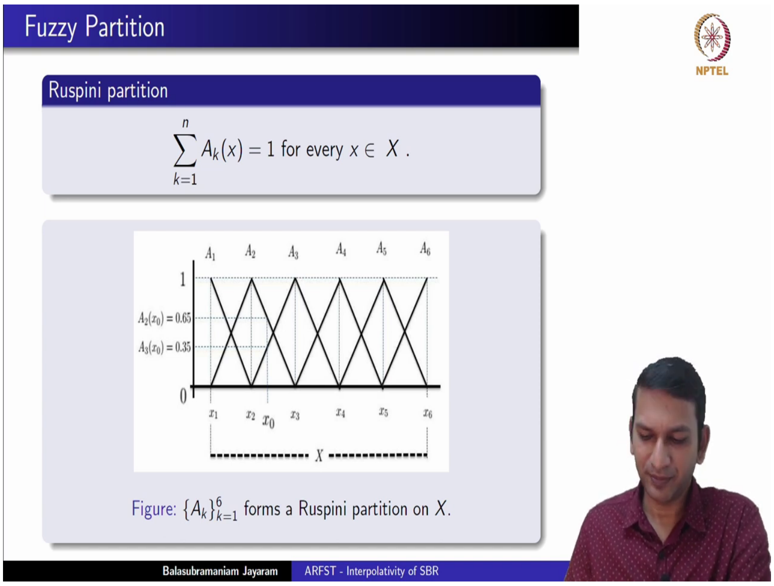


Ruspini partition is one such that every point if you consider the membership value of a point to all the fuzzy sets considered in the partition. The sum of the membership values should add up to 1.

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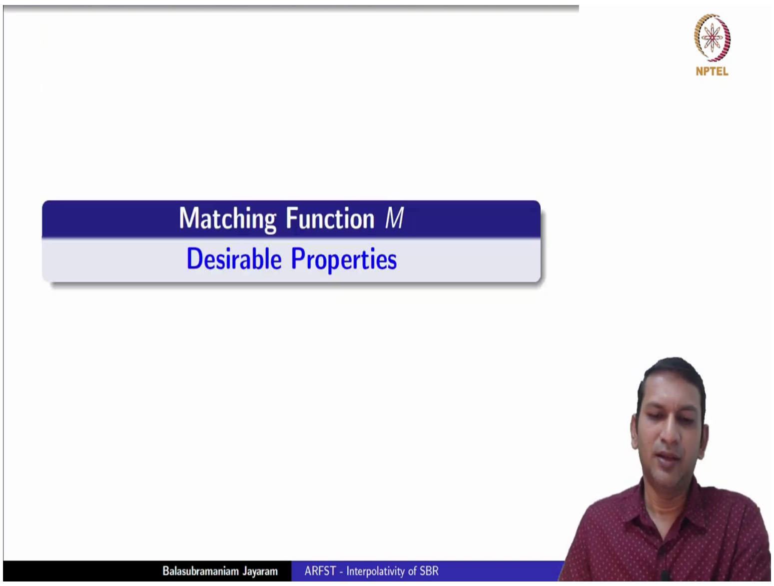


Now, we want to discuss the interpolativity of an SBR scheme. Let us look at some desirable properties that we would need on the matching function, so that we can talk about the interpolativity of the underlying SBR scheme.

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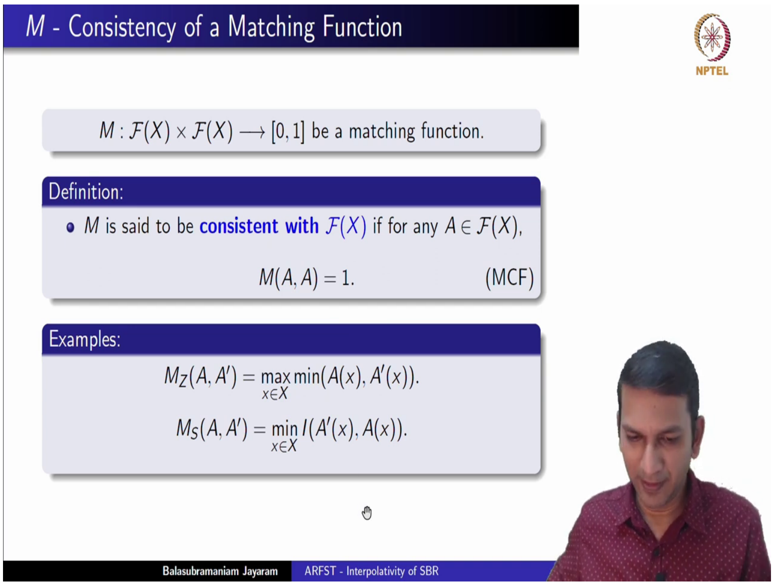


When do you say matching function is consistent note that matching function essentially takes 2 fuzzy sets from a particular domain and maps it to 0,1 interval. We say it is consistent with respect to the underlying space of fuzzy sets, if given any A M of A, A is actually 1. That means, if you match a fuzzy set to itself then it should take the maximum value which is that of 1. Now let us look at this particular example, which is the Zadehs matching function.

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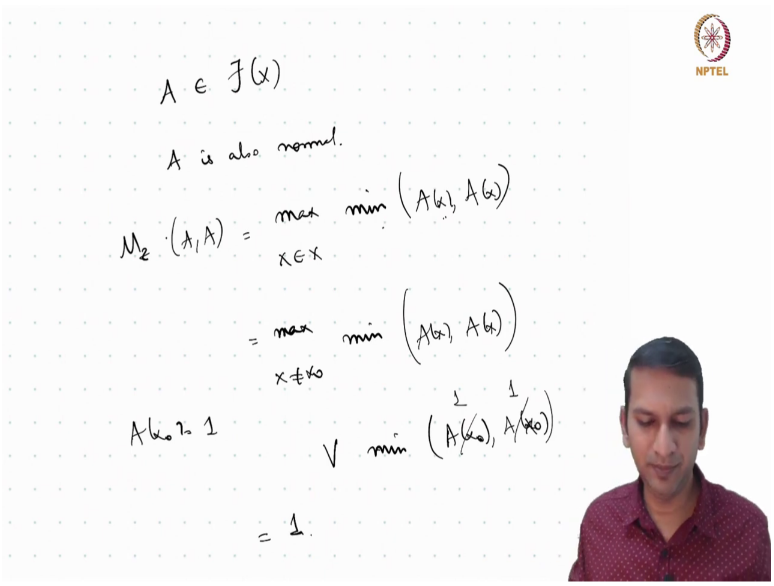


Now if you take any A from F of X does not matter F of X what it is? If A is also normal then M z of A, A is nothing but max X element of X minimum of A of X comma A of X we know that since a is normal there will exist a point X naught at which A of X naught is 1. Now at that point we could again once again write it as max X not equal to X naught minimum A X comma A X minimum A X naught comma A X naught.

We know that at these 2 points it is 1 minimum of these 2 is 1. So, the maximum taken over this entire thing will actually come out to be 1. So, clearly if we consider F of X to be the set of all normal fuzzy sets on X, then the Zadehs function is in fact consistent with respect to F of X. So now, we need to only consider normal fuzzy sets.

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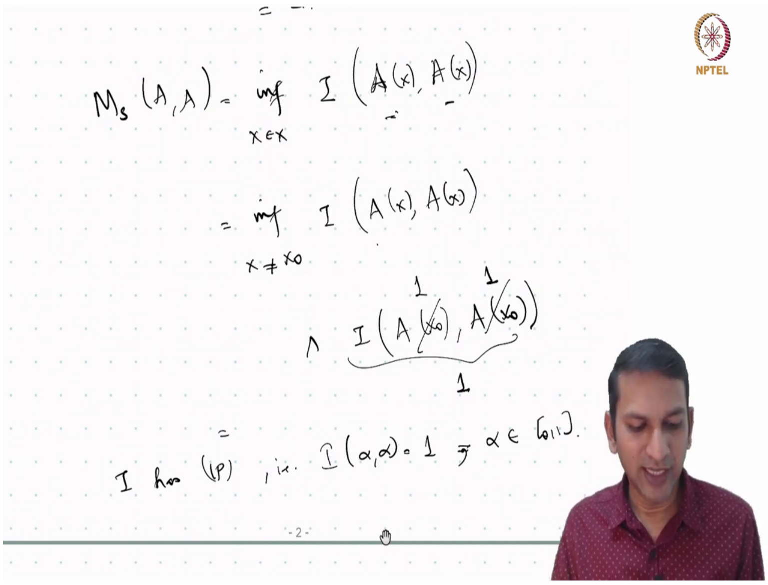


Now, consider the second matching function M S, M S of A dash is infimum over X element of X I is an implication. We have A dash of X comma A of X once again let us only consider normal fuzzy sets. However, in this case even if you consider normal fuzzy sets, if you were to split as infimum X not equal to X naught I of A dash of X comma A of X min I of A dash of X naught it should actually be A itself.

What we see is this quantity is 1 1, Since I is an implication this will turn out to be 1. But what happens here? We do not know what this value will be if this value goes down below 1 then the infimum would be less than 1. So that means, this matching function will not have it will not be consistent with respect to the space of normal fuzzy sets that we define on X. To ensure that what we need is that I has IP that is I of alpha comma alpha is in fact equal to 1, now this should happen for every alpha.

Now, if you insist on this immediately you will see we do not even need to consider fuzzy sets which are normal, because no matter what value you put here the same value you have to put here and for every X A of X is alpha I of alpha is 1 and infimum for all of them will be 1.

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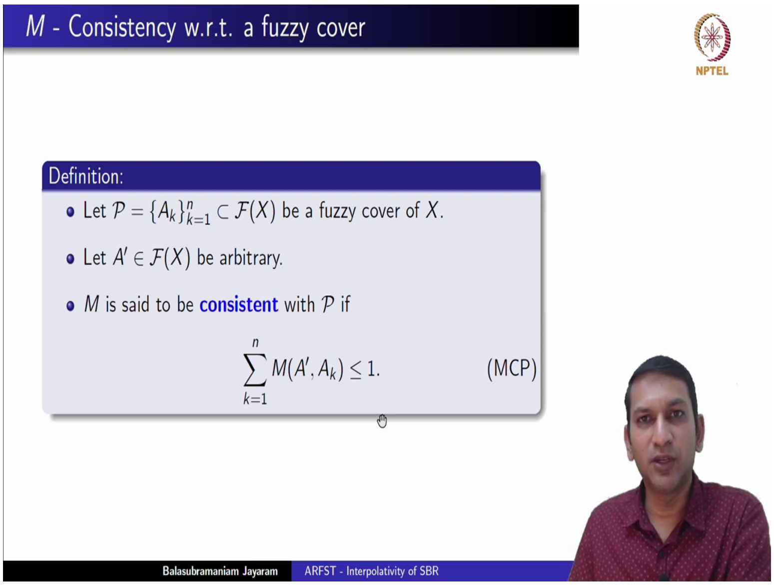


So, the second matching function that we have if you if we employ an implication which also has the identity principle the identity property then in fact this is consistent with any space of fuzzy sets. We can also define another property of a matching function called the consistency of a matching function with respect to fuzzy cover over X.

So, let us assume that we have a collection of fuzzy sets P which form a fuzzy cover of X and let us pick some A dash from the set of fuzzy sets this A dash need not come from P. We say M is consistent with respect to this fuzzy covering if the following property is satisfied; that means, when you match this A dash with respect to all of these A case then the summation of the similarity values should be less than or equal to 1.

This is more or like more or less like the Ruspini partition of course with subadditivity. In Ruspini partition we insist that it should be equal to 1. For example, if we consider A dash to be singleton then at the point where it is if I insist on equality here essentially this becomes the Ruspini partition, are there examples of matching functions, which are consistent with respect to fuzzy covering of X well.

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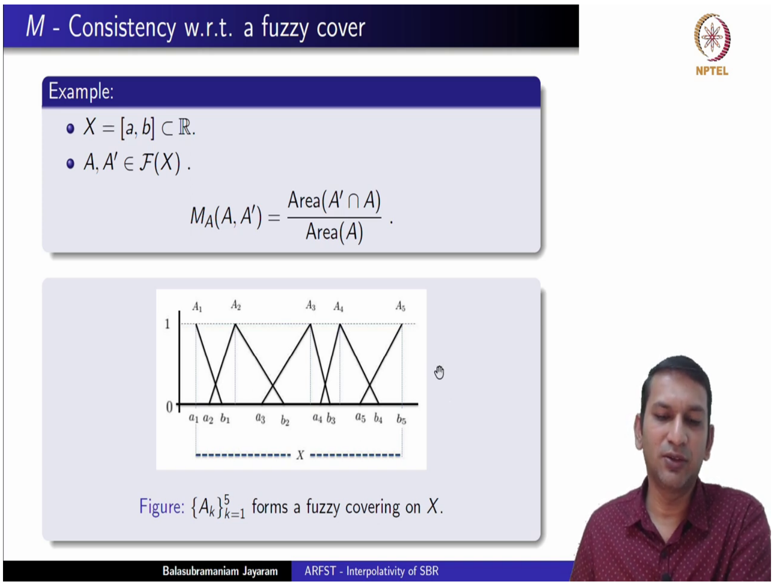


Consider X to be this interval which is a bounded interval and let us take A and A dash coming from set of fuzzy sets defined on this X. if we consider this function as the matching function all we are doing is taking the intersection of A dash and A and taking its area and dividing by the area of A clearly if A dash is A then M A of A, A is 1.

So that means, it is consistent over this F of X if we consider a fuzzy covering of this type it can be easily shown not without much difficulty that. In fact, this matching function is consistent with respect to this fuzzy covering, note that in this case the X the space X is actually the interval A 1 b 5. So, this is a matching function which is both consistent on F of X and also consistent with respect to such partitions on X.

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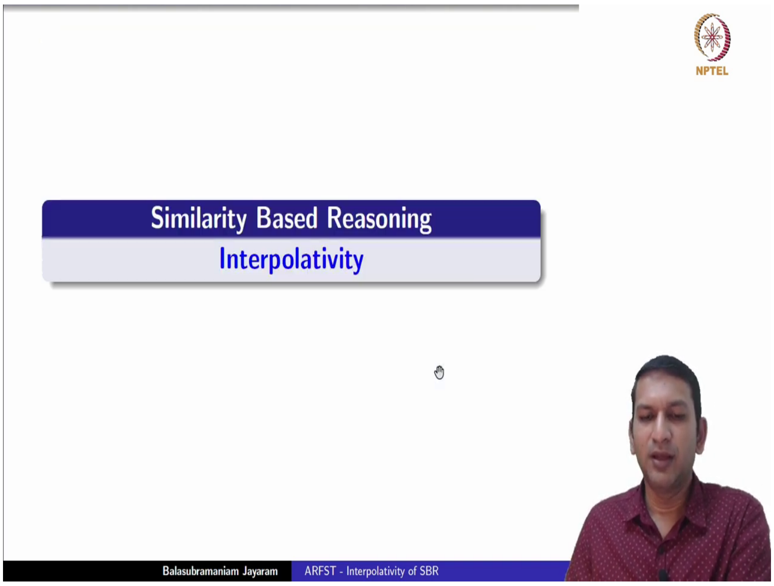


Now let us come to discussing interpolativity of similarity based reasoning.

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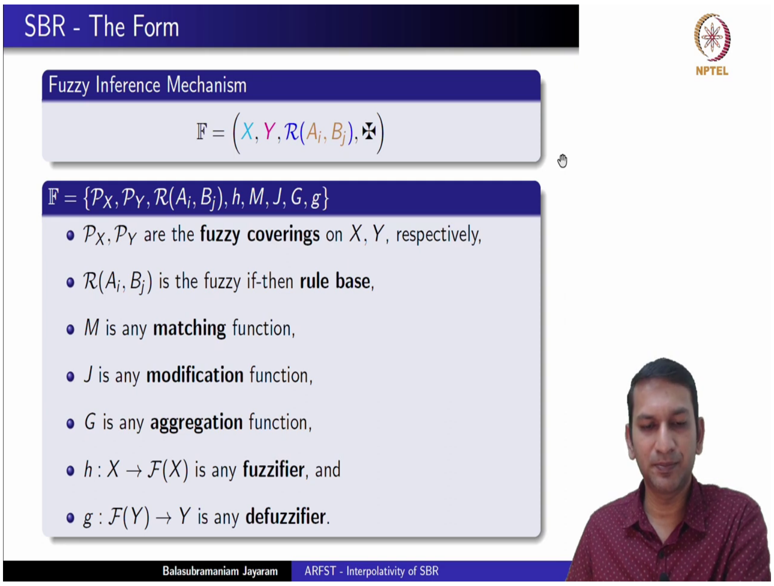


Note that the general form of a fuzzy inference scheme, we specify the input and output domains the rule base A i is the antecedents and B j are the consequence and the inference operators itself. When it comes to SBR we have these many degrees of freedom where P X and P Y are the fuzzy coverings on X and Y respectively, R of A i B j is the fuzzy if then rule base M is any matching function J is any modification function G is any aggregation function and h and g are the fuzzifiers and defuzzifiers if they are required.

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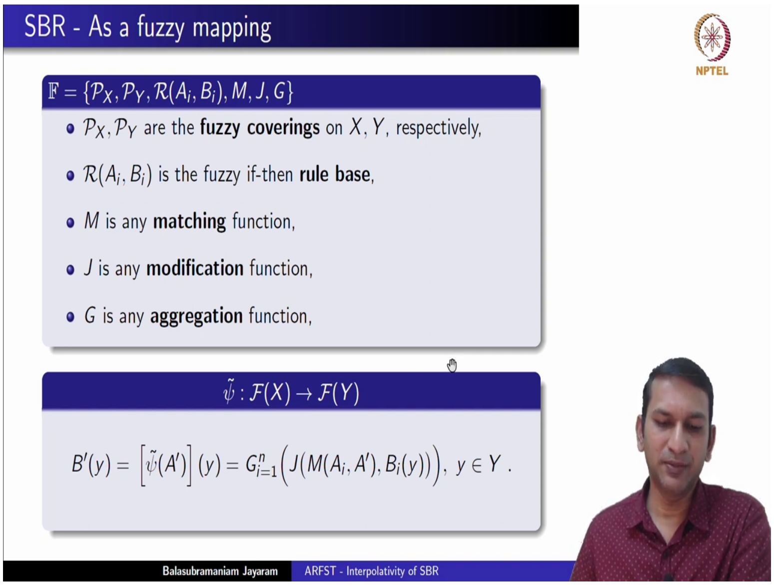


Now when we are looking at SBR as a fuzzy mapping, then h and g the fuzzifier and defuzzifier they do not play a role. So, we need to only consider these 5 factors. If we denote the fuzzy function that the fuzzy mapping that we derive out of specifying these factors these parameters in an SBR inference scheme, then the PSI tilde can be given like this of course, it is a mapping from F of X to F of Y it can be given like this.

We need to specify how an output fuzzy set will look like let us call that B dash B dash is a fuzzy set so that means it is membership value at every y needs to be specified. So, how does B dash of y look like it looks like this. So, A dash is a fuzzy set of X. So, psi tilde of A dash is the mapping that we get on F of Y it is the mapping the fuzzy set on Y; overall it looks like this given this A dash we match it with each of the antecedents of the room.

Take this similarity value use the modification function J to modify this B i the corresponding consequent and then finally, aggregate using G. So, this is the final form the formula of the inference that we obtain for a given image.

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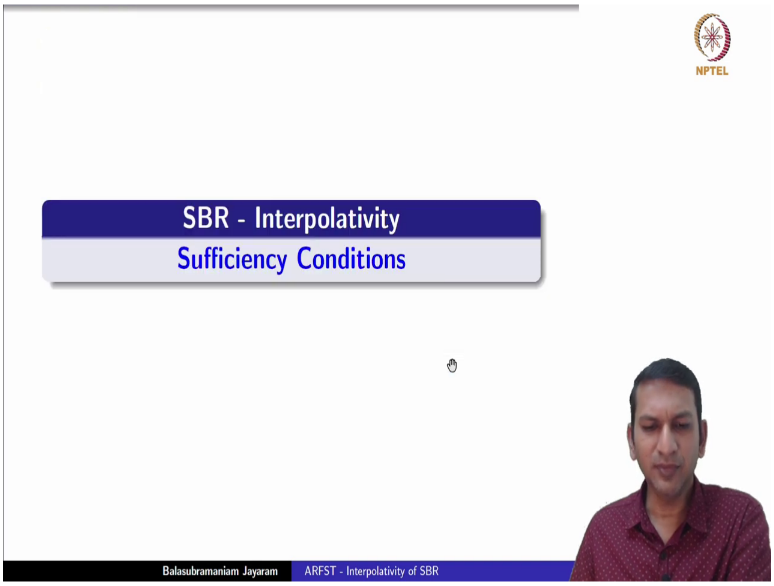


Let us begin by looking at some sufficiency conditions to ensure interpolativity in a SBR inference scheme.

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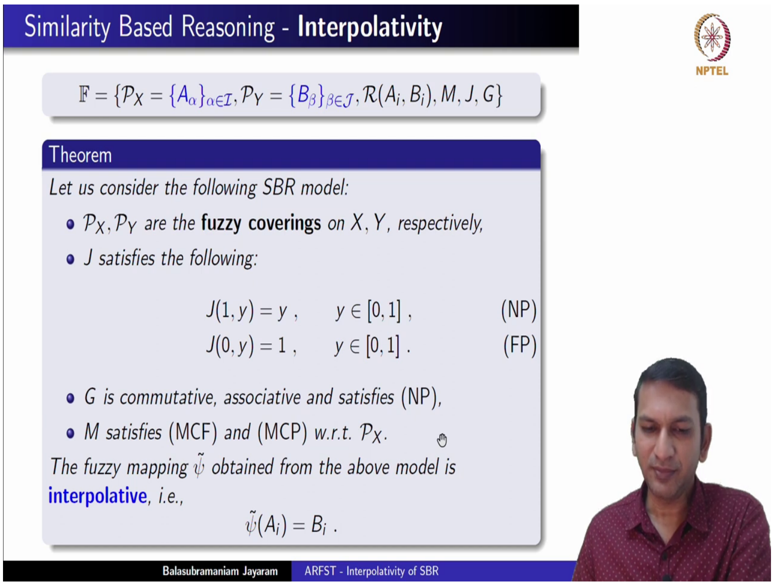


Let us fix the coverings to be actually specified in terms of A alphas where alpha runs over some index set of I and P Y again it is a fuzzy cover on y with respect to whether B betas are given. Where beta is running over index change the rule base is found by picking A i s from P X and B i s from this B beta M, J and G are the matching modification aggregation function.

Let us fix each of these and consider the following SBR model P X and P Y are the fuzzy coverings on x and y we do not insist anything more on it. But we insist that J actually satisfies the following condition, we know that J can be thought of as a binary fuzzy logic operation.

So, we want that J if J does satisfy neutrality property; that means, 1 is a definite element and 0 is in some sense the inverter. So, J of 0 comma y is 1 this also called the falsity principle in some logical context. So, we just denoted by FP, but what we want is J of 0 comma y is 1, if these 2 properties are satisfied by J and if G is a commutative associative operation that also satisfies NP.

So that means, 1 is both left and right neutral element and the matching function M is both consistent and also consistent with respect to the fuzzy covering P X that we have here. So, it satisfies MCF and also MCP with respect to P X, if we consider such a system the underlying degrees of freedom are chosen in such a way then the fuzzy mapping induced by the system psi tilde is actually interpolative.

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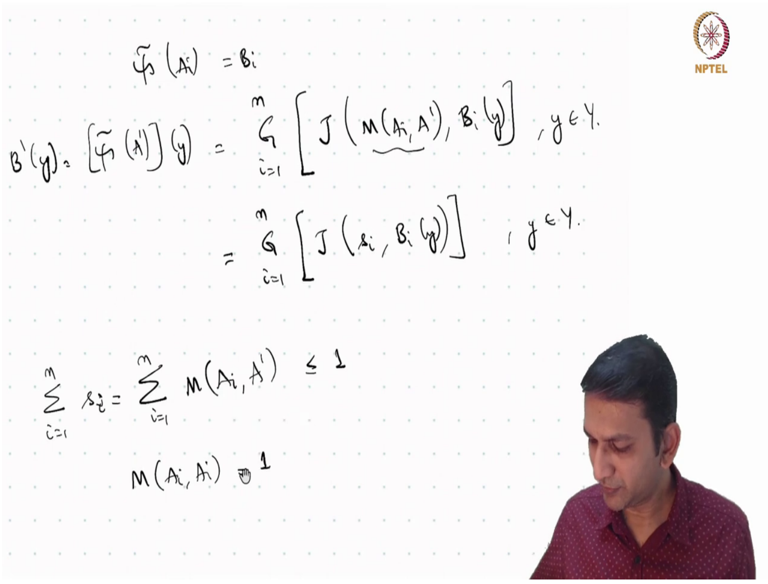


That means, what psi tilde of A i is actually equal to B i. Now how do we prove this? We want to prove that psi tilde of A i is equal to B i. Now what is psi tilde of A of any A dash at y this is what we wrote as B dash of y this is given as G i is equal to 1 to n the n rules J of M of A i comma A dash comma B i; note that, this is for an arbitrary y element of y.

This can also be further simplified like this G is the aggregation operator J of note that this is essentially the similarity over the s i B i of y. Now note that M satisfies MCP with respect to P of X this immediately implies sigma M A comma A dash is less than or equal to 1. Now this quantity essentially s i, so s i sigma s i is less than or equal to 1. However, note that M also satisfies MCF, which means M of A i comma A i is equal to 1, let us make use of all these properties.

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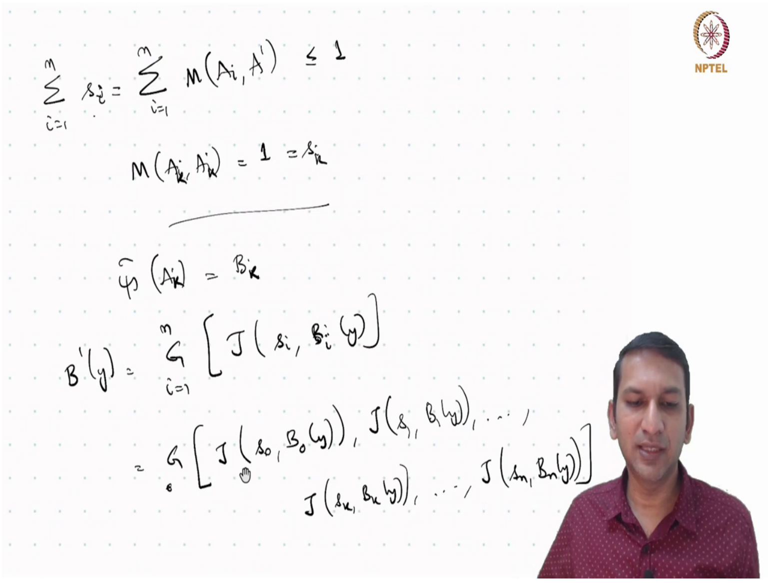

So, what we want to see is psi tilde of A i should be equal to B i. Now if we substitute it here B dash of particular y is actually equal to G i of i n J of s i comma B i of y for a particular y. But note that if you were to take this out when look at what is J of s 0 let us also fix this i to be k. So, it is (Refer Time: 20:14) J of s 1 comma B 1 of y. So, on going J of s k comma B k of y so on J of s n comma B n of y.

Now note here M of A i is 1 means s i is 1 now we have fixed k. So, M of A k, A k s k is 1; if s k is 1 rest of the s i s because M has MCP with respect to p the rest of the s i s will be 0.

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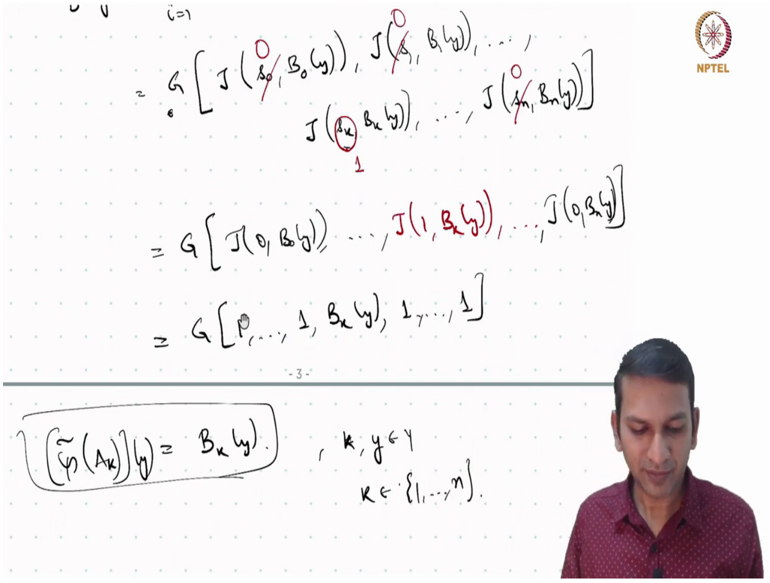

So, essentially what we get here is G of J of 0 comma B naught y comma. So, on only at s k we get the value 1. So, this is 0 this is 0 this is 0 only here you get 1. So, you get J of 1 common B k of y so on till J of 0 comma B comma. Now note that J of 0 comma y is 1 which means we have G of 1 comma 1 is here and J of 1 comma y is y.

So, this is B k of y and rest of them are again because they are all 0 is 1, finally we know that G is both associative and commutative and also satisfies NP which is J of 1 comma y is y you know that this turns out to be B K of y. So now what we took was psi tilde of A k of y is in fact B k of y.

So, now k was arbitrary y element of y was arbitrary k element of 1 to n was also arbitrary, which means we have shown that psi tilde of A i is equal to B i for any I. So, note that we have not put any conditions on the partition themselves it should only be fuzzy covering and on the modification and the aggregation function we have only asked for them to satisfy in some sense some kind of boundary conditions.

And of course, we put conditions on M such that it is consistent with respect to the fuzzy covering that we are considering.

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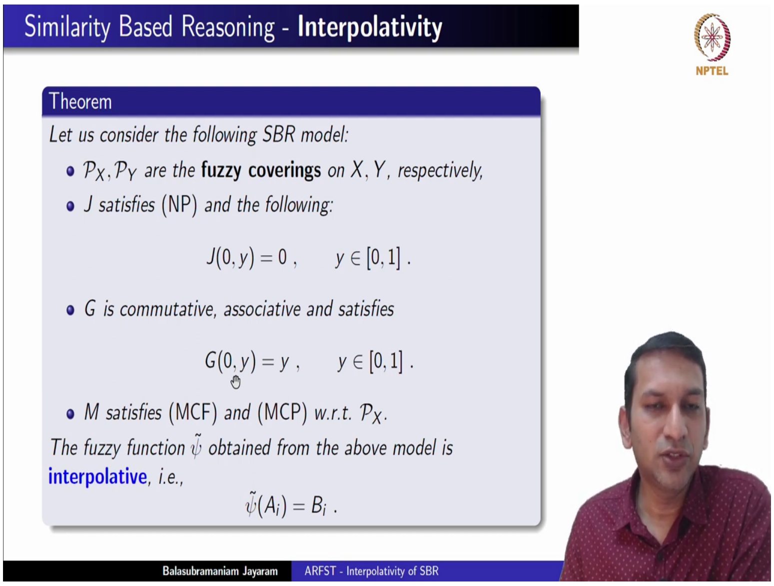


Now, a similar result another sufficiency condition can be given if we consider the following model, P X and P Y are fuzzy coverings on X and Y, J now satisfies NP that is 1 is a left neutral element and the following condition which is 0 y is in fact 0 instead of 1. So, more like a conjunction function earlier it looked more like an implication function and for G we consider a commutative associative operation such that G of 0 y is y.

So, essentially 0 becomes the neutral element here. So, more like a disjunction function and of course we insist that M satisfies both MCF that is consistent and also consistent with respect to the fuzzy covering P H that we have considered here. You can once again show that the fuzzy mapping that we get out of this SBR inference scheme psi tilde is in fact interpolative. Let us quickly work this out the proof follows almost along similar lines.

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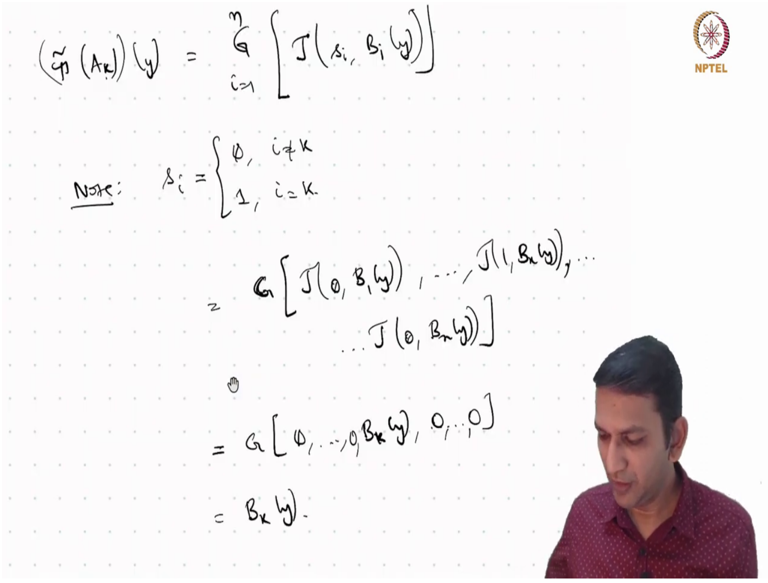


So, we are looking at psi tilde A k of y, now this is G i is equal to 1 to n J s i comma B i of y; note that from the properties on M we immediately see if you take a A k so s i is equal to 0 if i is not equal to k and s i is equal to 1 if i is equal to k. So now, if you use this then the above becomes G of J of once again 0 comma B i of y.

So, J of 1 comma B k of y rest of them will turn out to be J of 0 comma B n of y. Now let us look at the conditions on J of 0 comma y is 0; that means, G of 0 comma J of 1 comma B k y 1 is the left neutral element of J. So, this gives us B k of y and rest of them are again 0. Now we know that 0 is the left neutral element of G, which means we get B k of y.

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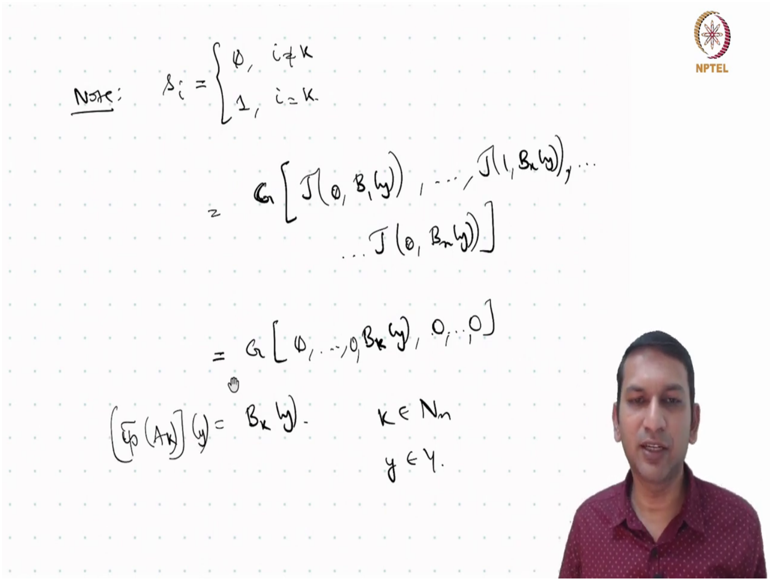


So, once again k element of N m was arbitrary y element of y was arbitrary which means for any A k that we take the output from this function psi tilde is in fact the corresponding consequent B k. Now these are some sufficient conditions where we specified conditions on G and J largely boundary conditions and of course we insisted on the M being consistent with respect to the fuzzy covering. That we are considering note that P X is the one that supplies the antecedents to the roots.

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Now are there some necessary and sufficient conditions well one such result is available recently.

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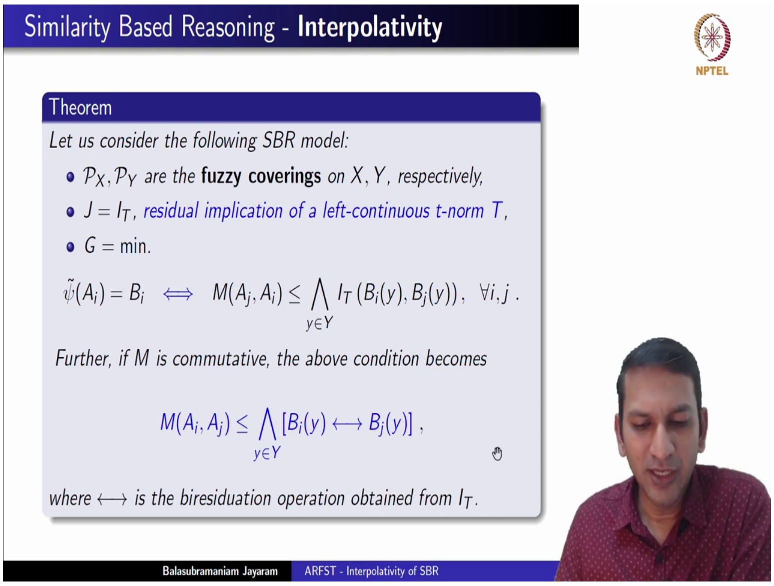


If we consider the following SBR model once again P X and P Y are fuzzy coverings on X, Y; J is a residual implication of left continuous t norm T. So, we consider J to be I T, G is min and there are practically no specific conditions on M; that means, we do not insist on MCF or MCP instead that condition is translated in a different way. We can show that if we consider such a SBR model then it will be interpolative if and only if the following inequality is actually valid.

What is this equality inequality say take any 2 i and j any pair of elements i and j, if they are matching values M of A j is less than or equal to this quantity on the right hand side then this system is in fact interpolative. Once again you will see the role played by residual implication; that means, this I T is actually coming from a residual lattice structure. The moment you pick your operations from residual lattice structure we are able to give much more stringent guarantees than what was available earlier.

Of course this is possible because of the rich the richness of the structure the myriad properties that the restated lattice has. In fact, if M is commutative for example if you consider the example M S it is not a commutative matching function, if M is commutative then the above condition actually looks like this.

Now it is been highlighted in blue because you might remember that it is similar such condition that we have seen was important to obtain interpolativity in an FRI scheme, because this is the by implication obtained from I T. We will have more to say in one of the upcoming lectures about how this equation is related to the interpolativity equation for an FRI; we will see that in one special lecture dealing with FRI s and SBR s fita and fati.

But for now the take home message is if you are choosing your operations from retroactive lattice structure, then often we are able to guarantee stronger results well.

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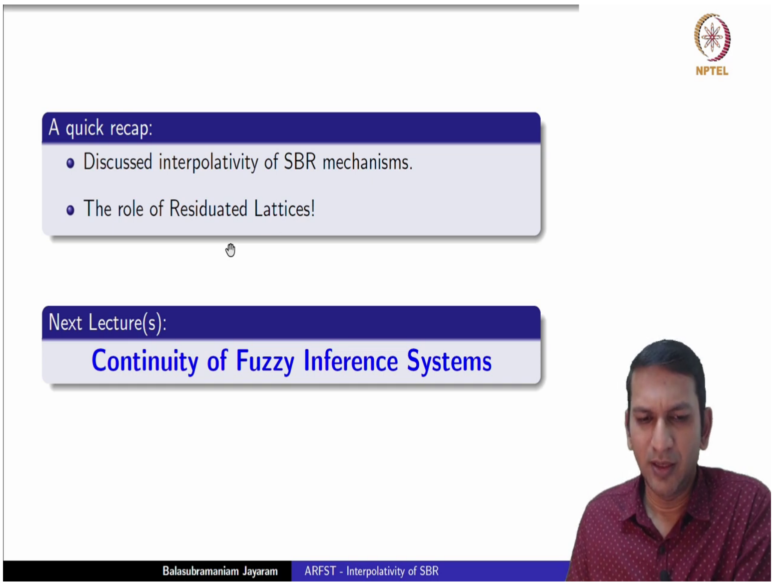


A quick recap in this lecture we discussed interpolativity of SBR mechanisms, once again we saw the role of residuated lattices. If we were discussing sufficiency conditions, yes sufficiency conditions only they are not necessary, but then we had leeway on the kind of operation we could use for J and G on matching function we wanted it to be consistent with respect to the input fuzzy cover from which we pick the antecedent.

However, if J were to come from the residual data structure if you consider it to be an implication, then we were able to give stronger results we were able to give a condition which was both necessary and sufficient. In the next week of lectures we will discuss continuity of fuzzy inference systems.

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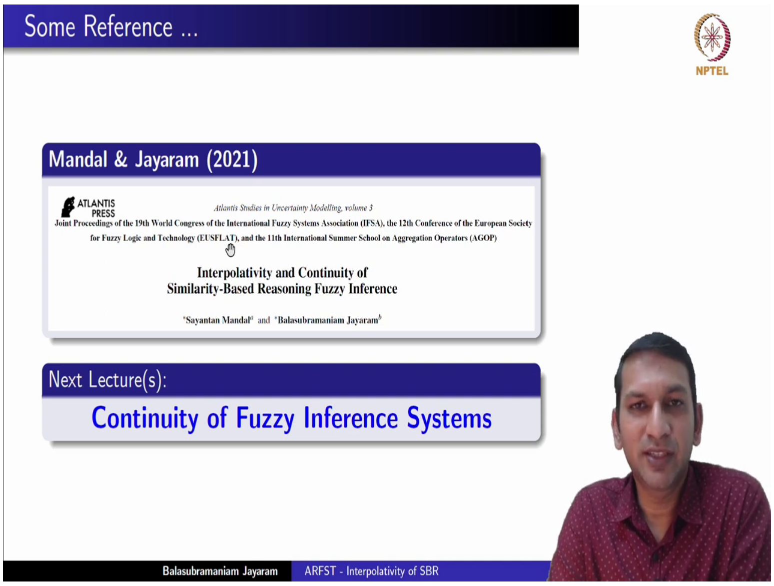


Many of these results that we have discussed in this lecture are relatively recent new it can be found in this work. We will meet in next week to discuss Continuity of Fuzzy Reference Systems. Glad you could join us for this lecture and hope to see you soon again in the next week of lectures.

Thank you everyone.