Approximate Reasoning using Fuzzy Set Theory

Prof. Balasubramaniam Jayaram
Department of Mathematics
Indian Institute of Technology, Hyderabed

Lecture – 27
On the Transitivity of Fuzzy Relations – II

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Hello, and welcome to the last of the lectures, in this week 5 of the course titled, Approximate Reasoning using Fuzzy Set Theory. A course offered over the NPTEL platform.

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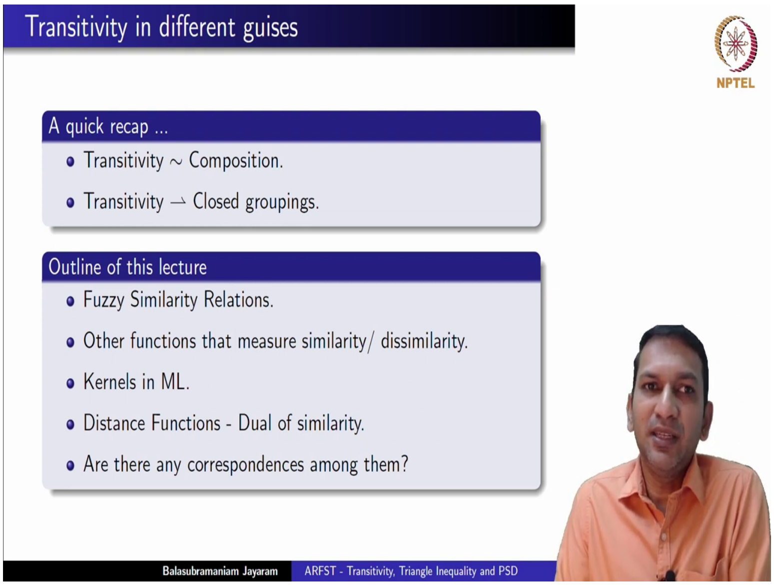

In the previous lecture, we have looked at the importance of transitivity so to speak. We have seen that, whenever you have a composition, there is going to be transitivity lurking behind and vice versa. And we have also seen, that transitivity is essential.

If you want to have closed groupings. Groupings which are non overlapping. What are we going to discuss in this lecture? So, far we have seen, fuzzy similarity relations; however, there are also other functions, that measure similarity or dissimilarity. For instance, the kernel functions in machine learning or the distance functions; which can be thought of as the dual of similarity. In this lecture, we would like to explore, some possible correspondences between these three functions.

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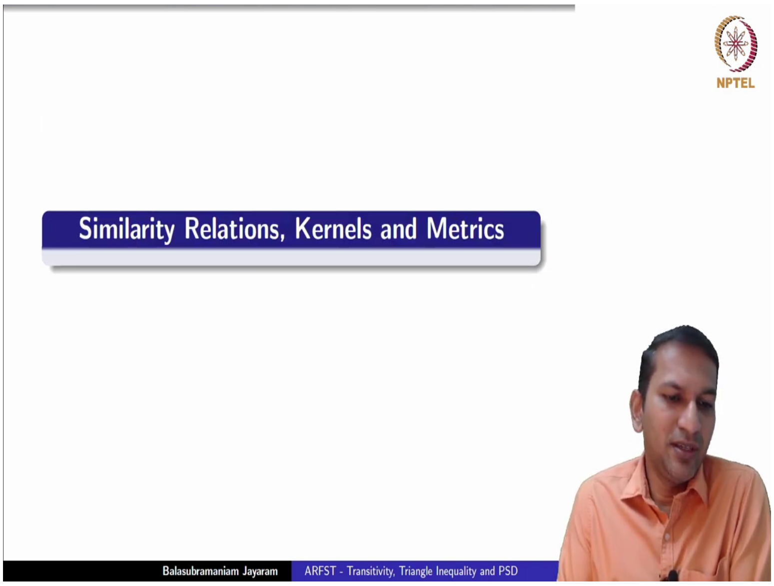

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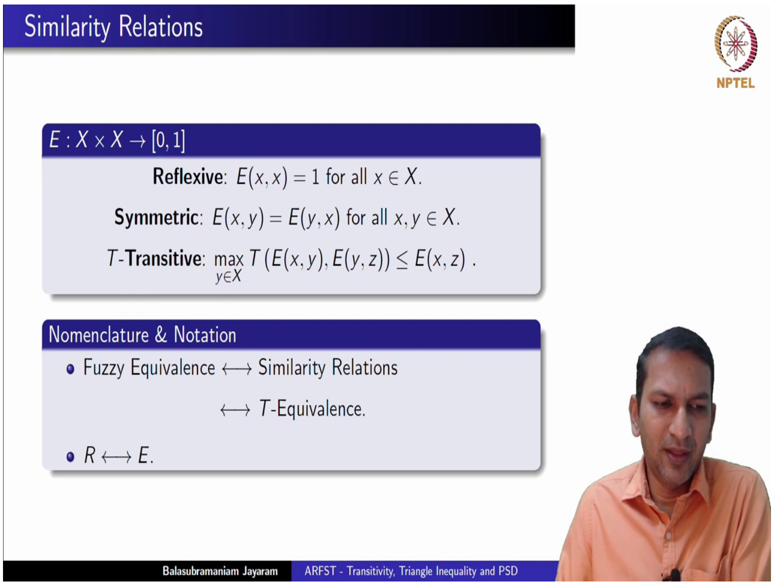

That that of similarity relations, kernels and metrics. We know what a similarity relation is; it is a reflexive, symmetric and T transitive fuzzy relation. Now note about the nomenclature and notation that we will follow in this lecture, we know fuzzy equivalence relations are also called as similarity relations. They are also called T equivalence relations. In the literature, this is also a term that is used to refer to fuzzy similarity relations; where it is clearly marked, what is the T and hence you talk about T equivalence where the T is the t norm.

Hence, just to stay true to the notation that you will find in the literature, we will also move from denoting a similarity relation by R to that of E. So, this is a small adjustment that we will make for this lecture alone.

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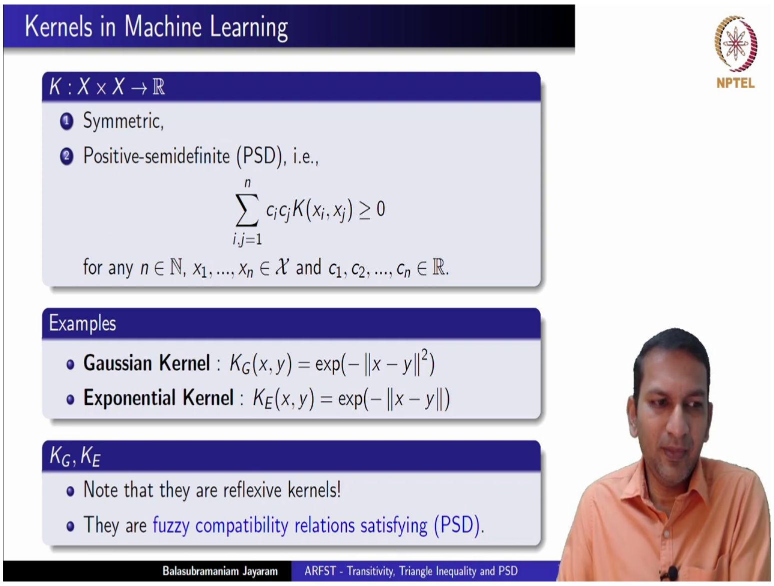

But, what are these kernels or kernel functions in machine learning? These are functions from given set X, X cross X to R, which are symmetric and satisfy positive semi definiteness. What do you understand by this PSD? That if you actually pick n points, any number of them for any value of n.

And construct the gram matrix, as K of x i , x j, we see that this inequality we will be valid for constants coming from R. Now let us look at a couple of examples. This is called the Gaussian kernel, which is e power minus norm x minus y whole square and there is another example, which is the exponential kernel, which almost looks like the Gaussian kernel, but its E power minus mod x minus y. Let us stay with these two examples a little longer. Firstly, we see that these are also reflexive kernels.

What do we mean by reflexive kernels? Just looking at them as binary relations, we see that K of xx is equal to 1 for both these kernels. This K G of xx is 1 and K E of xx is 1. Also, by their definition you see that, in facto, they are actually functions from x cross x to just the [0,1] interval; which means you can look at them as fuzzy relations and they are also reflexive and as kernels they are also symmetric.

So, essentially you could look at them as fuzzy compatibility relations, satisfying the positive this semidefiniteness.

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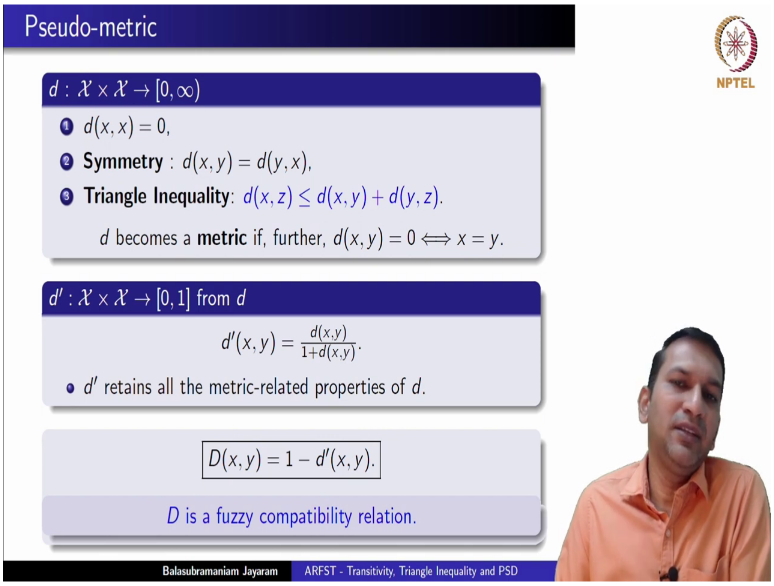

What is the pseudo metric many of us must already be aware of this. So, it is a function d from x cross x to 0, infinity such that, d of xx is 0, it is symmetric and satisfies the triangle inequality. We know that metrics are typically used where you would like to use a distance function. So, a pseudo metric is defined by these two properties a pseudo metric becomes a metric.

If it further satisfies this property; that means, d of x y is 0, if and only if x is actually equal to y. Now note that, this function d is actually from x cross x to r plus. Just like in the case of f generators or g generators, where we could find its non d generators; that means, normalize it, in case when f(0) is finite or g(1) is 1, we could do a similar thing, here too take this pseudo metric or metric d.

And we could obtain a normalized version of it, which we denote here by d dash as follows simply d dash is d by, 1 plus d. Clearly you can see that d of dash of xx is 0, d dash is symmetric and in fact, it can be shown, that d dash retains all the metric related properties of d. Now consider this function capital d, which is defined as 1 minus d dash. Once again this capital d is a fuzzy relation, clearly it is reflexive and symmetric.

Of course, we might have some property, related to that of transitive triangle inequality, through this 1 minus function, but currently it suffices to note, that this capital d is in fact, a fuzzy compatibility relation.

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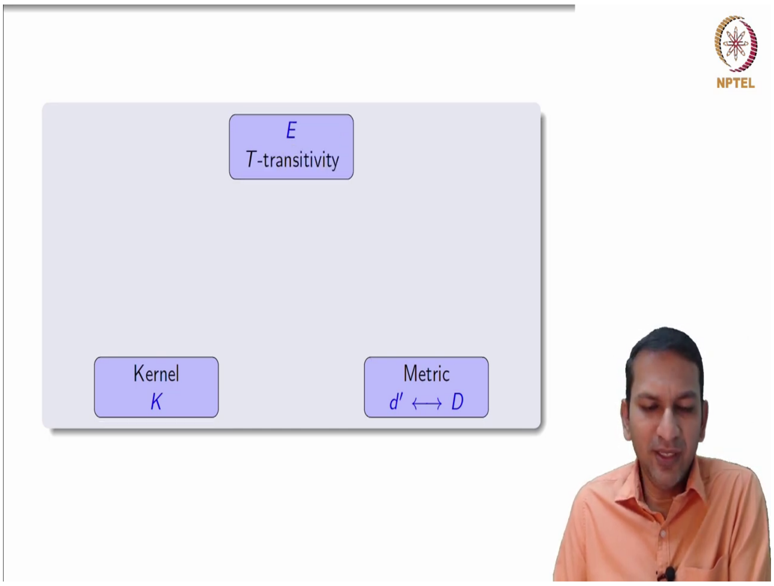

Now, moving forward, what is it that we are interested. So, we have a T equivalence relation, which is essentially a fuzzy similarity relation, satisfying T transitivity. We have the kernel K and the metric from which we can also obtain a dissimilar similarity kind of function a fuzzy compatibility relation.

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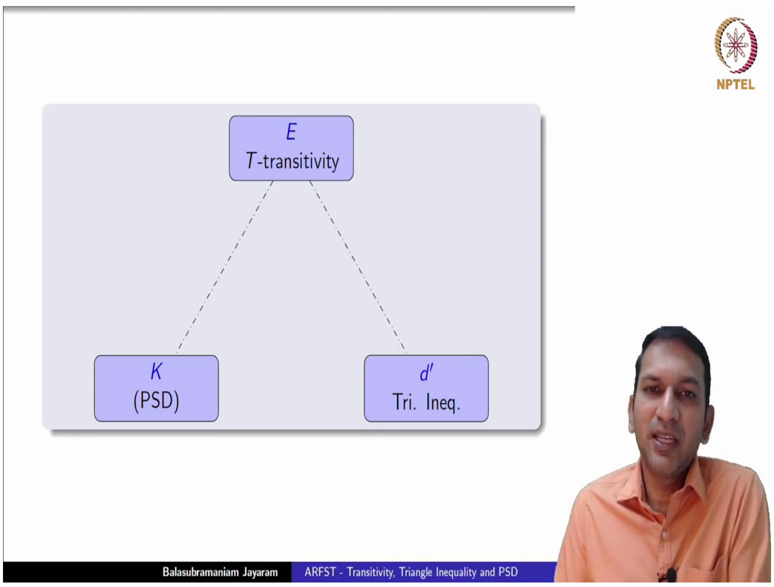

Now, if you look at it, in E what is important is T transitivity, in the kernel, what is important is the positive semi definitiveness, and if you are looking at the metric, what is important is the triangular inequality. We would like to see, if somehow these three properties can be related.

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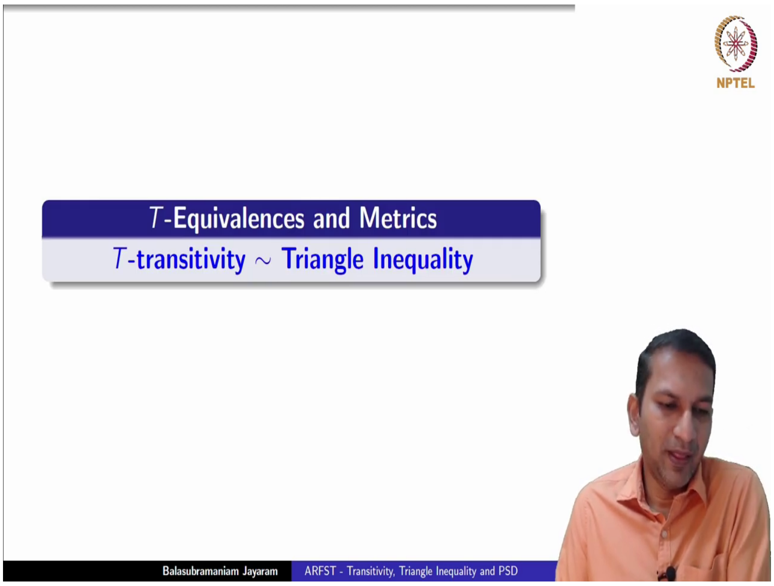

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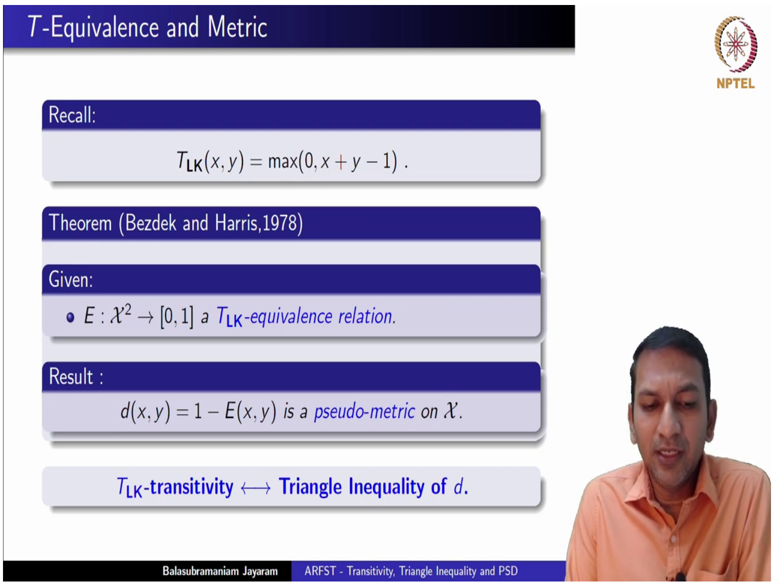

In this quest, let us start with T equivalences and metrics; and we would like to see if T transitivity can be somehow related to triangle inequality.

Recall, the Lukasiewicz t norm, which is nothing, but max of 0 , x plus y minus 1, Bezdeck and Harris, way back in 1978 proved the following this; they said, if you have a T equivalence relation, which is T transitive with respect to Lukasiewicz t norm. So, we call it a T LK equivalence relation; then, if you define d as 1 minus E, this d of x y is 1 minus E of x y, then this d is essentially a pseudo metric on x. There is a very interesting relationship between T LK equivalence relation and the pseudo metric.

So, essentially we are saying if it start with a T LK equivalence relation, you can in fact, get a pseudo metric ok.

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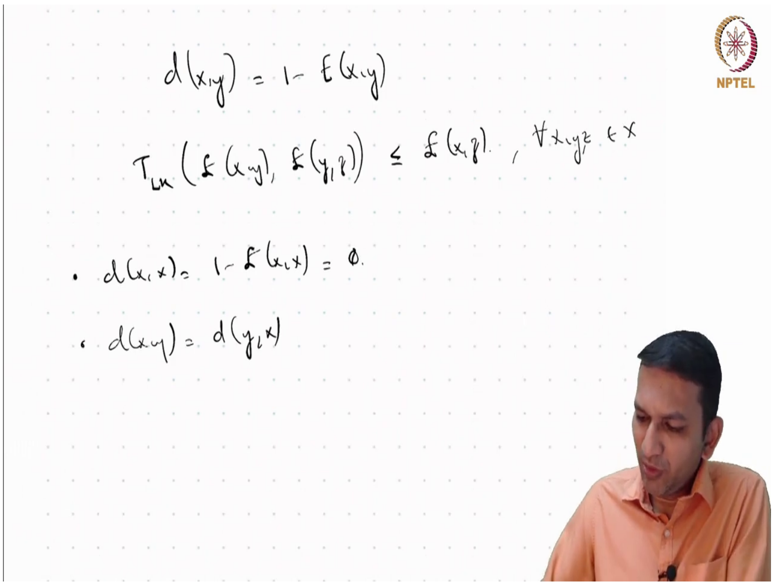

How do we show this? Remember we have defined d as 1 minus E of x y. We also know, E satisfies T LK transitivity; that means, T LK of E x y (Refer Time: 08:29) is less than or equal to e x z for all triples x y z coming from x. Now what we need to prove, is that this d is a pseudo metric clearly d of xx is equal to 1 minus E of xx.

Since E is reflexive with 0, clearly d of x y is also d of y x, because E is also symmetric, now what we need to prove, is the triangle inequality of the function d.

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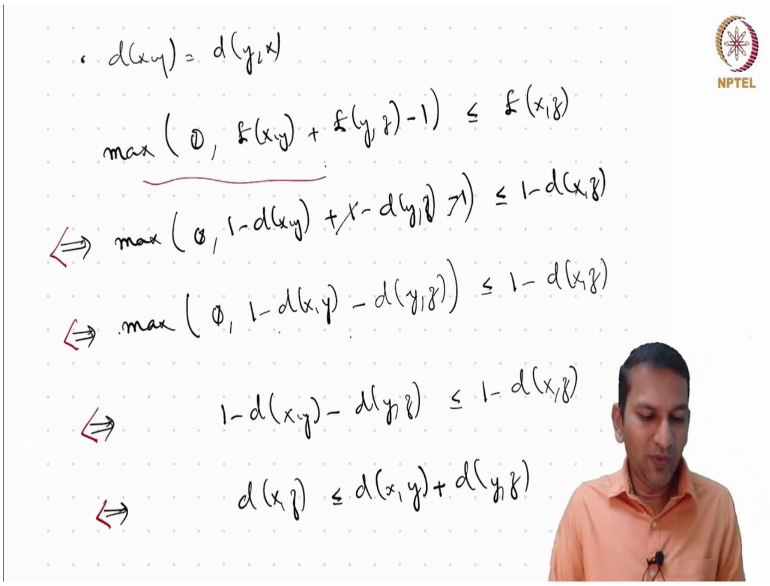

Now, let us look at this what are we given? We are given, that it is T LK transitive; that means, this property is valid. Now let us translate this what is this? This is nothing but, maximum 0 , E of x y plus E of y z minus 1, is less than or equal to E of x z. What does this imply?

This implies, by substituting for E, since d is 1 minus E. We could write E as 1 minus d. So, this is 1 minus d of x y plus 1 minus d of y z, minus 1, this less than or equal to 1 minus d x z. Now this implies, by cancelling out these two things, what we have is maximum of 1 minus (Refer Time: 10:31) less than or equal to 1 minus d x z, note that 1 minus d x z; d norm is 1 minus E. So, this is a function from X cross X to [0,1]. So, this is always greater than or equal to 0.

So, now here either it is greater than or equal to 0 or greater than equal to this value, which means from here we get, 1 minus d x y minus d y z is less than or equal to 1 minus d x z well. From here, we see clearly, that d x z is less than or equal to d x y plus this is essentially, the triangle inequality of d. Now, it is also very easy to retrace these steps.

So, now look at it, if you are given a d, which is which satisfies triangle inequality, then from there we obtain this inequality from there we could also add this, because d is actually a function from x cross x to [0,1], from here you can retrace this and we obtain this which is essentially the T LK transitivity of E. So, what we have now, is that if you have an equivalence relation, which satisfies T LK transitivity, that is a T LK equivalence relation, we have shown that, the T LK transitivity in fact, can be related to the triangle inequality of d where d is 1 minus E. So, this is one of the very first results, which related the transitivity T LK transitivity, with the triangle inequality of a metric a pseudo metric that you can obtain from E.

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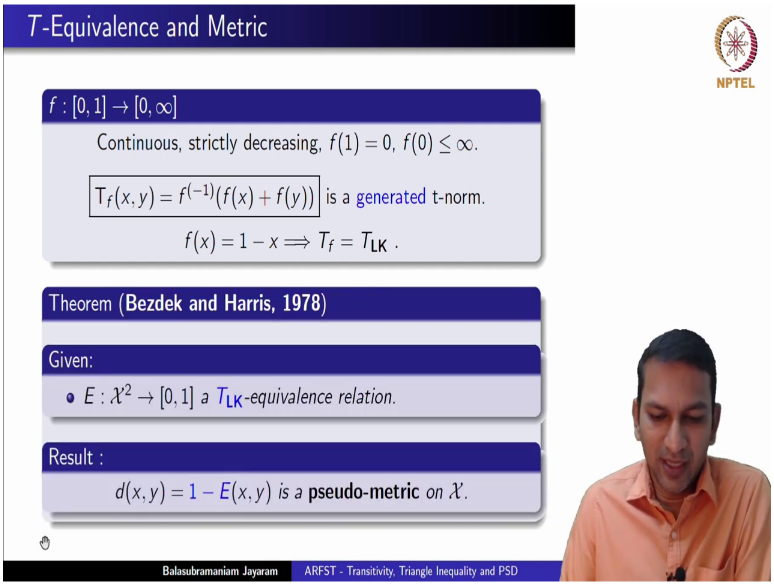

A quick recall. So, if you consider, the additive generators of a t norm, which are functions f from [0,1] to [0,infinity], continues, strictly decreasing, f(1) is 0 and of course, f(0) can either be infinite or finite, we know that, we can obtain a continuous t norm Tf in this fashion.

It is also called the generated t norm of f, now what is interesting is, when you look at f(x), is 1 minus x you can either look at it as a negation a fuzzy negation or also a generator of a t norm, and we know that this actually gives rise to the Lukasiewicz T norm. Note, this is the result obtained by Bezdeck and Harris in 1978.

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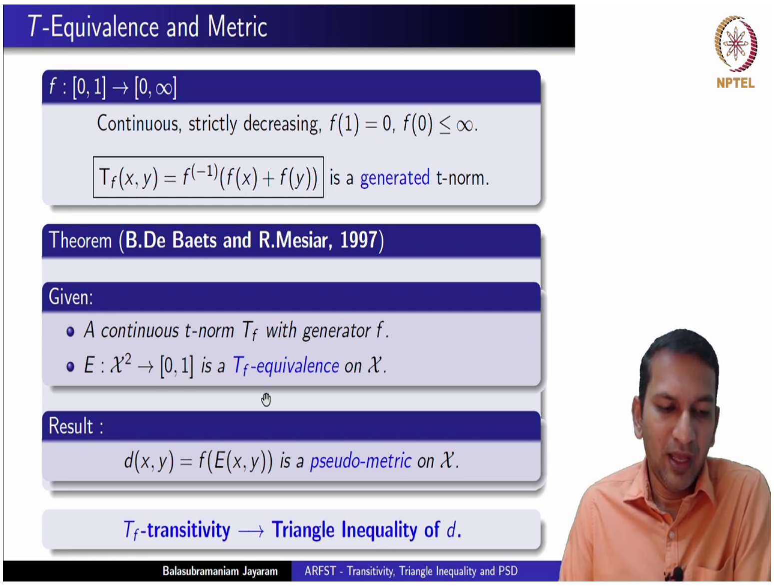

Bernard De Baets and Radko Mesiar they were quick to notice this and said look we are just using one particular generator, is it possible to generalize this.

So, instead of asking for an equivalence T LK equivalence relation, they took a continuous t norm TF with additive generator f and look at a T F equivalence relation on x. So, pick an E which is a T F equivalence relation on x, where T F is an f generator continuous t norm. Then if f were 1 minus x we are taking it as 1 minus c. So, now, change it f of E, and we were able to simply show, that if you construct T f equivalence relation, then by using this formula, d as f circle E, f composed of E, we where were able to obtain a pseudo metric on x. So, given a TF equivalence relation, you can obtain the pseudo metric they also showed. So, this also shows, that TF transitivity is again related to the triangle inequality of d as we have obtained here.

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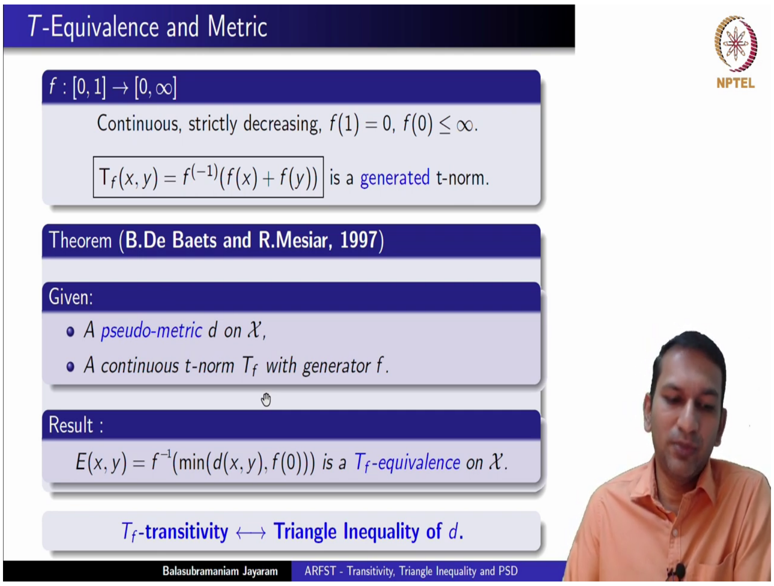

They were also quick to see that, on the converse, if you start with the pseudo metric d and pick any continuous t norm, which is generated from f and define a relation E as follows f inverse of min d x y , f(0), then they were able to show this is in fact, a TF equivalence relation on x.

So, essentially taking the triangle inequality and then be able to relate it to the TF transitivity. That is we clearly see, that this TF transitivity is somehow clearly it has a one to one correspondence to the triangle inequality of d as we have defined this pseudometry.

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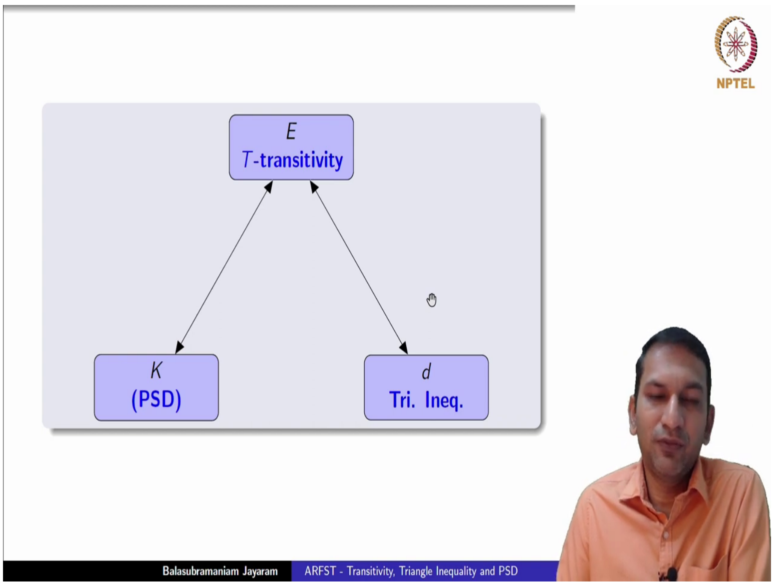

So, in our quest to relate these three concepts, what we have seen is, T transitivity leads to triangle inequality and in turn from triangle inequality we are able to obtain T transitivity. Now what about T transitivity and positive semi definiteness, are there some correspondences?

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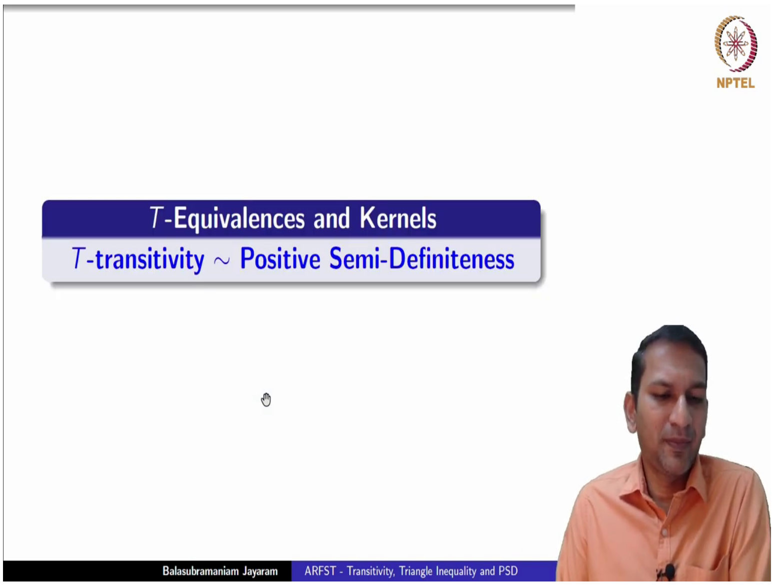

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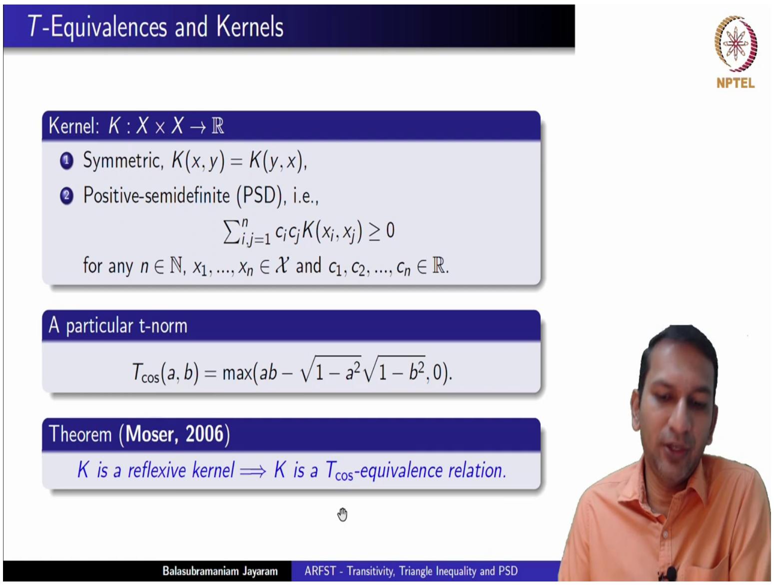

Let us look at them, recall this is the definition of kernel, let us look at a particular t norm; it is called the t cos t norm, because of the generators you will not get into the nomenclature, but this is a particular t norm. In 2006, Bernard Moser proved the following result. He showed that, if you have a reflexive kernel; that means, a kernel by kernel we understand it to be symmetric and positive semi definite.

If it is also reflexive, just like the Gaussian of exponential kernel, he proved, that In fact, that kernel looked at we have seen that Gaussian and exponential kernels can be seen as fuzzy compatibility relations with positive semi definiteness, was able to show such reflexive kernels. In fact, satisfy t cos transitivity and they become t cos equivalence relation; that means, with respect to this t norm, t cos they actually become a t cos equivalence relation.

So, first time we are seeing that kernel can also be at least some of these kernels can also be seen as fuzzy similarity relations or t equivalence relations for a particular t norm.

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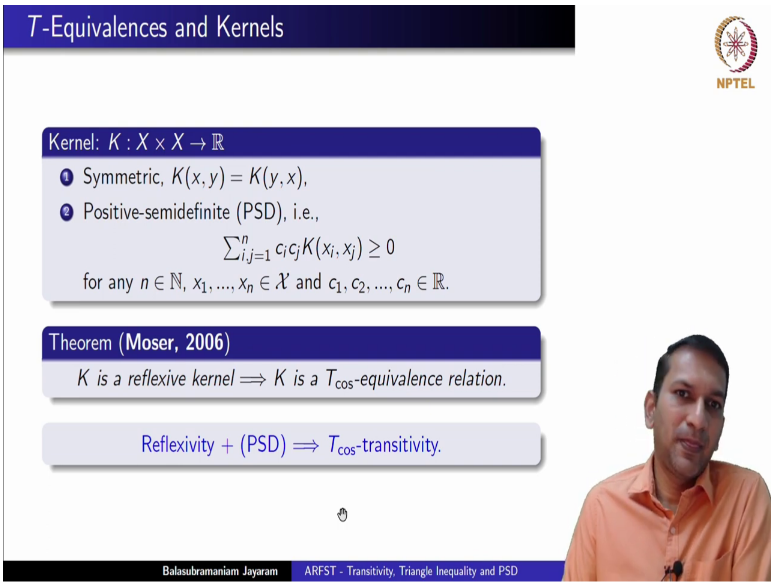

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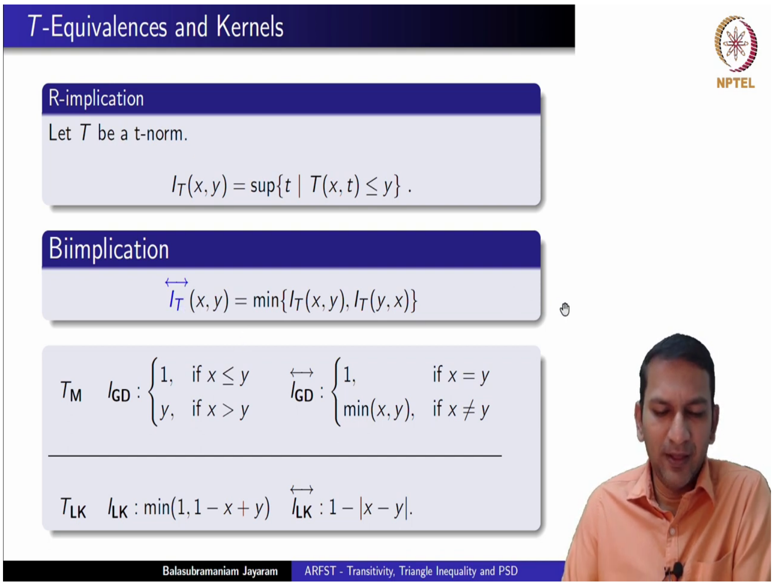

Now, what this means is somehow reflexivity along with positive semi definiteness leads to t cos transitivity, but we could also ask further questions in the quest to help us in answering them or exploring them lets revisit some of the things that we have seen with respect to their implications.

This is this must be pretty fresh in your mind, if you take a t norm, the corresponding R implication can be defined as follows, it is denoted as I T, IT of x y is nothing but supremum of all those Ts is that T of x t is less than equal to y. Now that is the R implication, we can also obtain a biimplication, from a given implication, but in this case we will just concentrate on obtaining this from the family R implications.

So, we will denote it like this, I T with double headed arrow on top, it is defined simply as min of I T of x y , I T of y x. So, this is called a biimplication; essentially it says minimum of a implies b and b implies a, if you consider the minimum t norm, we know that the corresponding R implication is the radon nik implication and the y implication obtained from the Godel implication is almost the minimum function except when x is equal to y it is 1.

Rest of the places it is minimum, if you actually consider the Lukasiewicz t norm, then you get the Lukasiewicz implication and what you get here as its biimplication is this function, which is 1 minus mod x minus y. Perhaps this rings a bell alreadyz, because in previous lecture we have seen this function also to be an equivalence relation. In fact, T LK equivalence relation. What does it mean presently, we will let just look at them as examples for biimplications. Now why did we introduce these.

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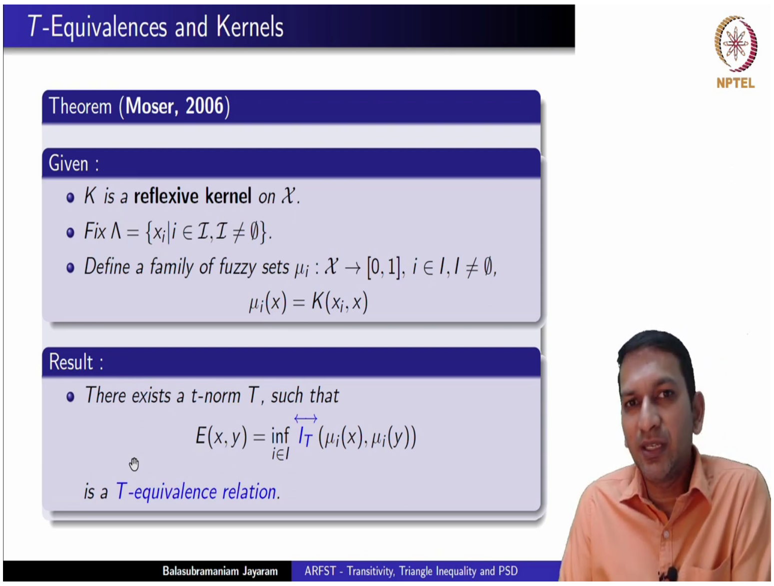

Moser, in the same paper also went on to show, that if you take a reflexive kernel on x and fix arbitrarily a few points from x, the set x say X i varying from 1 to n and for each one of these indices, if you define a fuzzy set. So, essentially you get a family of fuzzy set as follows, mu a x is equal to k of X i , x. So, now, if you are thinking of a K as a compatibility relation, because it is a reflexive kernel.

So, it is also symmetric, if you are thinking of it as some kind of fuzzy compatibility relation, all you are doing is we are fixing an Xi and you are looking at what happens with respect to that Xi when you use K, how are the other elements of x related to Xi under the similarity relation that you could get from the kernel K. So, that will give rise to these fuzzy sets here.

Now, if you consider this family of fuzzy sets, you are able to show that there will exist a t norm T, such that if you define e like this, infimum over this entire index set and over this index set. The biimplication is applied to these families mu i of x , mu i of y and these two point x , y, if you define the relation E like this, then it will be a fuzzy similarity relation which can satisfy T transitivity with respect to this T; that means, essentially its some kind of an existential reason where they have shown.

We have shown that, there will exist a t norm, such that, if you define the relation E like this, it will be a T equivalence relation. So, first time we are seeing that from reflexive kernel, you could also get T equivalence relations.

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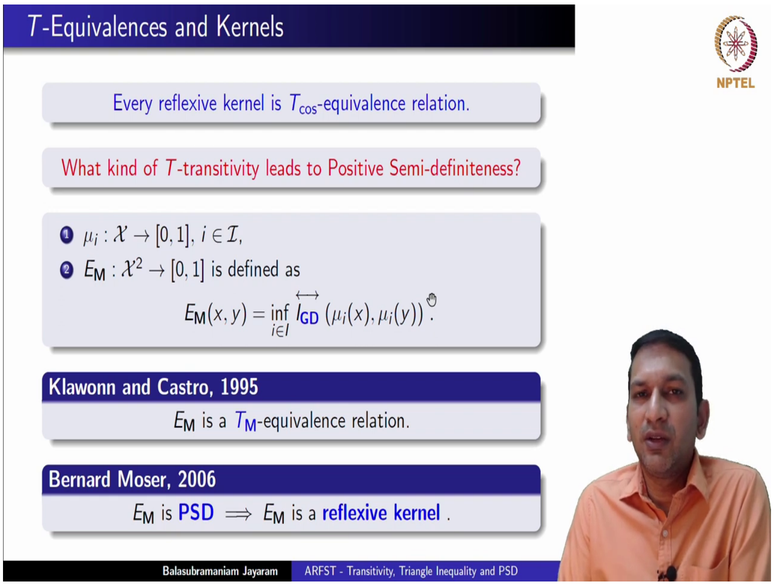

Well, we know that every reflexive kernel is T cos equivalence relation. The question is if you want the reverse, what kind of t transitivity will lead to positive semi definiteness. There is a question which has not been answered in general, but allow me to present you some existing reasons.

So, once again let us consider a family of fuzzy sets start with a family of fuzzy sets and define this relation, essentially as how we have defined earlier in the previous slide infimum over i this index set, but now we are using, the biimplication obtained from the Godel implication, what is interesting is if you consider this relation E M Klawonn and Castro showed that, this is in fact, a Tm equivalence relation. Moser, in the very same paper, showed that such a relation also satisfies positive semi definiteness.

Now, note that this reflexive and symmetric relation and it also satisfies PSD which means E M is in fact a reflexive kernel. So, there are T M equivalence relations, which are also kernels. So, it does not mean that kernel reflexive kernel we know that it leads to T cos transitivity, but it is not only T cos transitivity relations, equivalence relations which will give rise to kernels.

So, what have we seen so far, we saw earlier, that t transitivity can be related to triangle inequality of matrix defined in a particular way. And now we are seeing that, T transitivity can also be related to the positive semi definiteness property of the relation.

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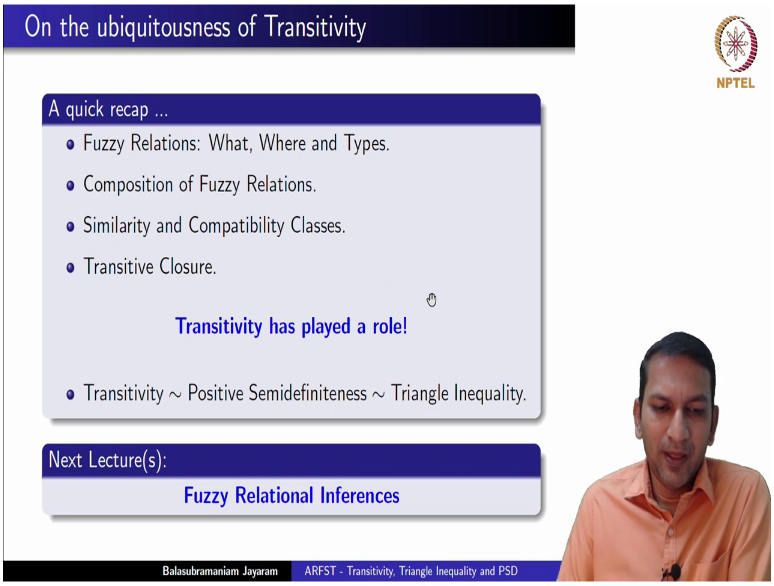

Well, the last two lectures including this, must have shown you the ubiquitousness of transitivity. In fact, if you were to take a look at the lectures that we have looked at in this week, the week 5 lectures, we have dealt with fuzzy relations, but everywhere transitivity has come up to the front it has had a role to play. For instance when we talked about different types of binary fuzzy relations transitivity was very much.

When we discussed about compositions, it was again there when we discussed about the utility or usefulness of similarity and compatibility classes in getting groups, again we saw the important role played by transitivity. And which led us to in fact come up with an algorithm to obtain transitive relations from given fuzzy relations. We came up with the concept of transitive closure and also proposed an algorithm for it and looked at an algorithm to obtain transitive closure.

So, all of this has shown us, the important role transitivity plays. And in this lecture, what we have seen is some correspondence between transitivity and positive semi definiteness and triangular equality. What next in next week, we will look at fuzzy relational inferences.

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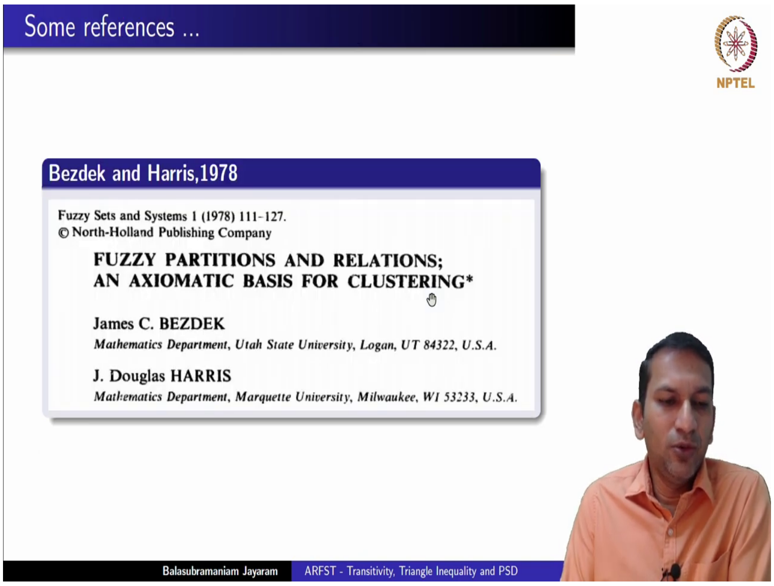

Meanwhile, allow me to present you the works that we have referred to in this lecture. This was the paper in which Bezdek and Harris showed that T LK equivalence is related to triangle inequality of the pseudo metric that you obtain as 1 minus E.

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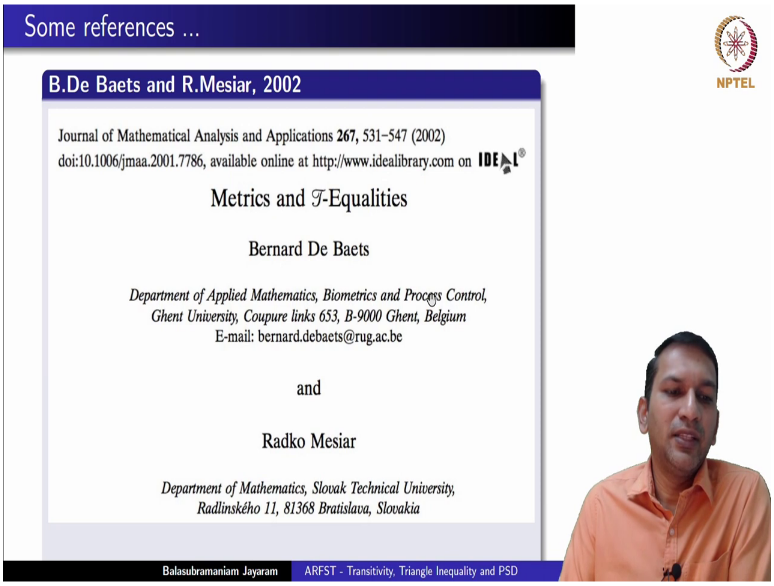

This was the first of the papers by Bernard Dae Baets and Radko Mesiar where they generalized the result of Bezdek and Harris and they also have further results exploring the same. It will be very interesting to look into this, they have a lot more results also about obtaining equivalence relations from given metrics in different ways.

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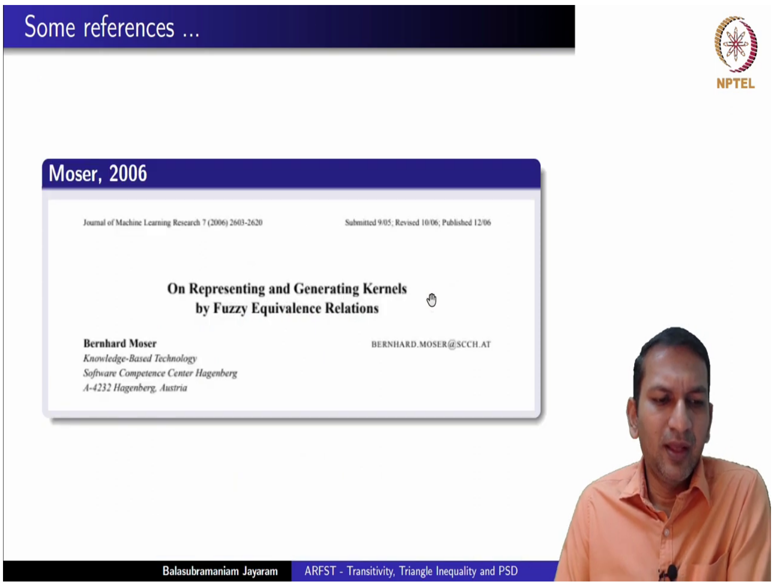

This is the paper, that we have referred to showing the equivalence between reflexive kernels or in some sense obtaining equivalence relations from reflexive kernels.

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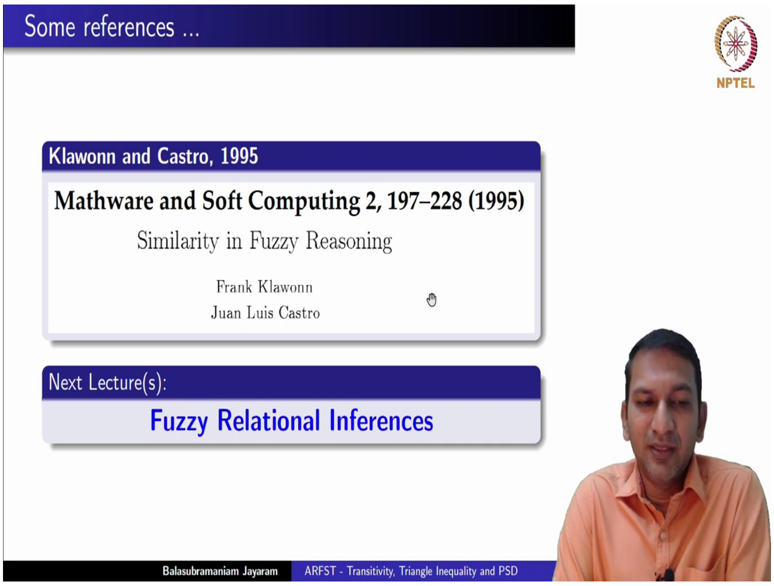

There is a work that has been published in the journal of machine learning research, this is the paper by Klawonn and Castro live back in 1995, wherein they had shown that the relation E M that you have come up with as infimum of the biimplication obtained from the Godel implication is actually E M transitive, which was shown later on to be also a reflexive by Moser right, once again in the next week, we will look at fuzzy relational inferences, glad that you can join us today for this lecture.

Looking forward to meeting you again soon, in the next lecture.

Thank you.