Approximate Reasoning using Fuzzy Set Theory
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Lecture - 42
Interpolativity of FRI - Multiple SISO Rules

Hello and welcome to the 4th of the lectures, in this week 8 of the course titled Approximate Reasoning using Fuzzy Set Theory. A course offered over the NPTEL platform. In this week, we have been discussing the Interpolativity of Fuzzy Inference Mechanisms. In particular, we are discussing the interpolativity of fuzzy relational inference schemes.

We began by discussing the interpolativity of FRIs in the presence of a single Input Single Output rule. We saw that a couple of functional inequalities played a role in ensuring interpolativity.

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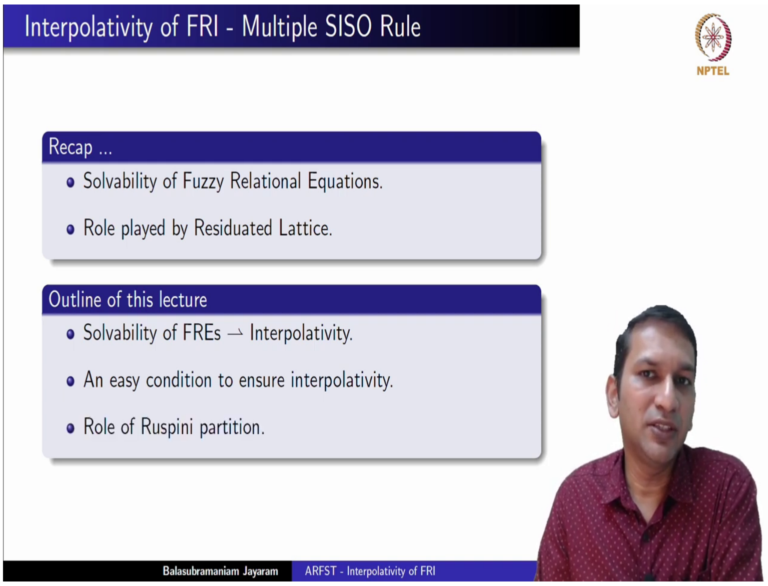

Of course, a single rule cannot capture the working of an entire system. We need multiple rules. So, of course, we need to discuss the interpolativity in the case of multiple SISO rules. The case of single SISO rule was only a stepping stone towards this. And towards our quest in exploring the interpolativity of an FRI in the presence of multiple SISO rules, in the last lecture, we have discussed the solvability of fuzzy relational equations.

We have once again seen the important role played by the residuated lattice structure that is available to us if you relate a left continuous T norm along to its residual implication. In this lecture, we will see how the solvability of fuzzy relational equations is indeed related and connected to the interpolativity of the underlying FRI mechanisms. We will also see an easy to ensure condition which will give us interpretability and we will see the role of Ruspini partition in this context.

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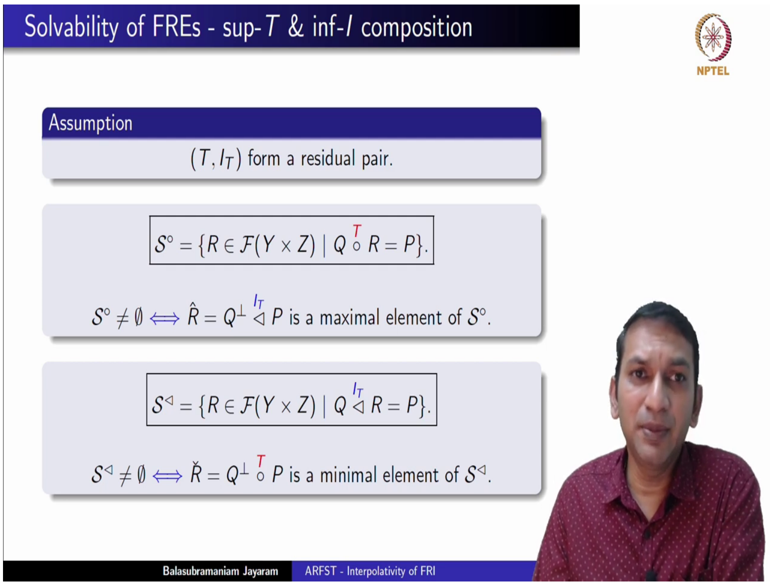

In the last lecture, we have seen that if you consider the underlying operations of either the sup-T or inf-I composition to come from a residuated lattice structure. That means, the pair T norm T and the corresponding R implication I T, if they form a residual pair essentially; that means, T is left continuous. Then, these are the results that we saw can be proved.

If given Q 1, P we want an R such that Q sup-T composed with R is equal to P, this solution space is non-empty if and only if, the relation which we denote by R hat or R cap which is given as Q transpose in phi T composed with P is not only an element of the space, but also the maximal element of that solution space.

Similarly, if we consider the solution space of the equation Q in phi T composed with R is equal to P, where Q and P are fixed, and we are looking for an R. This space is non-empty if and only if, the relation which we denote by R check which is given as Q transpose sup-T composed with P is not only an not only an element of S of this solution space.

That means, not only does it satisfy the solution, but it is also the minimal solution. It is also a minimal solution. There can be many other, but this is a minimal solution; that means, there is nothing below this. Now, the question is how is this related to interpolativity of an FRI, specifically CRI which uses sup-T composition and BKS which uses inf-I composition.

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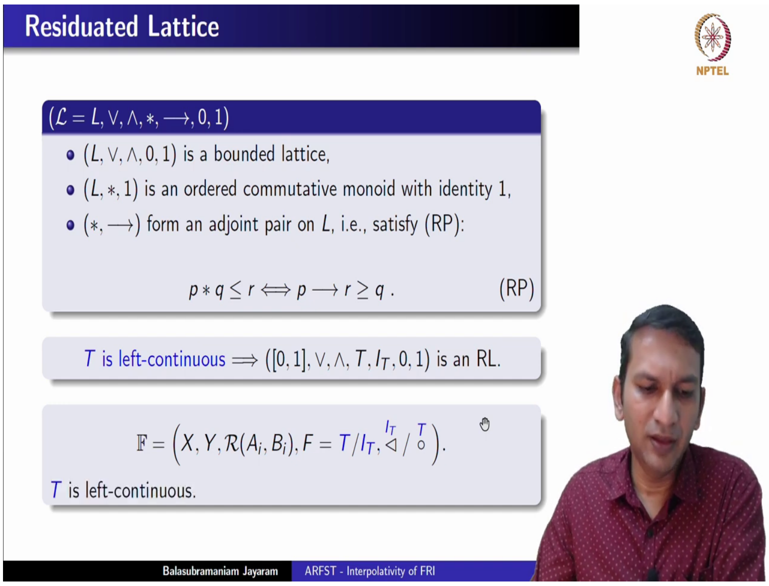

Note that we are in the setting of residuated lattices. If T is left continuous then we know on the unit interval, this entire structure becomes a residuated lattice. And so, we are considering FRIs, where if F is given by T, then we are looking at the corresponding composition to come from, either BKS with I T or CRI with T.

And similarly if F is given by the corresponding residual I T, then once again one of these two compositions is what we are looking at. Of course, T is left continuous.

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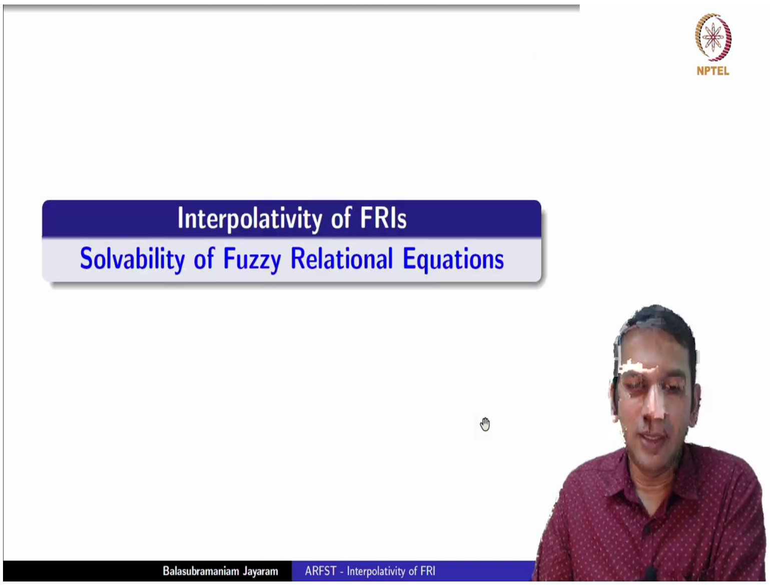

Now, let us look at how interpolativity of FRIs is in fact, ensured by solvability of these fuzzy relational equations.

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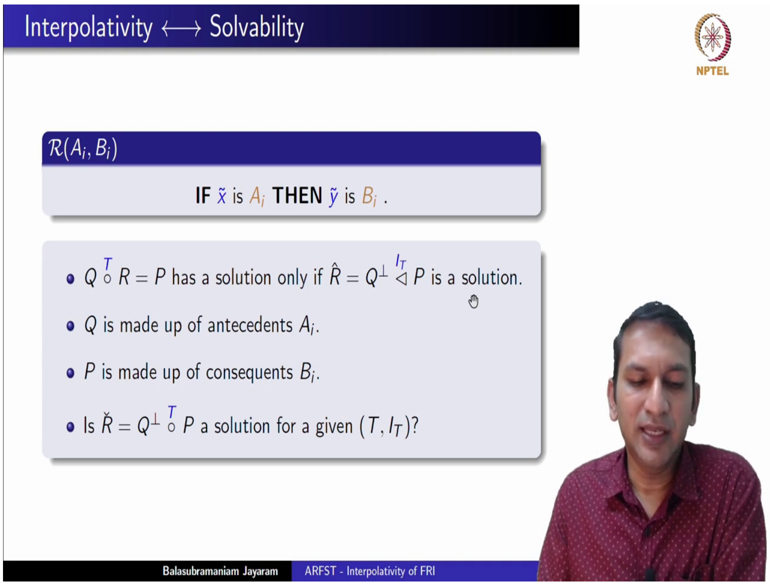

What do we have? We have the set of if-then rules single input single output rules n of them let us say, many of them. We are looking at the solve solvability of the equation Q sup-T composed with R is equal to P, where Q and R, Q and P are given. And we have seen that this is this has a solution only if R cap is a solution.

Now, what is Q for us? Q essentially is made up of antecedents A i and P is made up of consequents B i. So, in the setting of interpolativity, the Q matrix is made up of the antecedents and the P matrix is made up of the consequents. And now we are looking for an R, such that Q circle R is actually equal to P.

Now, the essential question the crucial question that we need to answer here is, if R check is a solution of this system provided T and I T are actually assumed to be coming from the residuated lattice structure.

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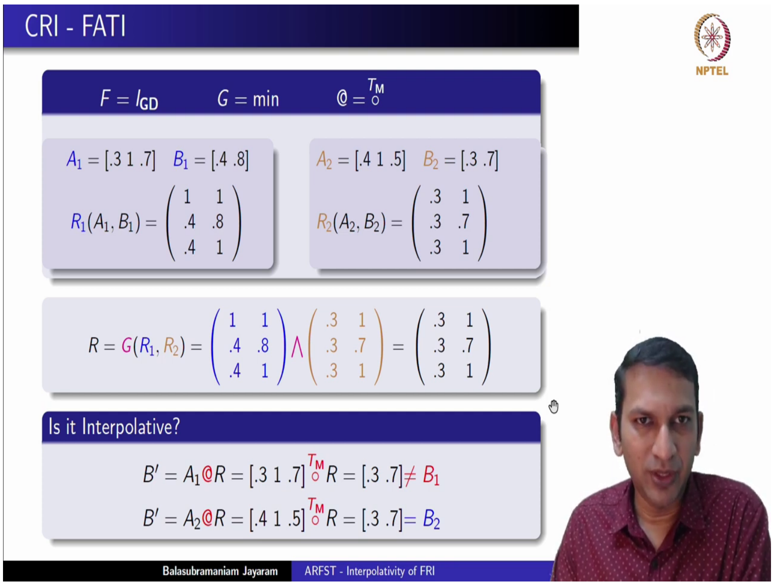

Let us revisit one of the examples that we have seen already in previous lectures. Note here the composition is sup-min and we know that minimum is a left continuous T norm in fact, it is a continuous T norm, so it is a left continuous T norm. And the corresponding R implication is the Godel implication.

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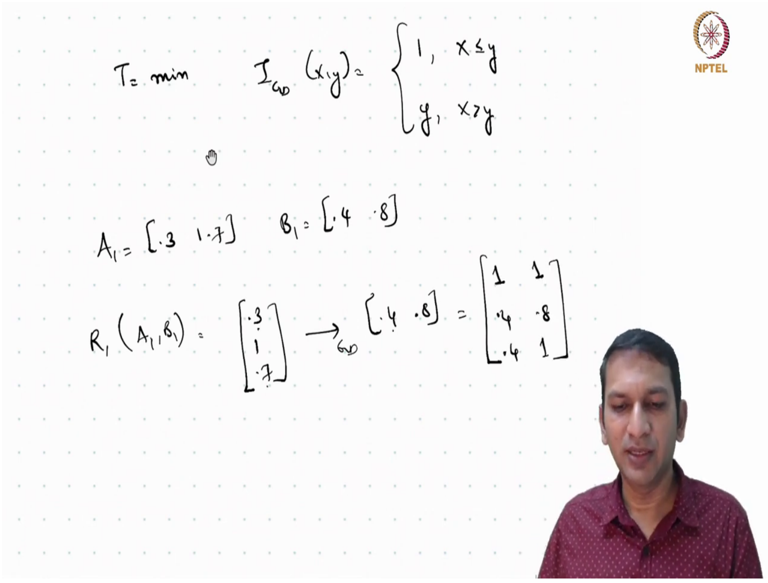

So, we have T to be min and the I Godel implication is given as follows. 1 if x is less than or equal to y, and y if x is greater than y. Now, the relations R 1 and R 2 are obtained from A 1, and A 1, B 1 and A 2, B 2. Let us look at this. So, A 1 is 0.3, 1, 0.7 and B 1 is 0.4, 0.8. We know that R 1 of A 1, B 1 is obtained as we write the transpose of A 1 and use the Godel implication 0.4, 0.8.

So, if we take 0.3 and 0.4, 0.3 is less than or equal to 0.4, so this is 1 here. 0.3 is less than or equal to 0.8, this is 1 here. 1 implies, 1 is left neutral element of Godel implications, so it is 0.4, 0.8. 0.7 is greater than 0.4, so this remains 0.4. 0.7 is smaller than 0.8, so this is 1. So, this is how we have obtained the corresponding relation and that is what you see here.

Similarly, we have obtained the relation R 2 using the Godel implication and the antecedent A 2 and the consequent B 2. Now, what do we do next? We are in the realm of FATI, the inference strategy that we are using is first aggregate then infer. So, we are taking these two relations and aggregating them using min.

So, these two relations aggregating them using min, it is a component wise aggregation, so it is clear to see that this is the final relation we get. Now, the question is, is it interpolating? Well, we have seen that if you compose A 1 sup-min composition because we have fixed this as the minimum T norm with this relation R, we get 0.3 0.7 which is not actually equal to B 1.

However, we have seen that with A 2 we are getting the corresponding B 2. But what we want is that for every input A i we should get the corresponding and consequent B i.

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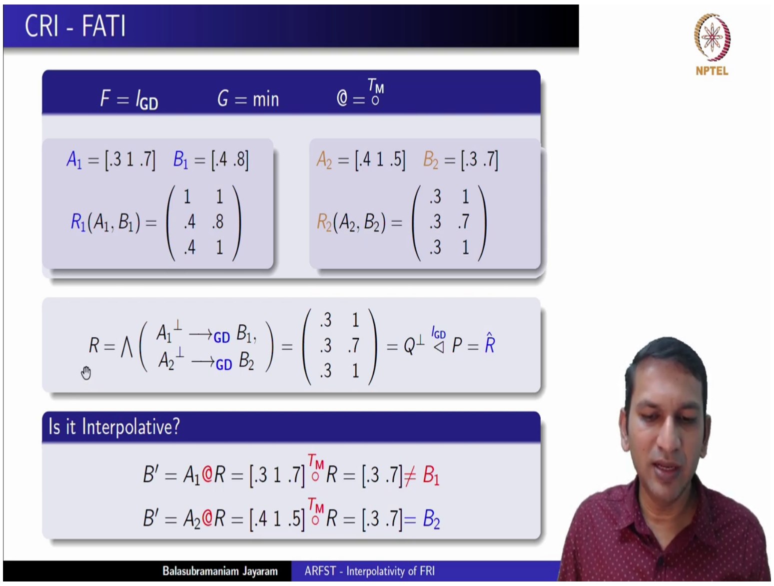

Now, in fact, if you look at this relation, what we have here is essentially R cap.

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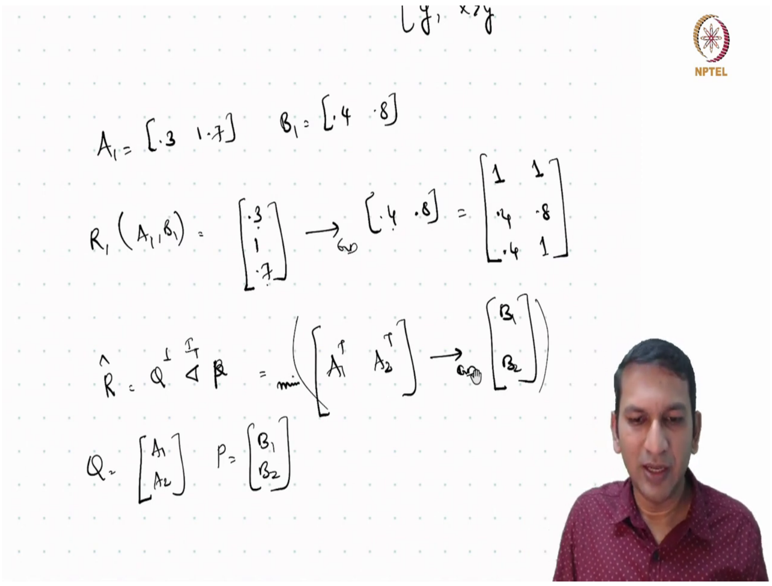

Because what is it, what is R cap? R cap is Q composed I T of R. And what is Q? If you look at it Q is actually A 1, A 2. P is B 1, B 2. So, Q transpose is nothing, but A 1 transpose A 2 transpose. And we are using the Godel implication here B 1 and B 2 and min of it. So, essentially what we have here, what the relation that we have actually got as R is in fact, R cap. Notice that is how we have got this and we are taking the minimum component wise.

So, in essence what we are saying is, not only is this R not a relation, but this R is in fact, R cap. Which means according to the solvability of FRIs for this given system, if R cap is not a solution, clearly it is not going to be the maximal solution. And that means, the solution space for this system is empty. That means, given these antecedents and consequents, we will not be able to obtain an R, such that this system is interpolative.

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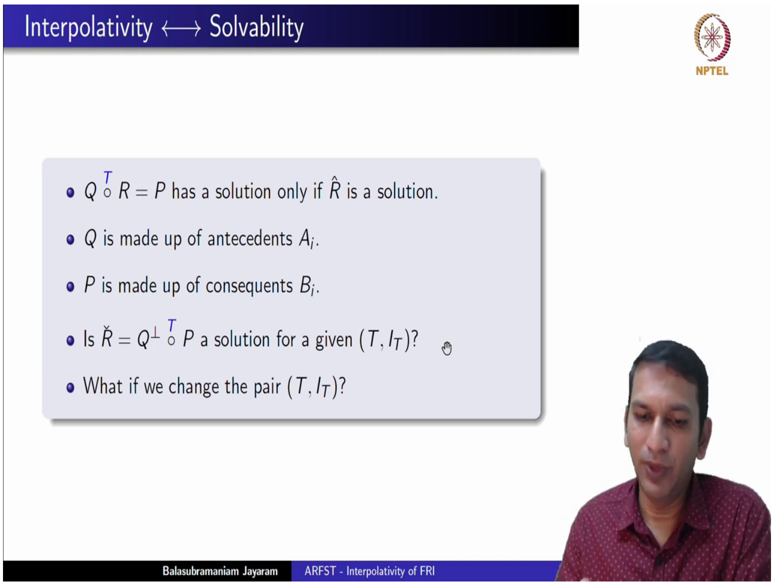

A quick recap so, what we have is Q made up of antecedents, P made up of consequents and we are looking at whether R cap is in fact, a solution for the given system. Now, for min and Godel, we found that it is not interpolative. What if we change the T norm and the corresponding residual implication? Would it work? Well, let us look at that.

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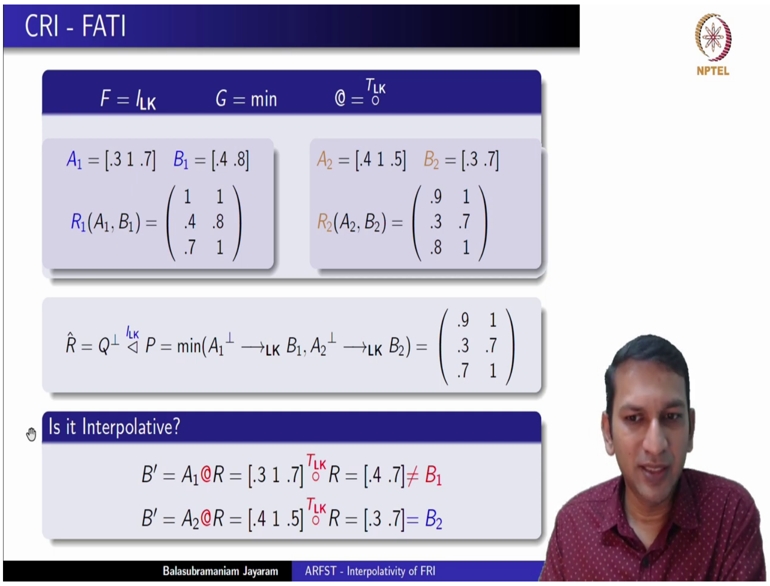

So, let us look at taking another T norm and the corresponding implication.

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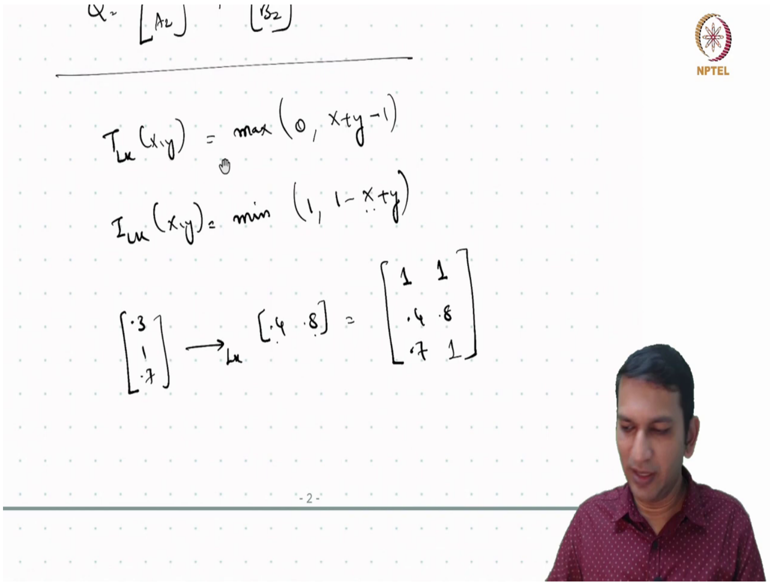

For that, for the sup-T composition, let us consider the Lukasiewicz T norm which is like this, is essentially we are taking by adding and then removing minus 1. So, this is max of 0 x plus y minus 1. And for F, we are considering the corresponding residual implication which is the Lukasiewicz implication. This is given as minimum of 1 comma 1 minus x plus 1.

So, now, for the same A 1, B 1, this is the relation that we will get. Let us quickly work this out 0.3, 1, 0.7; 0.4, 0.8. So, my R 1 is 0.3 implies 0.4. We know that Lukasiewicz implication has ordering property 0.3 is less than or equal to 0.4, so that means, this is 1. 0.3 less than 0.8, so this is 1.

Now, 1 is left neutral element, so that is 0.4, 0.8; 0.7 implies 0.4 when we put here 1 minus 0.7 plus 0.4, so it is 0.3 plus 0.4, so this will be 0.7. 0.7 and 0.8, 0.7 is less than or equal to 0.8, so this is 1. So, this is essentially the relation R 1 that we are going to get and that is what we see here. Similarly, we get R 2.

But this time we are wise we know that this is in fact, R cap itself because we are considering the corresponding R implication. And in the case of FATI what we would get is taking these two relations and then taking the min of it is in fact, taking the Q transpose of it and taking the inf I LK composition with P.

And the relation if you actually take the component wise minimum of these two R, and R 1 and R 2, is what we get. So, what are we trying to do? Keeping the same system, but only changing the pair of operations from min and Godel to the Lukasiewicz T norm and the Lukasiewicz implication. Is it interpolative? Let us look at it.

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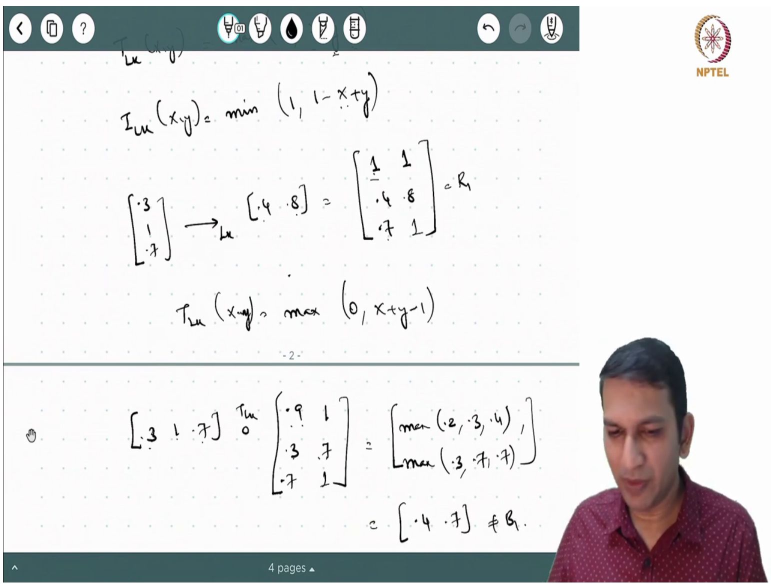

So, we have this R here 0.3, 1, 0.7 composed with R. R that we are going construct as 0.9 1, 0.3 0.7 and 0.7 1. So, note that the Lukasiewicz T norm is max of 0 comma x plus y minus 1. So, now, you look at this what we have is 0.3 and 0.9. So, add them up and you know 1.

So, it is maximum of 0.2 comma 1, 0.3 is 0.3, 0.7 0.7 is 0.4, same row, with the second column its max of 0.3 comma 1 for any T norm 1 as an identity element, so it remains 0.3. 1 comma 0.7 is 0.7 and 0.7 comma 1 is also 0.7. So, from here if we get this what we obtain is 0.4 comma 0.7 which clearly is not equal to B 1. And that is what we have there.

Now, with A 1 not giving us B 1, immediately it is clear it is not interpolative. If we do this for A 2 of course, we seem to be getting the corresponding B 2. So, the point here is even when we change the pair T LK which seem to be going nowhere.

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So, now, the question is, how is the solvability of FREs is really helping us in the interpolativity? To discuss this let us go from existence of a solution to determination of the solution.

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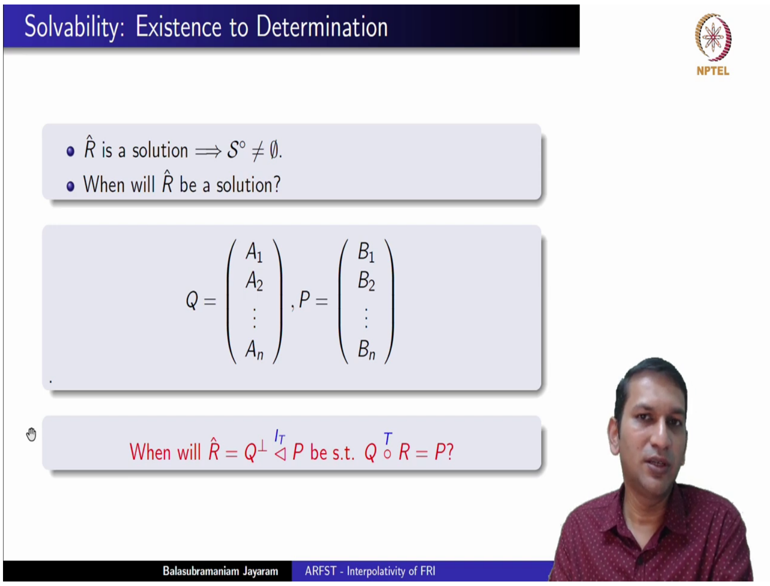

We know that if R cap is a solution, then for the sup-T composition that we are considering, the solution space is non-empty. But the question now is when will R cap be a solution. We tried different pairs, it did not seem to work, but for this result to be useful we need to know when R cap can be a solution.

Note that, Q consists of this antecedents, P consists of this consequents. So, the question now comes down to when will R cap which is given as Q transpose in phi T composed with P, be a solution for Q circle T R is equal to P.

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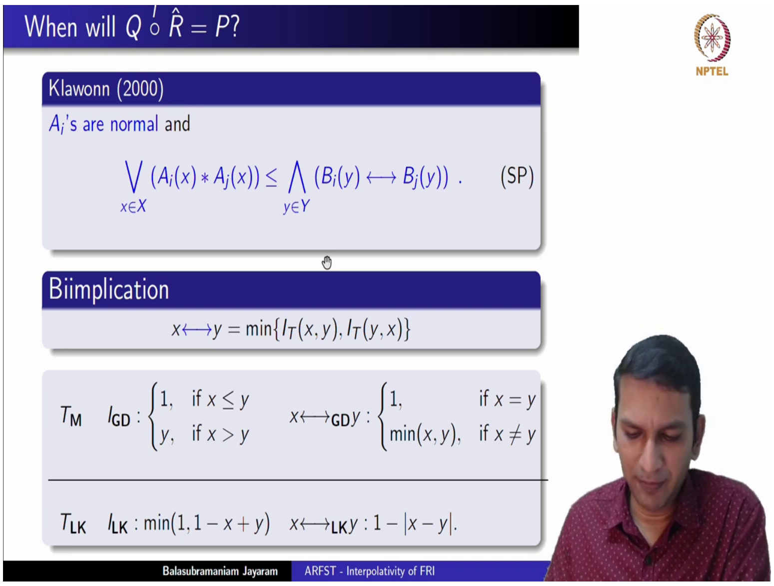

Well the first such result on this was presented by Klawonn, way back in 2000, in which he said if the antecedents A i's are normal, that means, there exists a point on the underlying space, the input space x such that a of x is 1, for every i. Of course, for different x perhaps with respect to A i.

If the antecedents are normal and if this particular inequality is held, now what does it say? When you take the supremum over x element of x, A i of x star A j of x for a pair of i comma j this should be less than or equal to the quantity on the right hand side. Now, what is this quantity on the right hand side? The infimum over the B i implications, B i implication applied for the corresponding membership values taken by y on the consequents.

What is this operation B i implication? You may recall we have seen this, this is nothing but minimum of any implication, but in this case since we are in the residuated lattice setting, it is minimum of x implies y and y implies x. Do recall that if you consider the T norm to be a minimum, we get the Godel to be the corresponding R implication, and the B i implication obtained for Godel will look like this, if x is equal to y it is 1, otherwise it is min of x y.

In the case of Lukasiewicz, we see that what we obtain is 1 minus mod x minus y. You may recall this particular function as a relation, as an equivalence relation from the lectures on similarity classes. Well, so this is the first result that appeared which say that if A i is a normal and if we if they satisfy this inequality for every pair of antecedents and the corresponding consequents i j, then it will ensure interpolativity of the corresponding system.

Now, you would notice that, we have tagged it as SP, because this is also called the semi-partition inequality. We will perhaps have more, we will see this in more detail in the lectures to come. But this is the first such result that gave us a sufficiency condition to ensure interpolativity.

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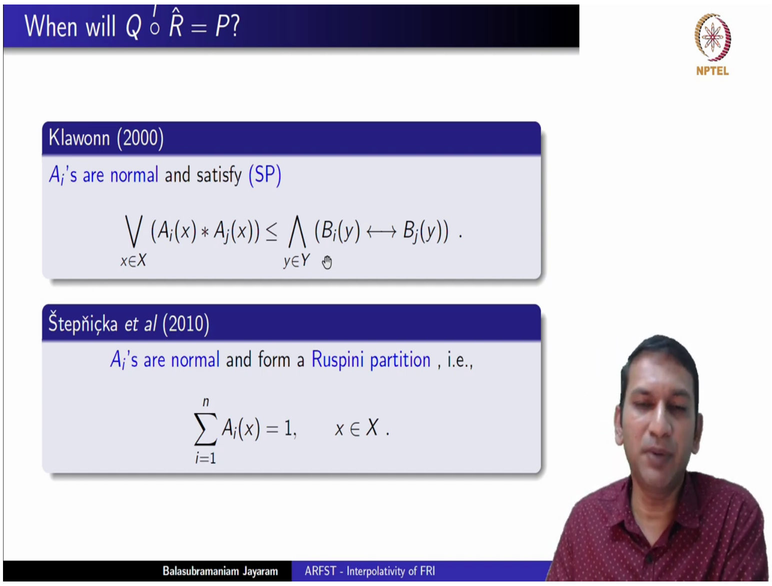

However, you will see that this is not so easy to immediately determine while being very useful. Later on almost a decade later, Stepnicka and others, they came up with a slightly modified result which say that if the antecedents are normal and also form a Ruspini partition, then we can ensure interpolativity. Now, this is a very simple kind of a sufficiency condition. All we want are that the antecedents are normal and satisfy the Ruspini partition condition.

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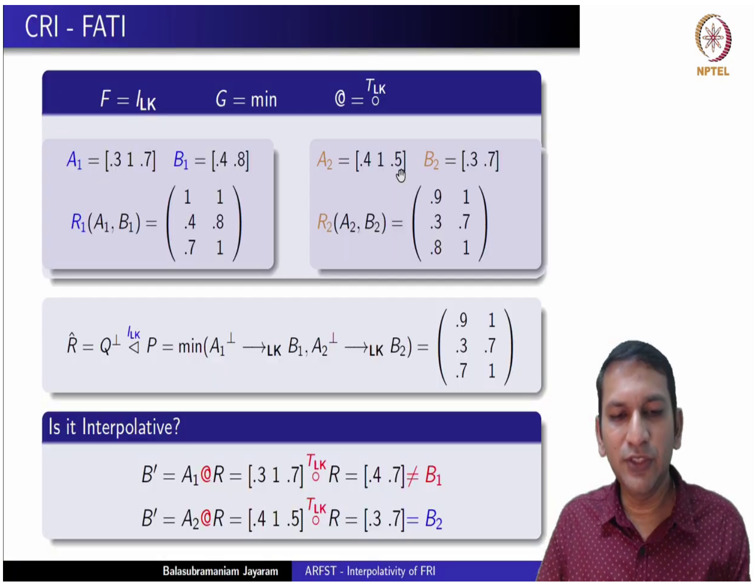

Let us look at this example that we have considered before. If you look at A 1 and A 2, they are normal. However, they are not in fact, they do not form Ruspini partition.

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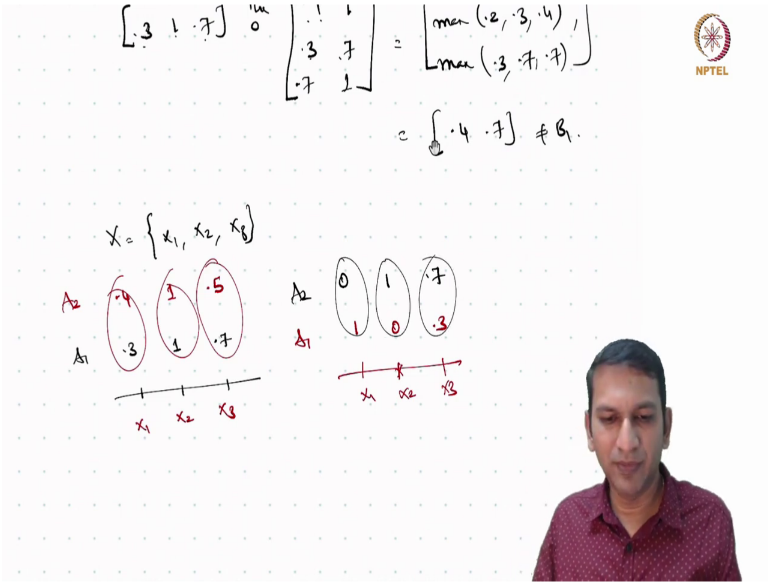

For instance, consider that x actually consists of x 1, x 2 and x 3. So, what we have is these 3 points. At this point A 1 is 0.3, 1 and 0.7. And if we consider A 2, it is given as 0.4, 1 and 0.5. Clearly, these do not add up to 1, these add up more than 1 and definitely these two add up more than 1. So, we consider this as x 1, x 2, x 3. So, clearly here the sufficiency conditions are not satisfied.

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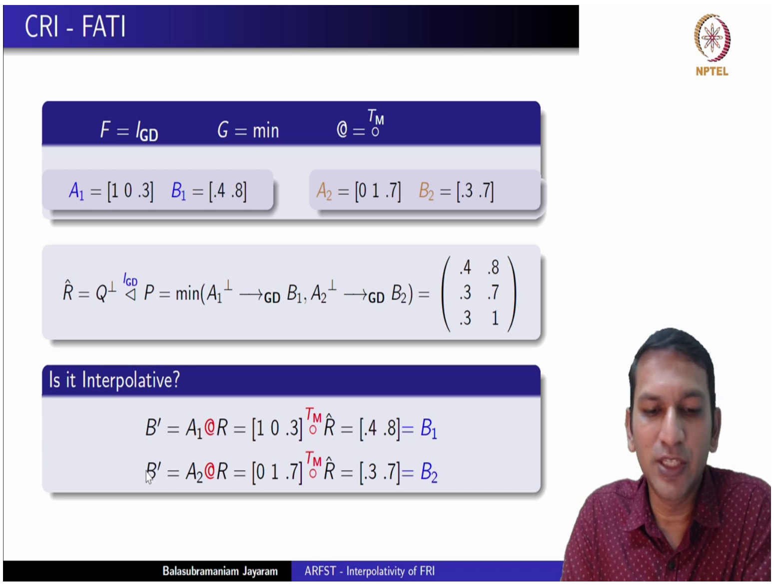

So, let us take an example where the sufficiency conditions are satisfied. So, now, what are we doing? We are taking the same 3 points x 1, x 2, x 3. Now, A 1 is given as follows this 1, 0, 0.3 and A 2 is 0, 1 and 0.7. So, you see here this is 1, this is 1, this is 1, so they add up to 1.

We have not changed the consequents because the condition is only on the antecedents and not on the consequents. We consider the same system; that means, sup-min composition and F used to relate the antecedents to the consequents to get the relation is the corresponding R implication of min which is the Godel implication. And let us look at R cap.

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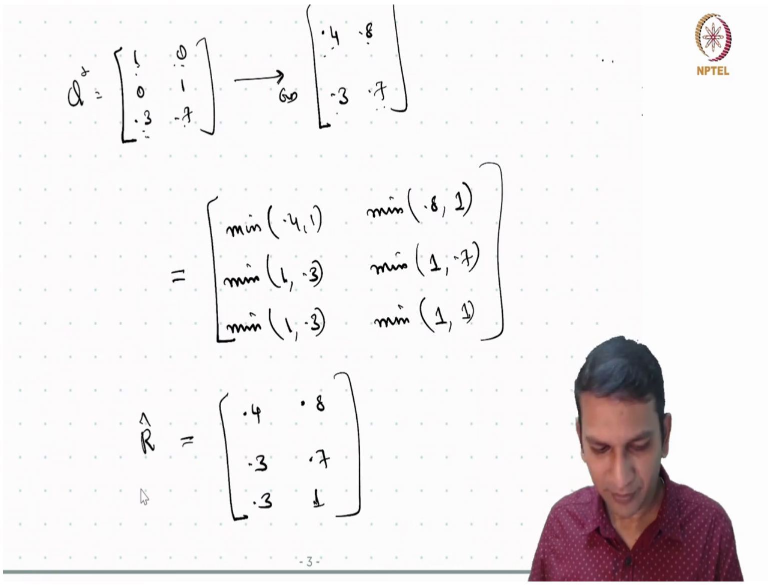

So, we know that what we need to do is actually take Q transpose which is 1, 0, 0.3, 0, 1, 0.7. Use the Godel implication with P which is 0.4, 0.8, 0.3 and 0.7. So, now, we can directly do calculation. Let us calculate this. We will take 1 against 0.4. Note that we are using an implication.

1 is the neutral element for Godel implication, so it is 1 comma 0.4. Let us calculate this. So, 1 implies 0.4 with minimum of 0 implies 0.3 is minimum of 0.4 comma 1; 1, 0 with 0.8, 0.7 is minimum of 1 implies 0.8 is 0.8, 0 implies 0.7 is 1. Now, second row with this matrix here.

So, minimum of 0 implies 0.4 is 1, 1 implies 0.3 is 0.3, 0 implies 0.8 is 1 again, 1 implies 0.7 is 0.7 with respect to Godel. Finally, minimum of 0.3 implies 0.4 is 1 because Godel implication has the ordering property 0.7 implies 0.3, when x is greater than y, it is y, so it is

0.3. Finally, 0.3 implies 0.8 again due to ordering property it is 1 and 0.7 implies 0.7 is 1. So, this is the matrix that we get.

So, essentially the R cap that we are going to get is 0.4, 0.3, 0.3, 0.8, 0.7, 1. You see that this is exactly the matrix that we have got. So, in one shot, we have obtained R cap which is essentially the relation that we would obtain if you aggregate the individual relations. Now, question is it interpolative. Now, according to the condition we know that A i's A 1 and A 2 are both normal and they form a Ruspini partition. Let us look at this.

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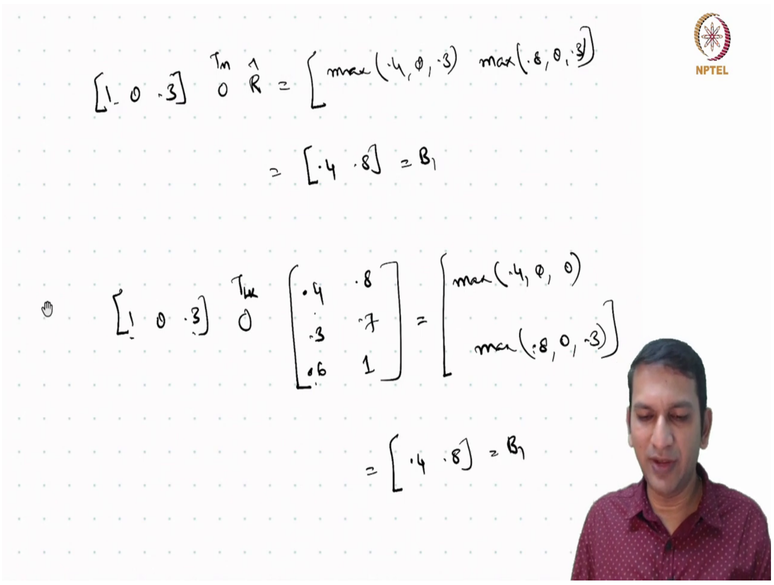

So, to this R cap we need to give A 1, so A 1 is 1, 0, 0.3. And note that sup-T m of R cap. So, if we take this and then compose it with this matrix here, what you get is max of 1 and 0.4 is 0.4, 0 and 0.3 is 0, 0.3 and 0.3 is 0.3. And in the second component 1 and 0.8 is 0.8, 0 and 0.7 is 0, 1 and 0.3 is 0.3. So, when we take the max it is essentially 0.4, 0.3 which is our B 1.

So, when you see here we have obtained for A 1 when we compose it with this R cap, we actually obtain it is B 1. A 1 was the difficult antecedent, always we will not obtain the corresponding consequent. So, we seem to have obtained a first success here. But let us also check for A 2. In fact, if we check once again we will see that we will get 0.3, 0.7 which is B 2. So, for this system, it seems to be interpolative. But what if we change the pair T and I T.

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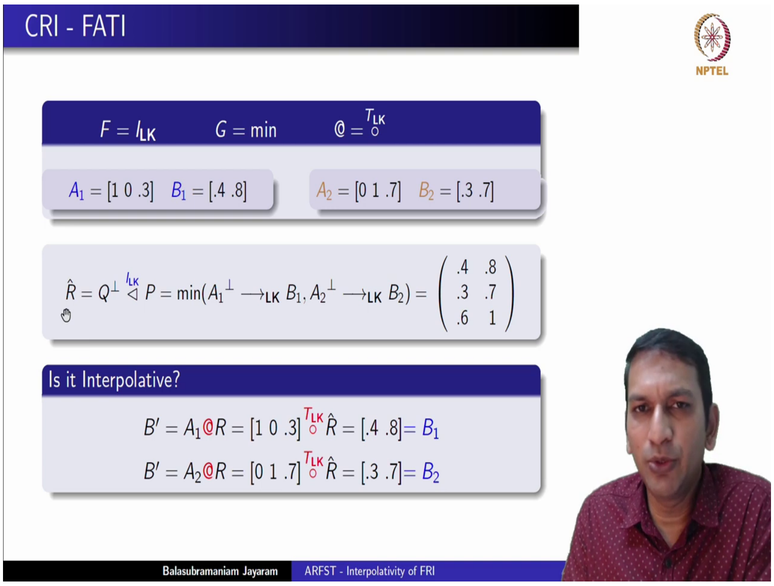

Let us try that let us keep the A i's same, but let us change the T norm to be the Lukasiewicz T norm, and of course, the corresponding implication to be the Lukasiewicz implication. Now, once again if we do the math, what we should get is in fact, this matrix. So, note that we are using essentially the Lukasiewicz implication here. Aand this is the R cap that you would get. Is it interpolative? Let us check.

So, that means, we want to give A 1 to this. So, we have 1, 0, 0.3. Now, remember we have to use the Lukasiewicz T norm and the R is given like this 0.4, 0.3, 0.6, 0.8, 0.7 and then 1. So, now it is max of 1 and 0.4 is 0.4, 0 and 0.3 is 0, 0.3 and 0.6. Now, it is Lukasiewicz T norm which is max of 0 comma x plus y minus 1. If it is 0.3 and 0.6 it is 0.9 which is less than 1, so it is 0. And the second component will be 1 and 0.8 is max of 0.8 comma 0 and 0.7 is 0, 0.3 and 1 is 0.3.

So, the output that we get is 0.4, 0.8 which we seen is in fact, equal to B 1. So, once again success with A 1 and also success with A 2. So, we see here, the condition, the sufficiency condition that we have that the antecedent should be normal and form a Ruspini partition, seems to be sufficient no matter which pair of residual implication of the corresponding T norm that we consider.

So, that in itself shows the reach and the potency of the sufficiency condition that we have got. Note that to ask for normality of antecedents is normally present even in practical applications and all we want is the Ruspini partition. We know that, typically, we have a fuzzy covering, but if it also forms a Ruspini partition, then we are easily able to ensure interpolativity.

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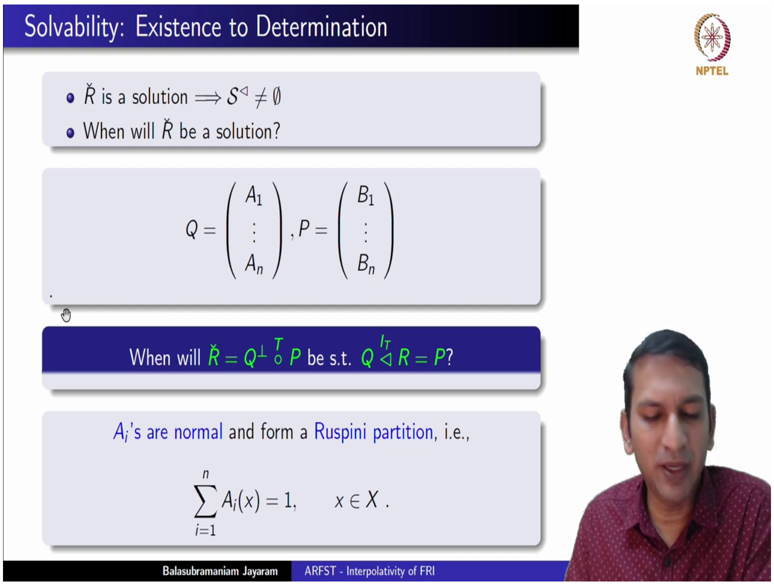

Well, now this was the case with CRI when we use sup-T composition. Now, what about the corresponding BKS inference scheme? There we know that if R check is a solution then the solution space where we use the inf- I composition is non-empty. Once again we have the same question, when will this R check be a solution where Q and P are given as antecedents and or found based on antecedents and consequents?

This is the question that we have. Once again the same sufficiency condition seems to hold. That means, the antecedents should be normal and form a Ruspini partition. Is this really true? Let us check this out with one simple example.

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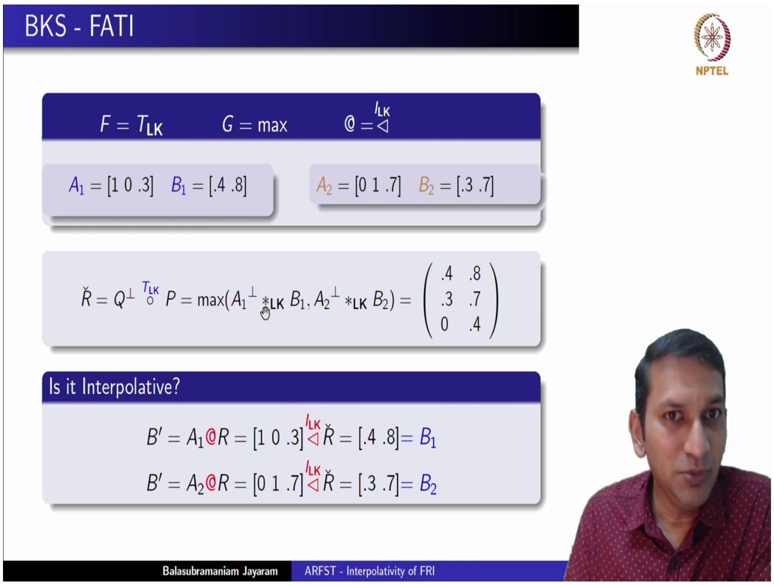

Note that we are considering the same set of rules the antecedents and consequents are same, but now for the composition we are using BKS with Lukasiewicz implication and. So, in this case the F becomes the corresponding Lukasiewicz T norm. So, now let us work out the math. Note here the R check. So, now, we are actually taking the transpose, I am using sup-star composition.

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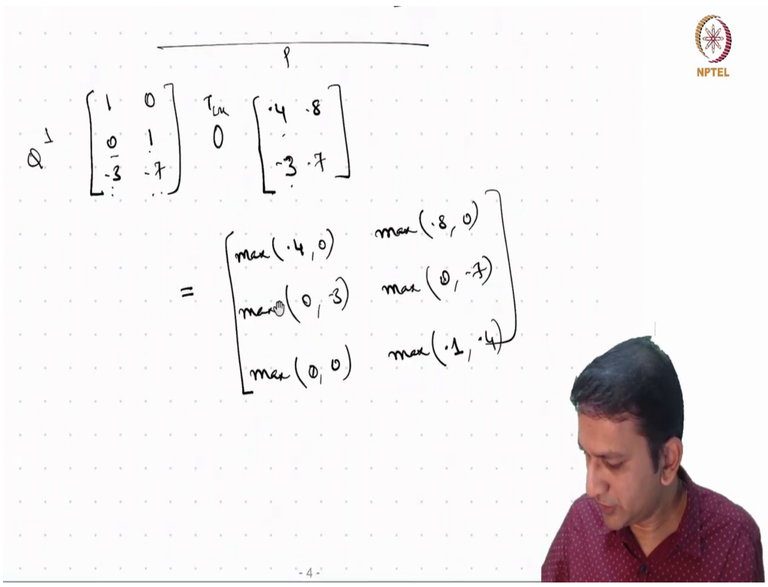

So, we have A 1 which is 1, 0, 0.3. A 2 which is 0, 1, 0.7; it is a and we are going to use this with the corresponding B 1 B 2 which is 0.4, 0.8, 0.3, 0.7. Note that we take the Q transpose here, but we take P itself. So, what is how do we get the relation R? We are going to apply max of the Lukasiewicz T norm. So, it is max of 1 0 with 0.4, 0.3. So, 1 with 0.4 is 0.4, 0.3 is 0, 1, 0 with 0.8, 0.7 is max of 1 and 0.8 is 0.8. Lukasiewicz T norm applied on 0 and 0.7 is 0.

Next, the second row was the first column it is max of 0 comma 0.3 next component will be 0 and 0.8 is 0, 1 and 0.7 is 0.7. Finally, the 0.3 and 0.4 is 0, 0.7 and 0.3 is exactly 1 which is again 0. With respect to the Lukasiewicz T norm now 0.3, 0.7 with 0.8 and 0.7, we see that 0.3 plus 0.8 minus 1 which is 0.1, 0.7 0.7 is 1.4 minus 1 is 0.4. So, this is the final R that we obtained here.

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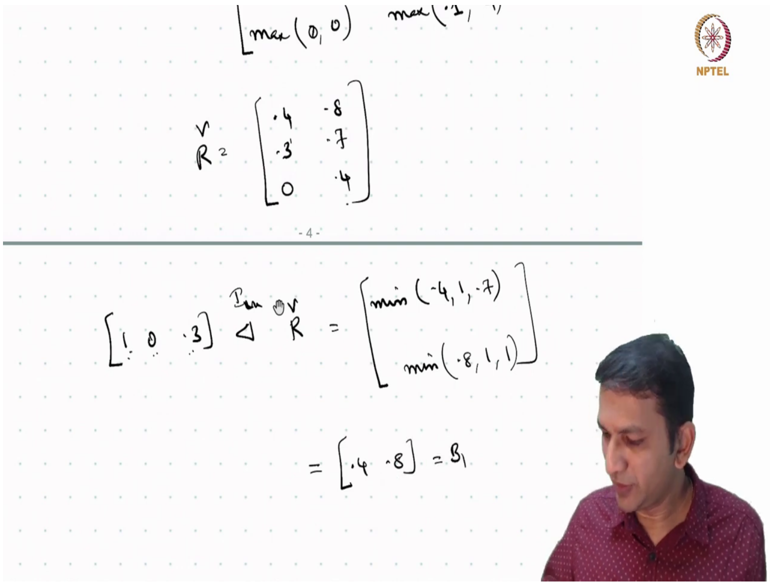

Now, that R check is now given as what? 0.4, 0.3, 0, 0.8, 0.7, 0.4. So, this is the R check we have got. Only question is, is it interpolative? Well, let us check once more. 1, 0 and 0.3, note that we have to use the Lukasiewicz implication here with R check. Now, if you do this remember this is the BKS inference; that means, you are looking at minimum of 1 implies 0.4 which is 0.4, 0 implies 0.3 is 1, 0.3 implies 0 is in fact, 0.7.

In the Lukasiewicz implication, 1 implies 0.8 is 0.8, 0 implies 0.7 is 1 and 0.3 implies 0.4 is 1. So, the output that we get is 0.4, 0.8 which is clearly this T norm. Similarly, if we give A 2 as the input, we can easily check that we get B 2 as the output. So, it seems to work even for BKS.

So, it does not matter the sufficiency conditions are applicable both for BKS and CRI, when the T and I T are actually coming from residuated lattice. And it does not depend which T that we consider. So, it is valid for any pair any residual pair that we take.

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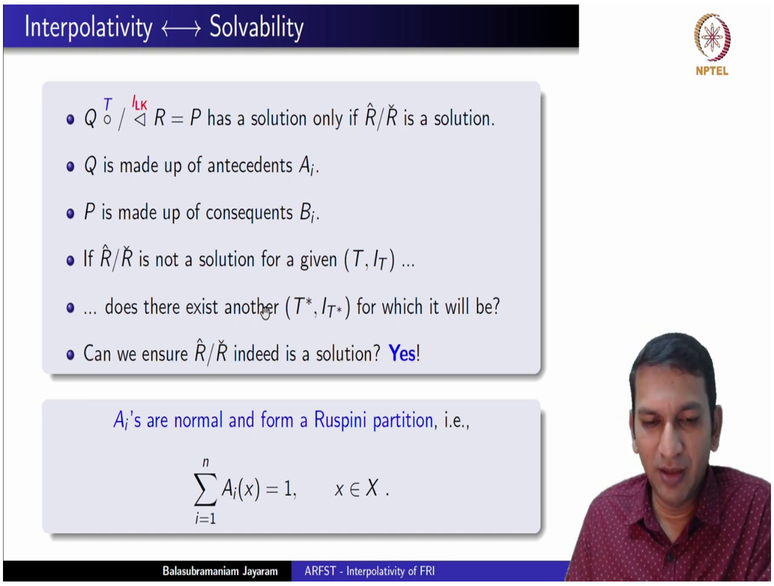

Well, what is it that we started with? We have this equation and we know that it has a solution only if either R cap or R check is a solution, where Q is made up of antecedents A i and P is made up of consequents B i. The question we asked was if R cap or R check is not a solution for a given pair, does there exist another pair? We saw that. We could not find. We were on a wild goose chase.

So, we modify the question and ask the question, how can we ensure that these two relations are indeed a solution for the corresponding fuzzy relational equation? Is there a simple condition that we can check? It so happened yes, and that condition was simply that the antecedents should be normal and form a Ruspini partition.

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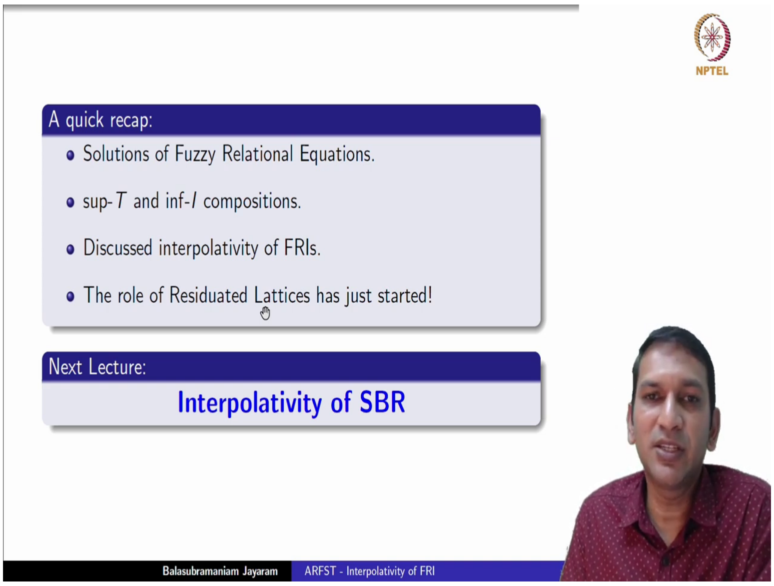

In this lecture, what we have seen is how the solutions of fuzzy relational equations involving sup-T and inf-I compositions lead to interpolativity of the corresponding FRIs. Namely, that of compositional rule of inference and the BKS inference. It is no exaggeration to say that the role of residuated lattices has only just started in the weeks to follow.

We will see that the rich structure and the myriad properties that residuated lattices have, they play an important role in ensuring a lot of desirable properties of fuzzy influence mechanisms. In the next lecture, we will discuss the interpolativity of similarity based reasoning scheme.

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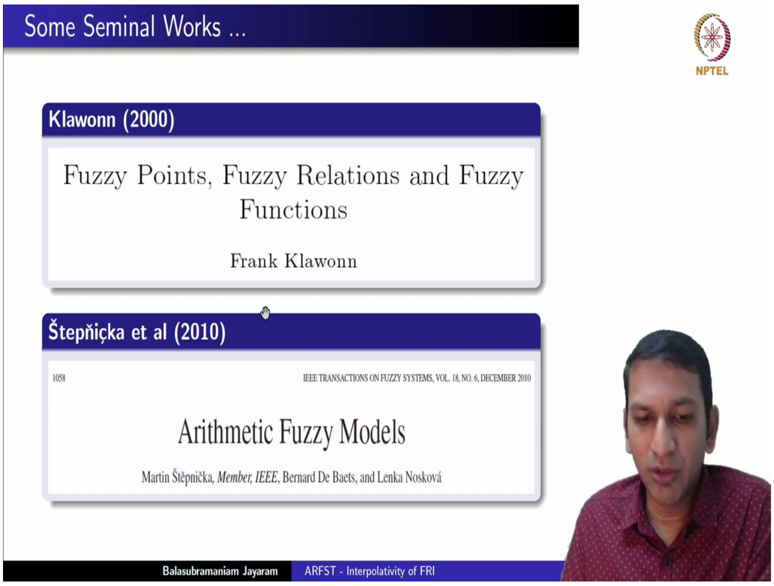

Let us point out some seminal works that have been done in this context. The first sufficiency condition was proposed by Frank Klawonn in this paper way back in 2000. It was further modified, later on in twenty 2010 by Stepnicka and his co-authors in this work.

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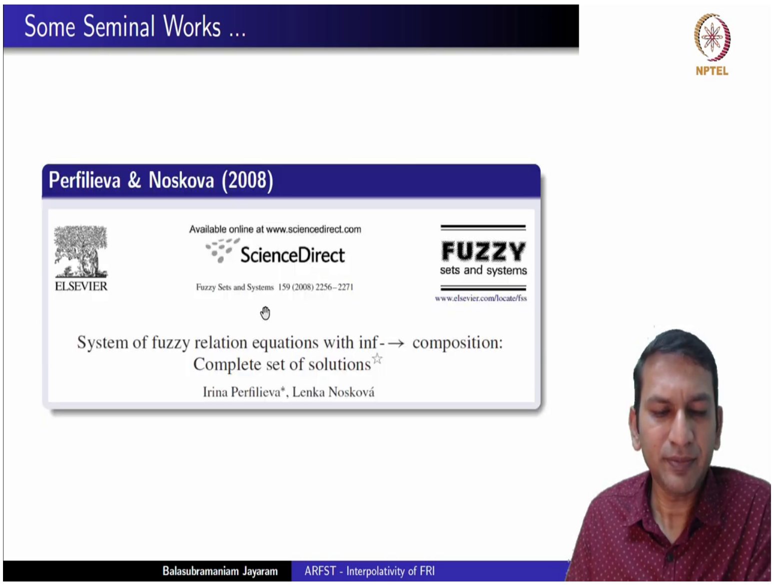

Some related works also have to be pointed out by that of Perfilieva and Noskova.

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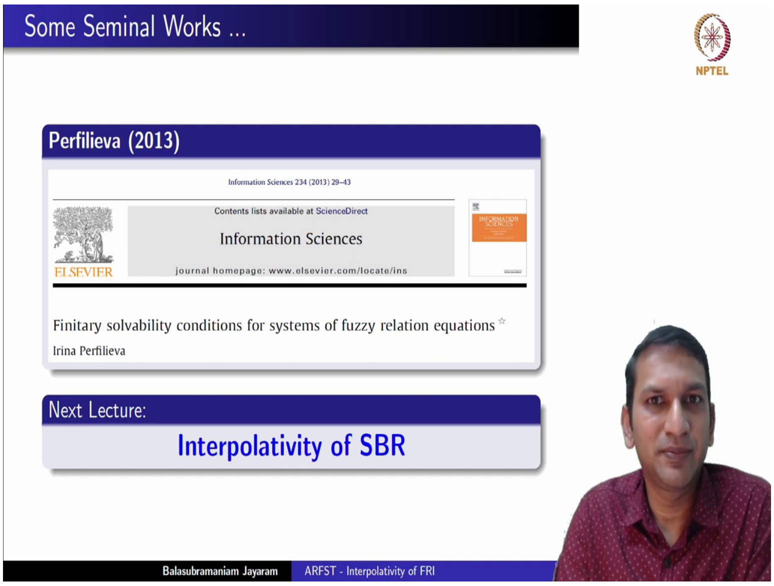

And this is also another paper a work dealing with solvability of fuzzy relational equations. As I said, we will meet in the next lecture to discuss Interpolativity of Similarity Based Reasoning scheme. Glad, that you could join us for this lecture. Hope to meet you soon in the next lecture.

Thank you again.