Approximate Reasoning using Fuzzy Set Theory
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Lecture - 03
Fuzzy Sets - Representations

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Hello and welcome to the 3rd of the lectures in this week, under the course titled Approximate Reasoning using Fuzzy Set Theory. In this lecture we will see an appropriate way of representing fuzzy sets.

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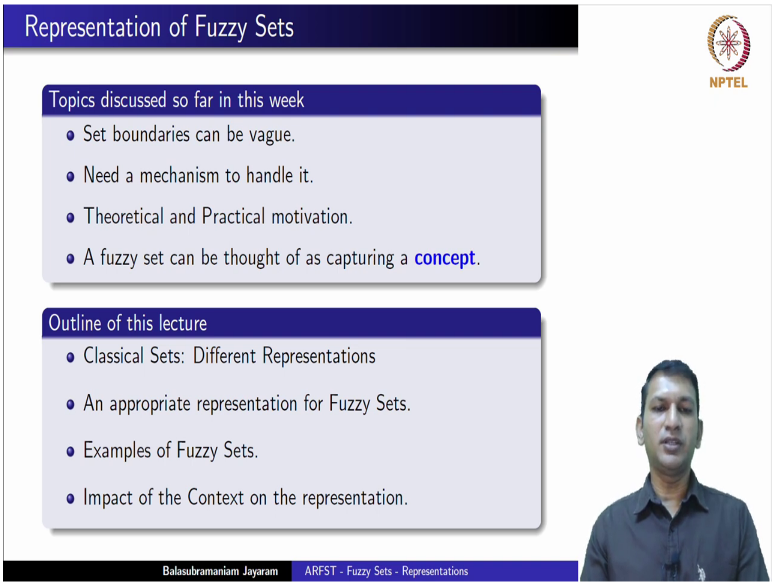

So, far in this week we have seen that set boundaries can be vague, but need a mechanism to handle them. We have also seen some theoretical and practical motivations for going towards fuzzy sets and now we are aware that a fuzzy set can be thought of as capturing a concept. In this lecture, we would like to see how this concept can be represented using a mathematical construct.

Towards this end we will visit a few representations of classical sets and pick one which can be suitably generalized to represent the concept that a fuzzy set represents. We will see a couple of examples of fuzzy sets, specifically we will see the impact of the context on the representation.

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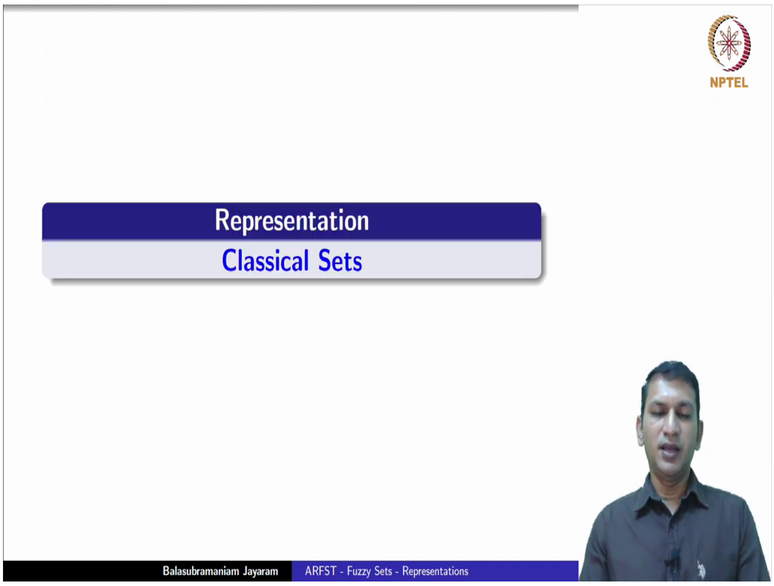

Let us begin by looking into some of the representations that a classical set can have.

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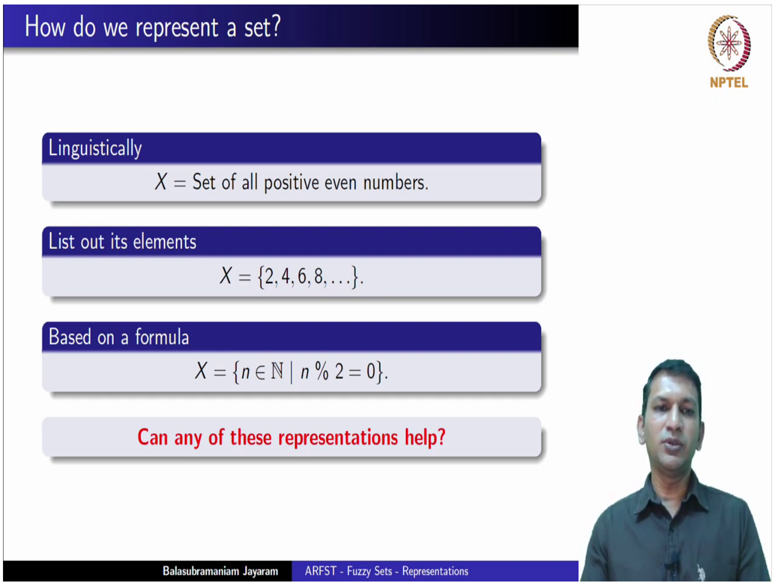

If you ask a question how do we represent set, it could be represented linguistically. For instance, we can say X is a set of all positive even numbers, we could represent the same set by listing out its element when they are finite. And in the infinite case when a pattern emerges also this way, we could also list out a set by characterizing it in terms of a formula.

For instance, the same set X can be written like this. All those natural numbers which are divisible by 2; however, on closer scrutiny it can be seen that none of these representations actually help us when you are trying to capture a vague concept. Thankfully for us there is yet another representation of a classical set, that of characteristic function which helps us in our code. If you have a subset A of X it can also be represented by its corresponding characteristic function.

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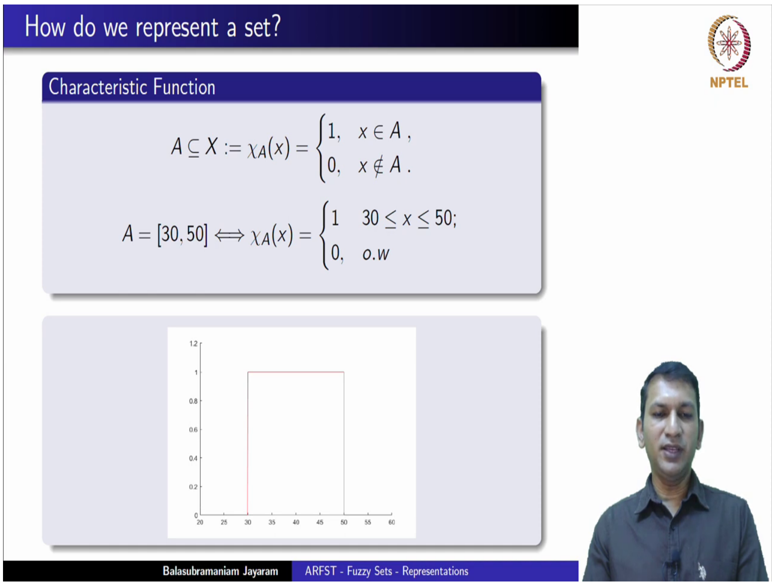

A function which is nothing but a map from the universe of discourse X to the set {0, 1}. Now, the values that this characteristic function of the subset A takes is 1, if element belongs to the subset A and 0 otherwise, more like an indicator function. Now, if A is the interval [30, 50], the closed interval [30, 50] it can be represented in terms of its characterized function as follows.

A graphical representation of that would be like this, if you restrict your X to be between 30 to 50 then you see that the step function or a rectangular function, if you would like to call it looks like this. Everybody between 30 and 50 belongs to the set and takes a value 1 and everybody else between 30 and 50 they are 0.

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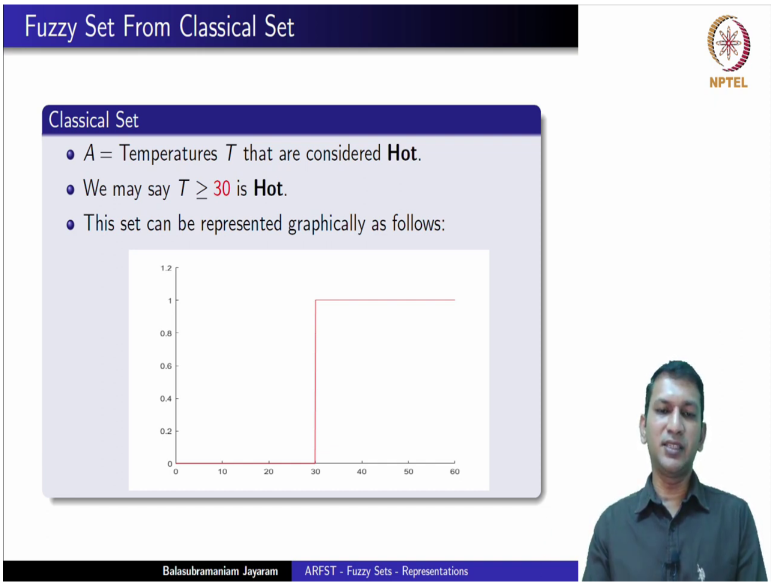

Now, we want to represent a concept. Let us look at how to do this using a classical set. Let us pick up temperatures that are considered hot and call it the set A. Now, how may we proceed? Of course, from the previous lecture we are aware that hot is actually a concept, we may say that any temperature which is greater than or equal to 30 is hot.

So, a graphically it can be represented perhaps like this. So, if you fix the domain to be between 0 and 60, then everybody till 30 or less than 30 is not hot. So, the characteristic function is the value that it takes on the co-domain is 0 and everybody from 30 onwards takes the value 1.

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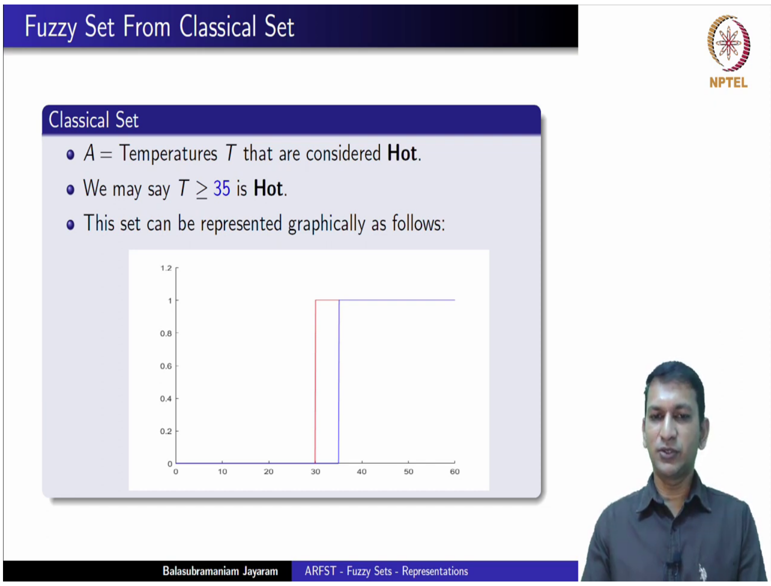

Somebody might contend this and say no perhaps 35 is the one that we should be looking at; that means, we should threshold at 35. So, in the case of representing hot with a threshold at 35 the characteristic function would look like this. Yet another person might come and say no for me 40 degrees is hot and nothing less than that.

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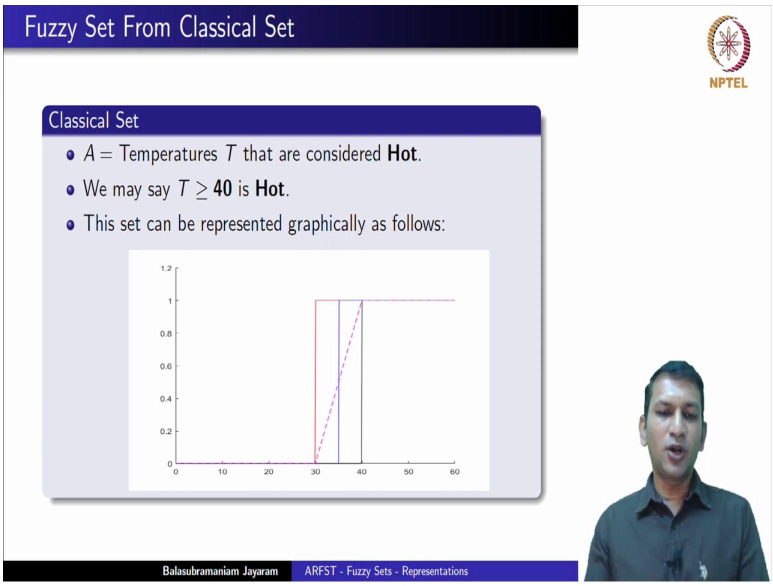

In that case the characteristic function would look like this. Now, you see here this hot as we have seen in the previous lecture is subject to different interpretations. Of course, contextual also, but largely subjective and allowing different interpretations of itself. Now, if you put a heart threshold, then we would get different interpretations for this and different corresponding characteristic functions.

However, if you look at these three characteristic functions based on the subjective opinions expressed by people, one thing perhaps is clear if we were to go with this as the ground tool that all of them seem to agree that anything below 30 degrees is not hot and every temperature above 40 is can be considered hot.

So, a better representation or better capturing of this concept which has been expressed through their opinions could be having a function of this type, that is till 30 it is 0, above 40 it is 1, but between 30 to 40 it seems to be linearly interpolating the values between 0 and 1. Now, this is the idea of a fuzzy set.

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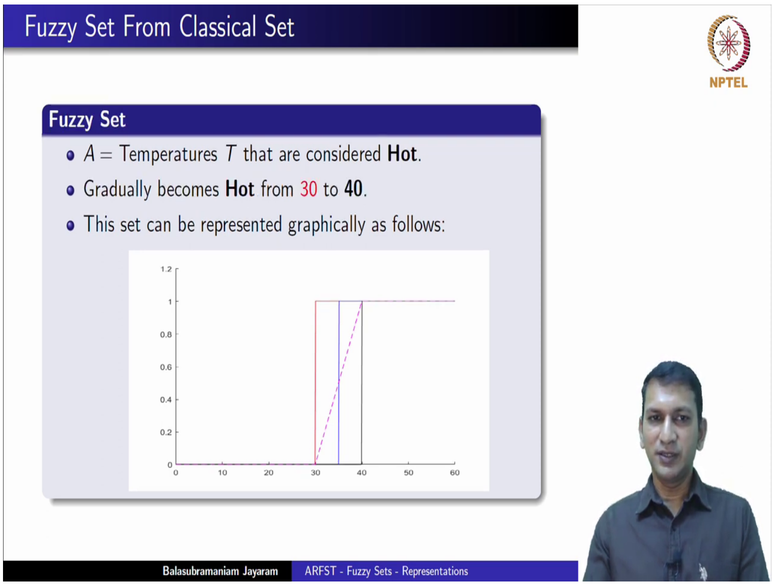

So, this is how we would like to generalize a classical set to a fuzzy set. So, in the case of hot when we move from a classical set representation to fuzzy set representation, you see that the concept of hot gradually gets membership values from 30 to 40 and peaks at 40.

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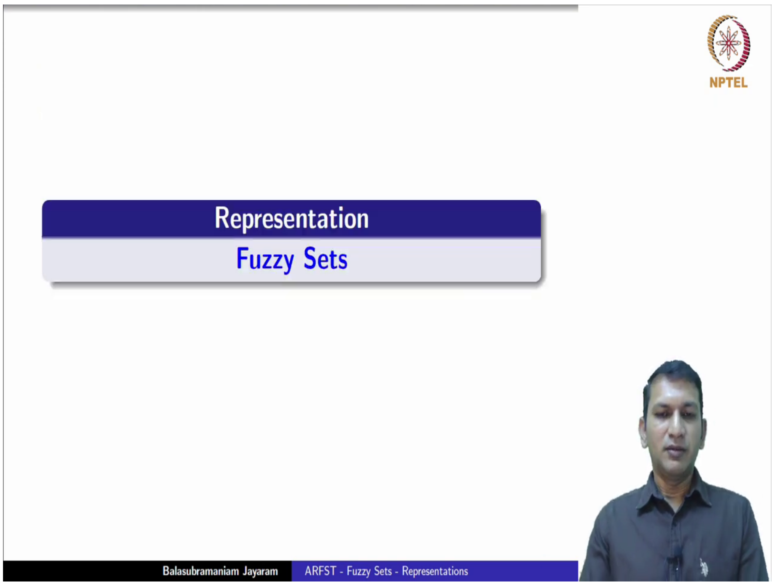

Whereas, everybody below 30 degrees is considered not hot.

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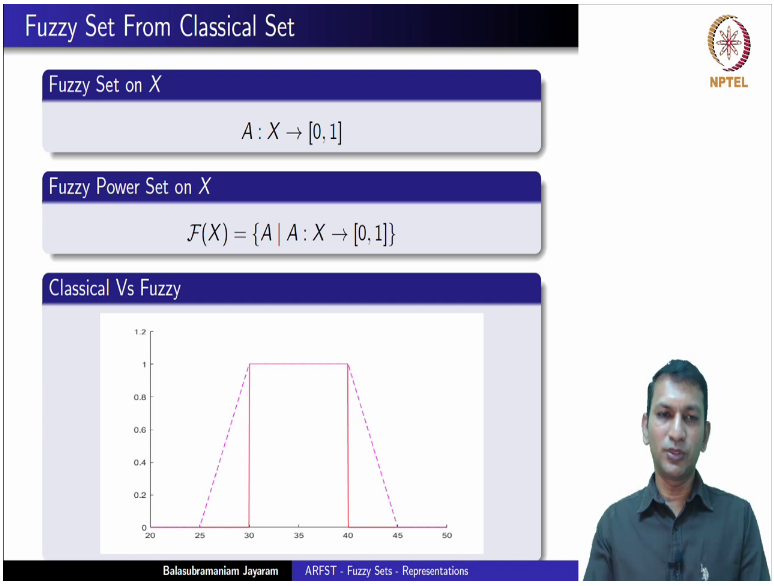

Abstracting from here, if we were to look at the representation of fuzzy sets, you can look at a fuzzy set as a function from X to the entire interval [0, 1]. Whereas, the characteristic function was a function from the underlying domain X to just the set {0, 1}. Through this course by the powers fuzzy power set of on X, we will denote that by F of X, script F of X which is a collection of all possible fuzzy sets on X.

So, in a nutshell classical set can be represented by its characteristic functions, which look like stepped functions typically rectangular functions. Whereas, fuzzy sets are represented by membership functions, that is what we call them because every value between 0 and 1, we look at it as a membership value.

If it takes 0 membership then we say it does not belong, if it takes membership 1, it totally belongs, but it could also have varying membership values. So, in that sense it is a common parlance that given a fuzzy set every element belongs to it is not a question of whether an element belongs to the concept, but it is only question of to what extent to how much it belongs.

So, this is the representation that has been found useful appropriate and also amenable for processing, which will become amply clear as we go through the course.

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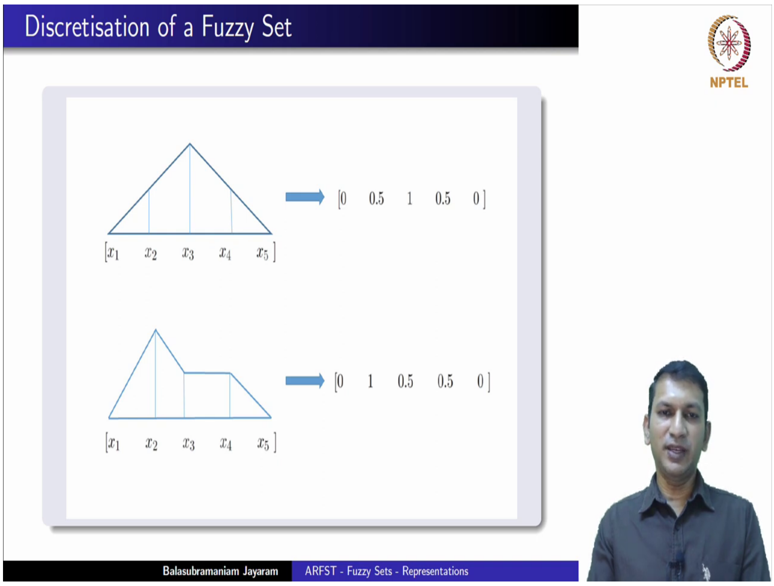

Often when we employ this, as was seen in the previous lecture the reason one of the motivations for coming up with fuzzy sets is from the application side, where we wanted to come up with machines that can be made to think. And anything that you implement using a computer often always ends up needing discretisation.

So, while in abstraction you can look at it as a function from X to 0, 1 often while implementing it goes through a discretisation. So, let us look at what is discretisation of a fuzzy set. So, now, imagine this is the domain X and you have a fuzzy set which is of triangular in shape, but we discretised the domain into just 5 points. So, now, this entire continuous function will can be represented by a mere vector of 5 components. And how do we choose these components?

We find the membership value at these 5 points of discretization, in this particular case it would turn out to be 0.5, 1, 0.5 and 0. Look at another fuzzy set which is not so symmetric, but again with the same points of discretisation the representation of this fuzzy set can be seen as follows. Its a 5 component vector, a vector in $[0, 1]^5$.

Now, that we have discussed about discretization so in essence we are looking at fuzzy set still from the point of view of a function, mapping the underlying domain X to [0, 1]. The X could be finite or infinite.

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Let us look at some examples of fuzzy sets.

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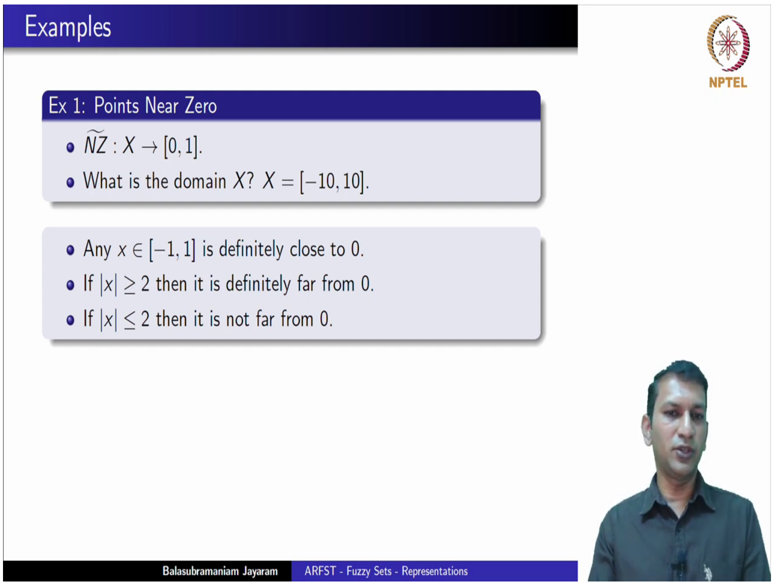

Consider this phrase points near 0, clearly it is the concept. So, we would like to represent it using a fuzzy set and as you have seen for a fuzzy set we need a domain X and be able to define a function on that domain. So, the moment you want to define the fuzzy set, all you need to find is identify the domain and find a suitable mapping where every element of X is mapped to some element of the [0, 1] interval. For this example let us choose our domain to be the interval [-10, 10].

How should we define this mapping? Let us say that any value that is belonging to this [-1 1] interval is definitely close to 0. And let us also agree that any value of x whose absolute value is greater than 2 that is essentially all those points at like below -2 or above 2, they are definitely far from 0. And what about the points in between it is between -2 to 1 and 1 to 2.

Well, let us say that if x mod x is less than or equal to 2 then it is not far from 0. So, with this basic idea which we believe captures the concept points near 0. One way of defining the fuzzy set could be as follows.

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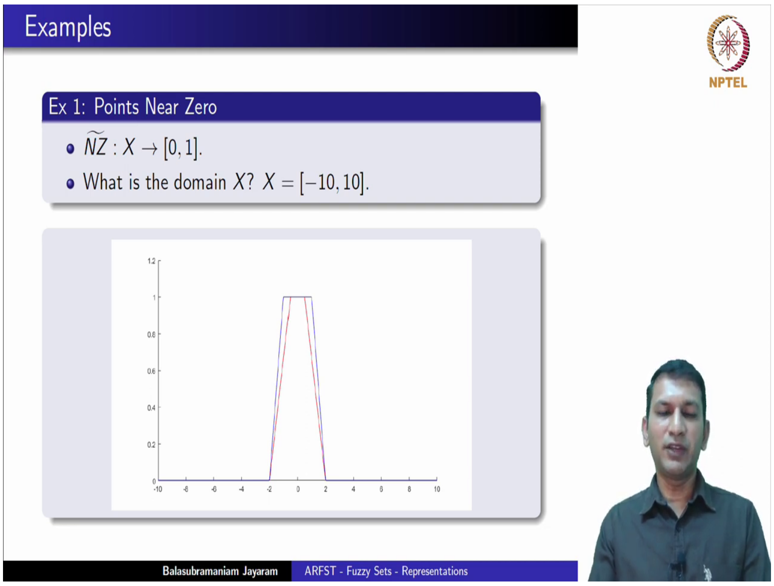

So, you see here that everybody between -1 and 1 has been given the full membership value of 1. And everybody up to -2 and beyond 2, we felt they are not close to 0, they are definitely far from 0. So, they have been given 0 membership and points that are between -2 and -1 and 1 and 2, we know they are not far from 0, but perhaps they are not very close to 0 in that sense we have given them a graded membership into this concept.

So, this is one way of getting fitting a fuzzy set to this concept. Of course, you could also think of another way a similar kind of a fuzzy set; however, the points that we consider close to 0 are now have now become enlarged from the earlier interval of [-1, 1]; now we have moved towards [-1.5, 1.5]. However, we still continue to consider points whose absolute value is greater than or equal to 2 as being far away from 0.

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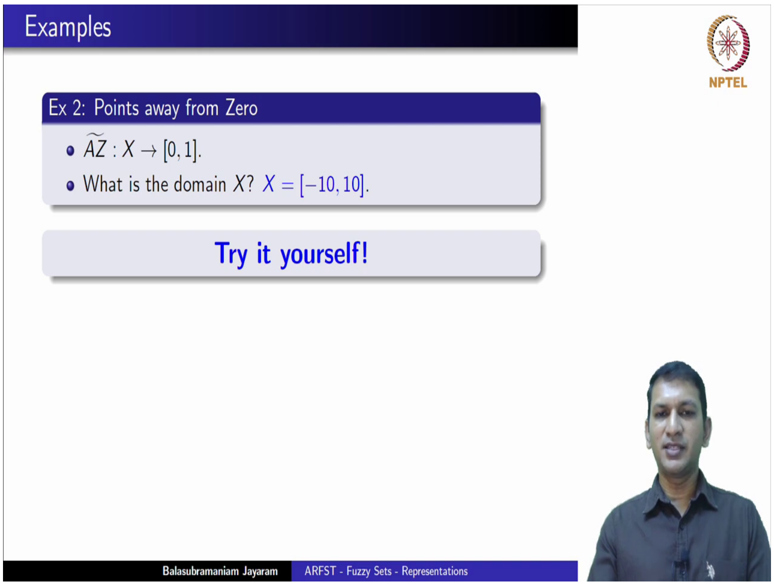

Let us consider another example. Now, consider this phrase, points away from 0, once again it is a concept. So, we can take the help of fuzzy set to capture this concept. Once again the moment we think of a fuzzy set, we think of a function from a suitable domain X, to the 0, 1 interval. Now, the question is what is the domain X?

Once again it is a synthetic example. So, let us assume that the domain X is still [-10, 10]. Now, we would like to now come up with a mapping that captures this concept of points being away from 0. How, do we do this? Well just to break the monotony of my speech, I would like you to try it yourself and we will compare notes soon enough. Perhaps take 45 seconds to a minute, we will wait here before we progress proceed. May I request you all to take a pen and paper on this domain of -10 and 10.

What is your idea of the concept points away from 0? Please try it we will compare notes soon enough maybe in little more than a minute. Welcome back, if you would like more time please feel free to pause the lecture here, try it for yourself and then continue. For those of you who have done it, let us move ahead.

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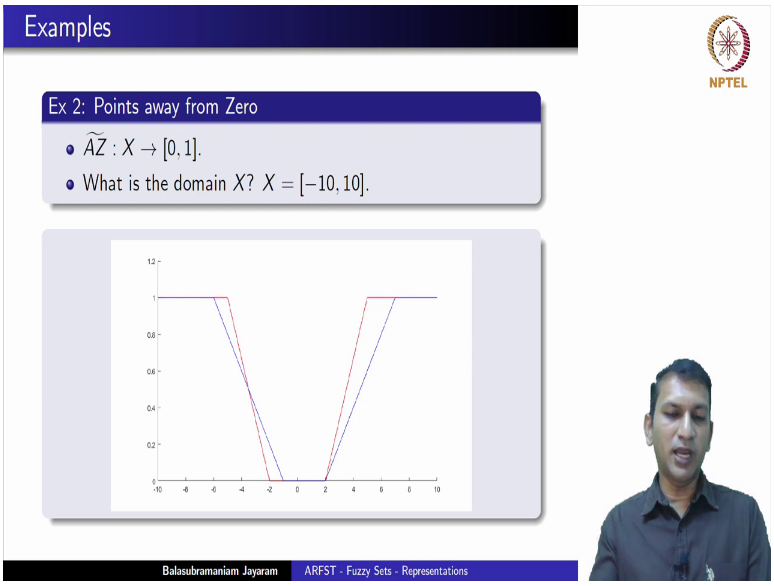

The way I have conceptualized points away from 0 and captured it as a fuzzy set is as follows. Keeping in mind the way we have captured points close to 0, in my graph all those points that are between minus 2 and 2 have not been given any membership value in this concept. That means, if it is between minus 2 and 2, they are not points away from 0, but those points whose absolute value is greater than or equal to 5.

That means, all those points which are less than minus 5 and greater than 5 they have been given a membership value of 1, indicating they are actually prototypical elements of this concept that is they definitely are away from 0. Now, the points between minus 5 and minus 2 and 2 to 5, they have greater membership value to this concept. This is one way to represent, perhaps how you have represented may not look like this, but as long as you have a justification it should be fine.

We could also draw another graph like this, another function from minus 10, 10 to 1 which might capture according to you, the concept points away from 0. Please note here, that this function is not symmetric; however, if this is warranted in the context that you are trying to capture this concept then, so be it.

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We have always already seen two different representations for just these two concepts, points near 0 and points away from 0.

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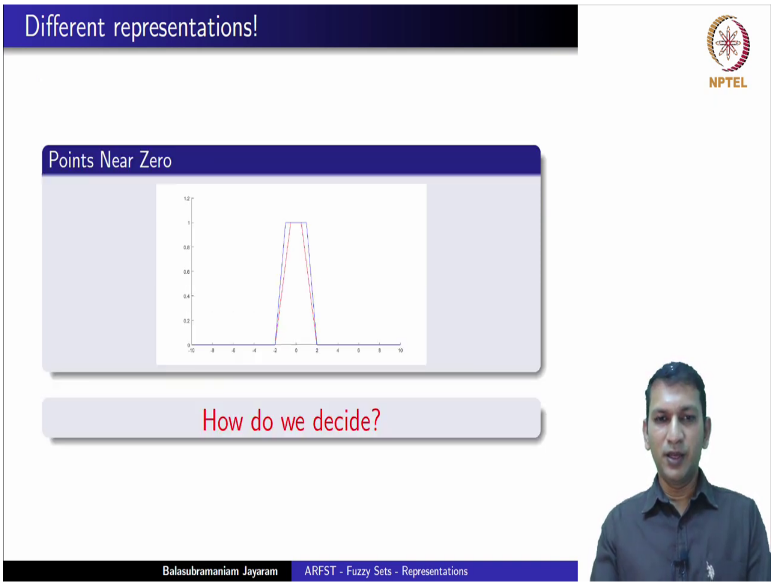

Now, the question is how do we decide and this is where we need to be aware of the context.

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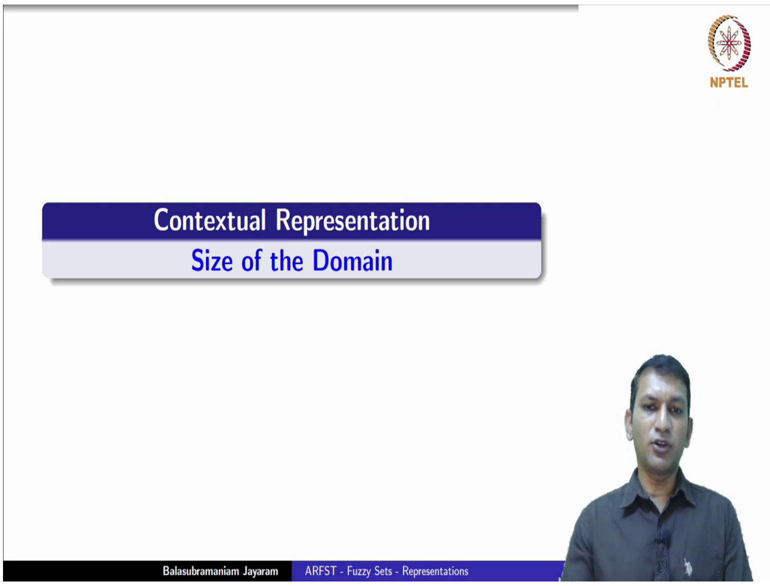

Let us look at how the context has an impact on the representation. The context can be determined by various factors.

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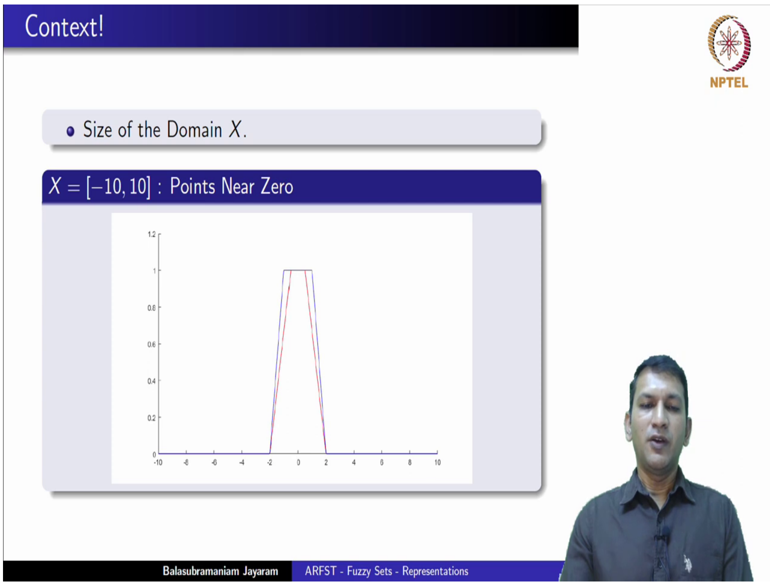

Let us began with the size of the domain itself. Now, we have seen the concept points near zero being represented by two different types of fuzzy sets. This was when the domain was the interval minus 10, 10.

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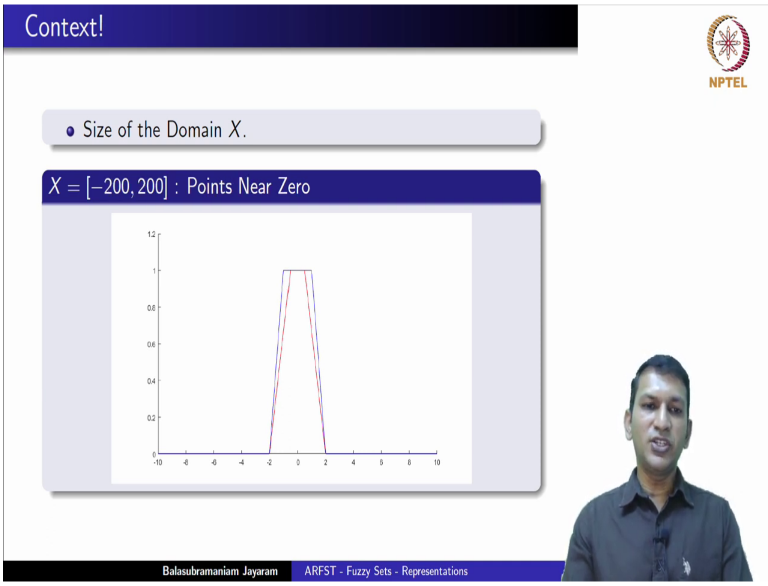

What if we change the domain to the interval minus 200, 200? In that case perhaps we would say a better or more appropriate representation of this concept can be captured through this fuzzy set.

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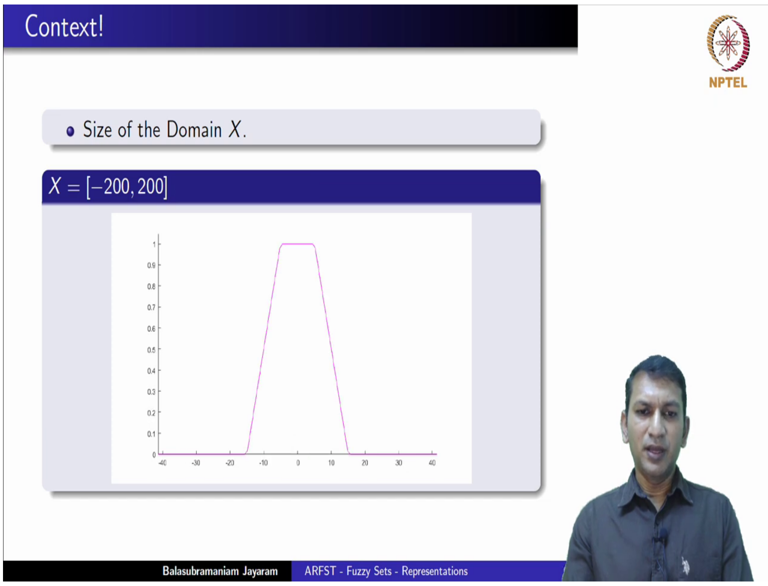

Please note here, the points which belong to this concept to the fullest extent that is to membership value of 1 are no more within the interval minus 1.5, 1.5, but now we have allowed them to extend between minus 10 and 10. You may question is this warranted? Well imagine yourself out shopping for a pen which would normally cost 20 rupees, but if you find the shopkeeper tells you that the cost of that pen that day is 40 rupees, price difference of 20 perhaps you would think twice before buying it.

On the other hand, on the same day, you go shopping for a dress perhaps a branded shirt and whose likely prices around 20 or so, that is what you have in your mind. And you go there and find that the price is actually 2020, 20 rupee difference between what you had in your mind and what you see on the bill, clearly in the latter case you probably would not think too much before buying the shirt.

This is once again related an important aspect of reasoning, which we will visit perhaps in the 8th or 9th week of our lecture series, under the concept of robustness of inference. But it suffices to say that with the change in domain, of the domain and the size of the domain the representation is also likely to change.

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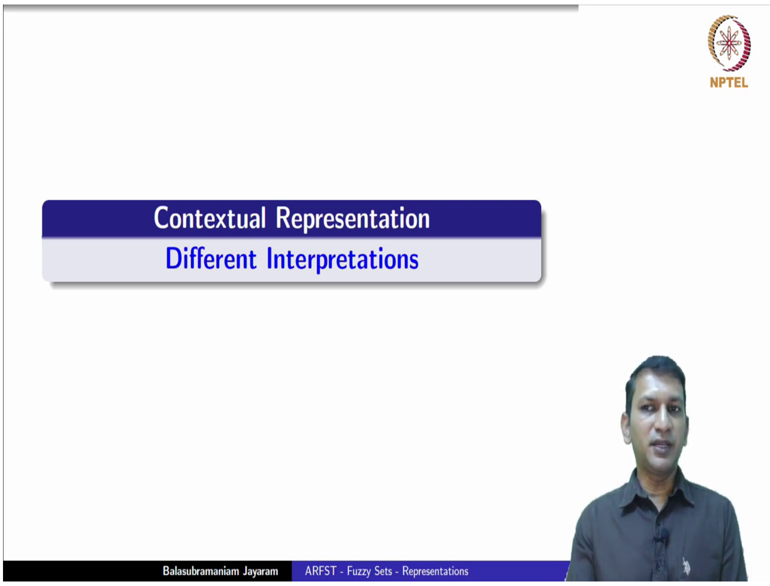

And another factor that we need to take into consideration is the interpretation that we have over the domain.

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Let us for the moment consider three different interpretations, but all on the same domain. Consider the concept hot, now what we understand by somebody by hotness or being hot varies depending on weather we are talking about weather, like in the climate day to day weather or if you are referring to somebody's body temperature or if you are referring to the temperature of water.

Let us look at this. Let us fix the domain to be the interval 20 to 60. So, now, we are going to consider temperatures in centigrade between 20 and 60. Now, if fact referring to how hot the weather is as we have seen to remain consistent with what we have seen earlier in this lecture.

Let us say that anything above 40 degrees is definitely hot anything below 30 is definitely not hot and every value between 30 and 40 is hot, but to a certain degree. So, this is how we would represent a fuzzy set which captures hot when the interpretation is with respect to the weather. However, if you are looking at body temperature normally we say 98.4 degree Fahrenheit is the normal temperature, when you convert it to centigrade it is around 36.8 or 36.9.

Now, we feel that anything above 100 degree Fahrenheit is already hot the body is hot. So, which means it translates to something like 39.7. So, if you were to plot this, this is how the corresponding fuzzy set would look like. So, for a body temperature to be hot this is the fuzzy set representation that you might come up with. Now, on the other hand if you are talking about water being hot, perhaps till 45 degrees we perhaps would not be able to feel the heat in water.

If you were to check the water that you would get normally out of your water heater I think up to 45 degrees or so you may not consider it as hot water. So, now, the domain remains the same, we are talking about the interval 20 to 60 and interpreting them as degree centigrade; however, the same fuzzy set, the same concept of hot varies in with respect to the interpretation that we assign and accordingly the fuzzy set representation also varies.

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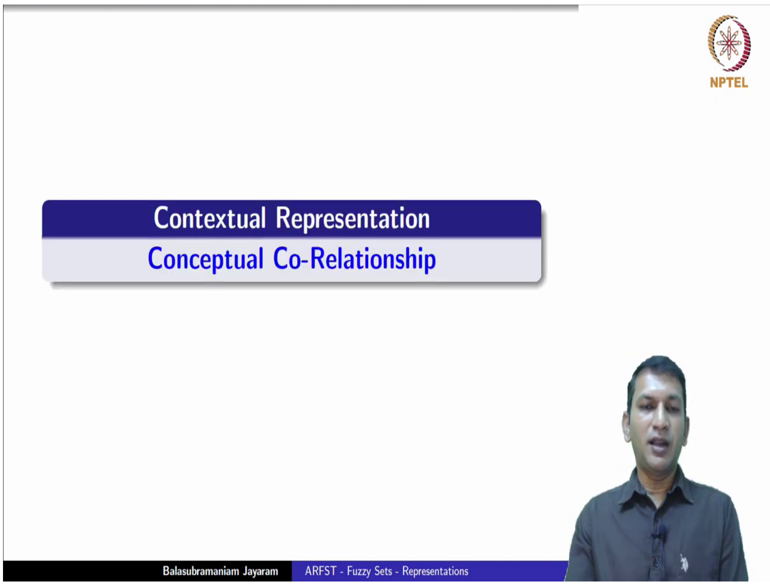

At times we should also be aware of the co-relationships that can exist between the different concepts.

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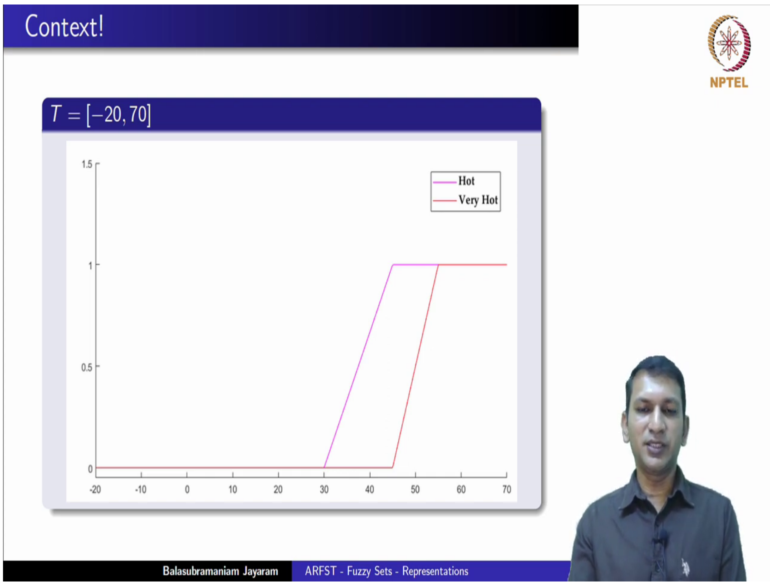

Now, let us look at this. The domain is between minus 20 and 70 and as we have seen, when we talk about hot let us interpret it as we are talking about the hotness of the weather. We have seen that this is the fuzzy set representation that we perhaps might want to have. So, between 30 and 45, we say it gradually increases and it becomes at 40 or 45 it becomes hot completely.

Now, we may also have another concept called very hot and how would you want to represent this? Perhaps to take an arbitrary threshold let us say that anybody above 45 degrees is slowly becoming very hot and once he reaches 55, he definitely is very hot. Now, this is the interpretation that we would like to give for very hot. Now, it is interesting to compare these two interpretation hot and very hot.

If they were to be represented like this, look at a temperature like 60 degree centigrade, it is not only very hot its a prototypical element of the concept very hot, but it is also a prototypical element of the concept hot. That means, you in fact, you could even say that every temperature degree which is completely very hot belongs to very hot to membership value 1, belongs totally to very hot also belongs totally to hot.

However, the converse is not true, there are temperatures which are considered totally hot, but they do not find that kind of a membership value in the concept very hot. Now, this is one interpretation of hot and very hot.

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You could also have another interpretation like this. Wherein, those points which are the prototypical elements of hot, the concept hot and the concept very hot these points do not actually intersect. So, they are actually disjoint. Now, in which context do we use this representation and in which context do we use the previous representation?

Well, this is something that we will see as we go along, there is making things work they say is not only science, but also an art. But slowly as you work with examples which we will do during weeks 6 and 7 of this lecture series, we will slowly get the hang of how to represent them, how to position them and how to interpret them with respect to the context.

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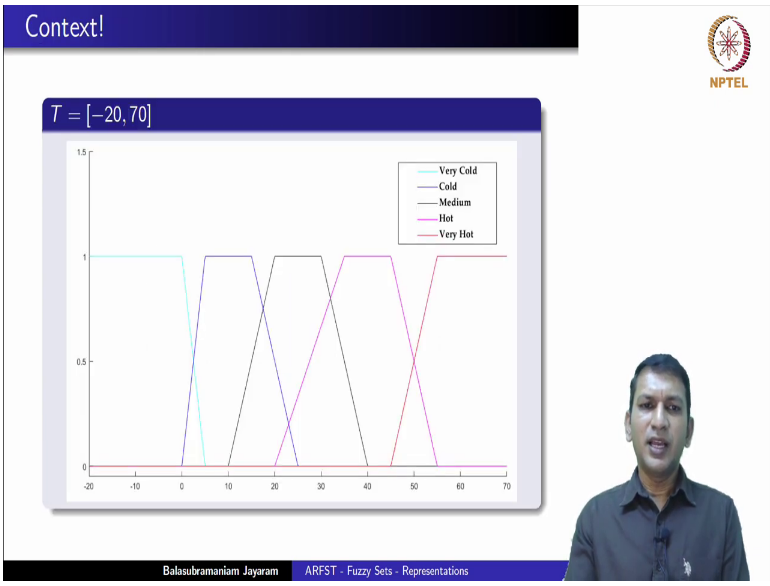

So, if you were to have other concepts related to this, weather you could perhaps have something like this. You could have very cold, cold, medium, hot and very hot represented like this.

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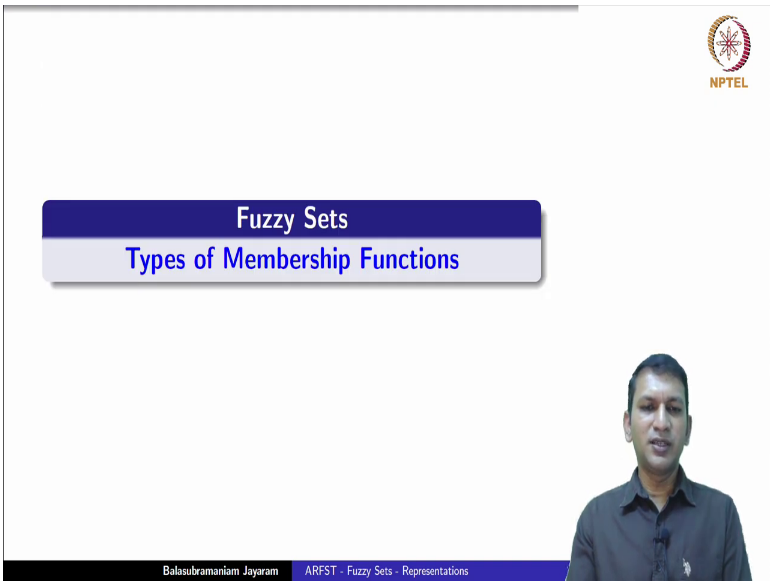

Finally, let us look at some of the typical membership function that we use.

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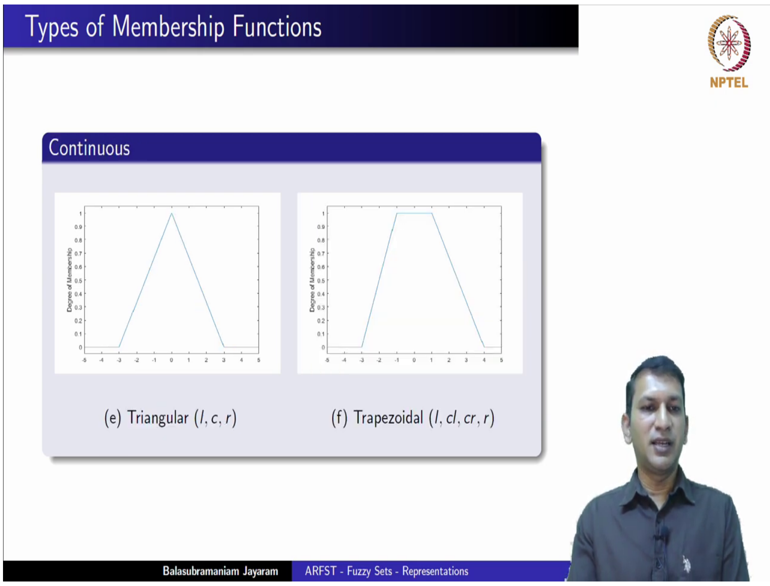

The first of them that you have already seen is the triangular membership function, it is characterized by 3 parameters, the left end point, the center point and the right end point. Clearly any value less than l and greater than or greater than r, will have 0 membership values. Only the point c will have full membership value and everywhere else between l and c and c and r you are linearly interpolating.

You could have a trapezoidal membership function, where the points which assume full membership value of 1, they form an interval. Unlike in the case of triangular membership function where the point the set of point that take the value 1 is singleton, here you will have an interval.

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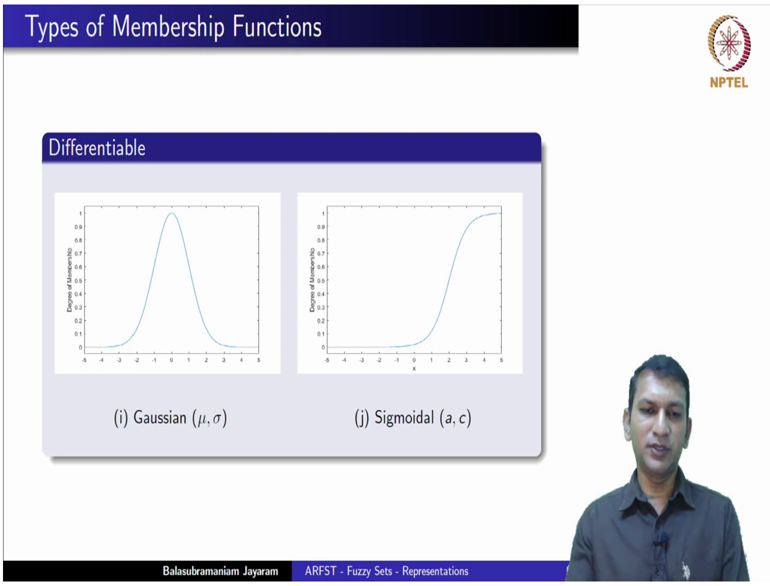

These are linear and continuous, but you could also have non-linear membership functions also with differentiability. There is your typical Gaussian membership function, wherein you specify the center and the width of the Gaussian using mu and sigma. You could also have sigmoidal membership function, where c represents the inflection point, a controls both the steepness and also the direction of the slope.

With this we could also alter this representation where we had only linear and continuous membership functions, we could change them with these non-linear but differentiable continuous function membership functions like this.

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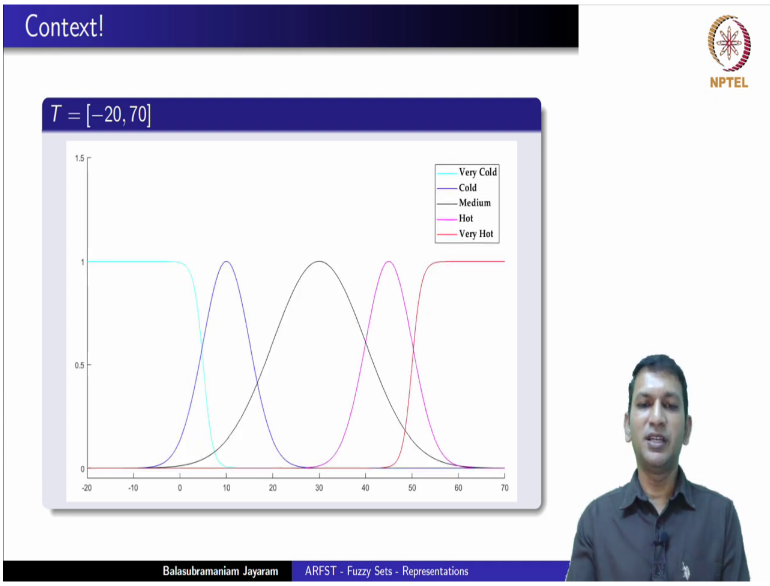

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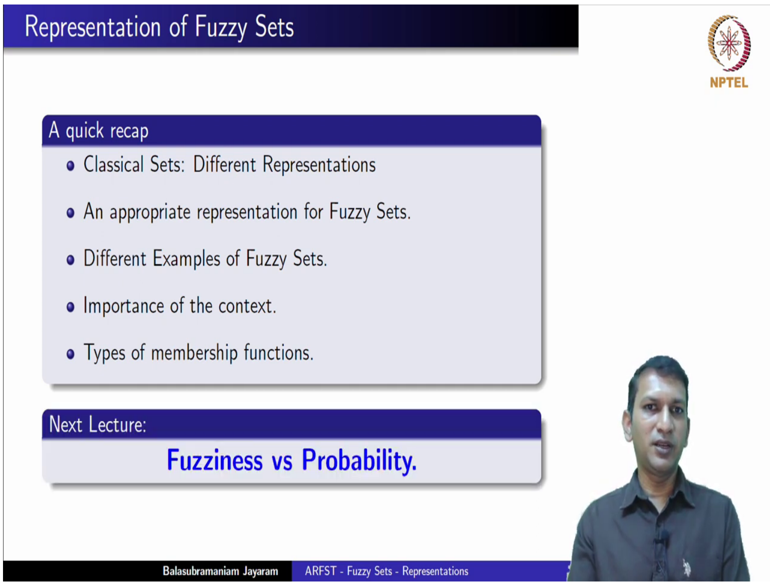

With this we come to the end of this lecture, allow me to give you a quick recap. We saw different representations of classical sets and chose an appropriate one to generalize to a fuzzy set. We have seen different examples of fuzzy sets, most importantly how the context in which we are trying to capture the concept has a bearing on the ultimate representation of the fuzzy set in terms of its graph. We have also seen some typical or regular membership functions.

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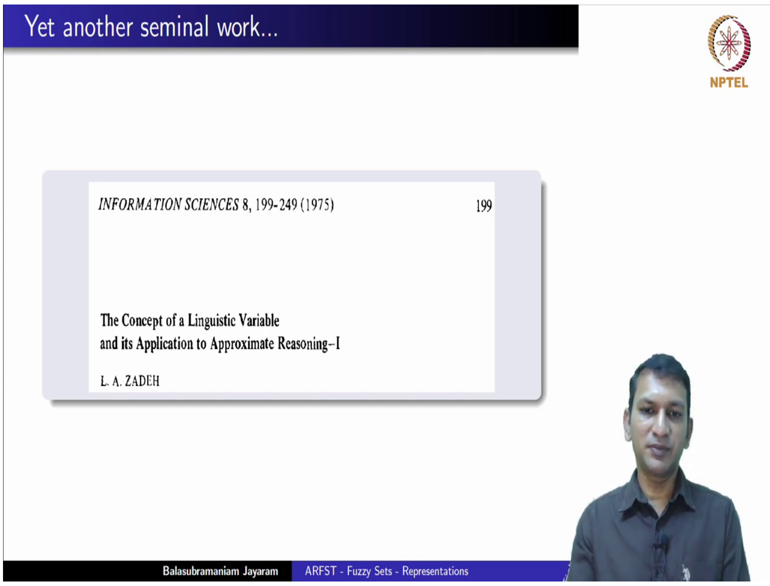

What next? In the next lecture we will have a quick look at the difference between fuzziness and probability. Let me leave you by just only pointing out another seminal work published by Professor Zadeh himself, which once again might resonate with you having gone through this particular lecture, based on the content that we have covered in this lecture. We believe that this is a reference which can be useful in both complementing and supplementing the knowledge you may have gained from this lecture.