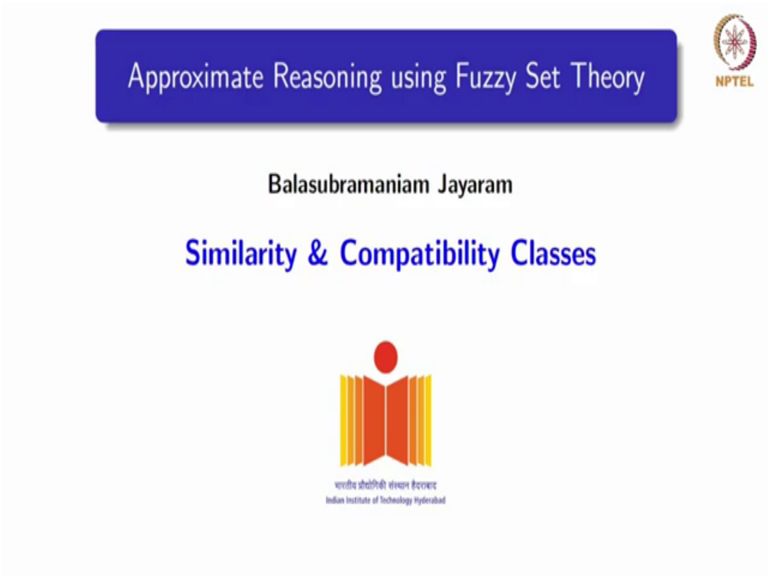
Approximate Reasoning using Fuzzy Set Theory
Prof. Balasubramaniam Jayaram
Department of Mathematics
Indian Institute of Technology, Hyderabad

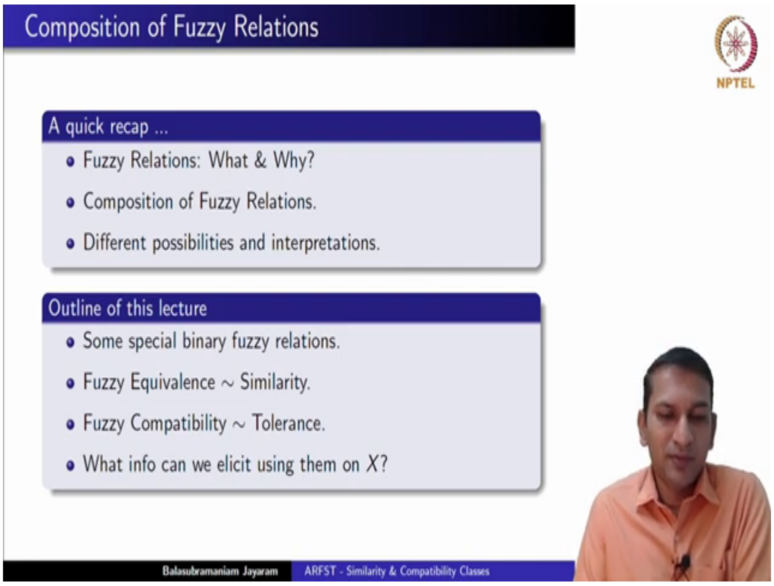
Lecture - 25
Similarity & Compatibility Classes

(Refer Slide Time: 00:15)



Hello and welcome to the third of the lectures in this week 5 of the course titled Approximate Reasoning using Fuzzy Set Theory a course offered over the NPTEL platform.

(Refer Slide Time: 00:31)

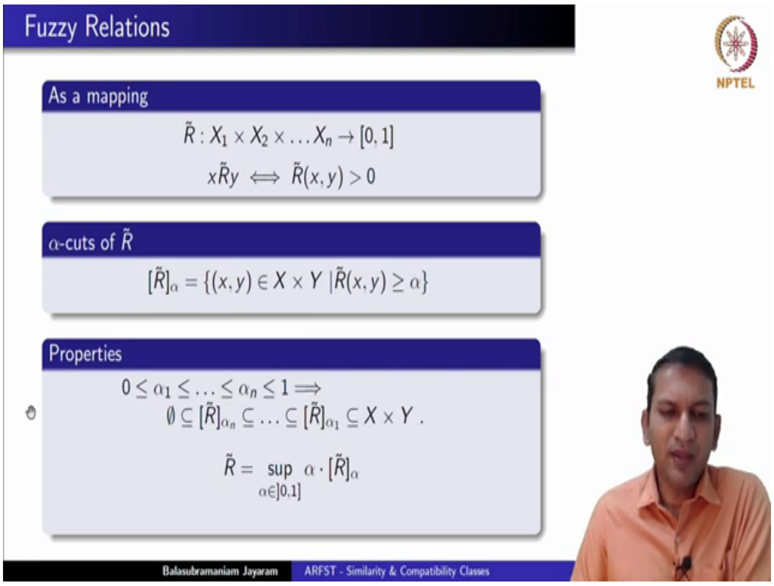


In the last two lectures we have seen what are fuzzy relations and why we need them. We also looked at composition of fuzzy relations and we saw there were different possibilities and interpretations on how we could compose given two fuzzy relations. In this lecture we will look at two special binary fuzzy relations that we have introduced that of similarity and compatibility relations. So, you may recall that fuzzy equivalence relation is called the similarity relation and fuzzy compatibility relation is also called as tolerance relation.

Now what is our interest in looking into these special binary fuzzy relationship relations in depth? We would like to know if based on these relations whether we can unearth some useful information about the elements of the set x this is going to be our quest in this lecture.

(Refer Slide Time: 01:43)

(Refer Slide Time: 01:48)

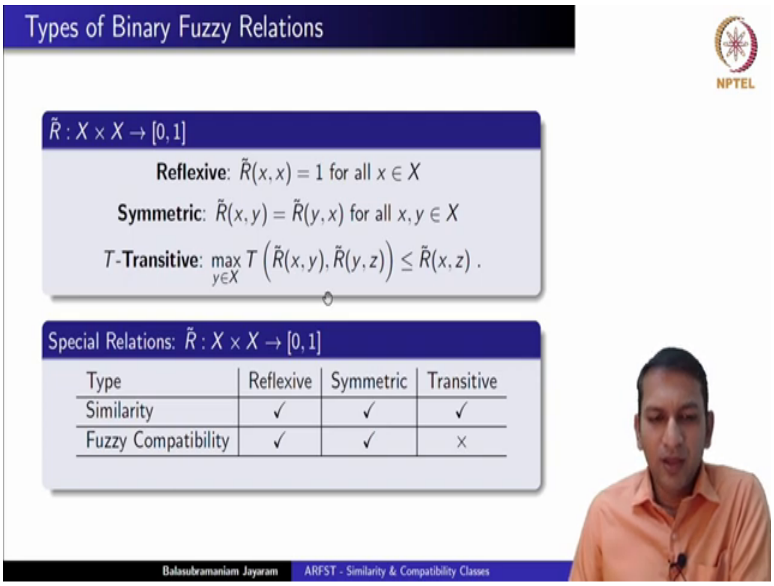


Let us begin with a short recap of what fuzzy relations are what we have seen so, far under fuzzy relation. We know a fuzzy relation is a mapping from the Cartesian product of sets to the unit interval [0,1] and in the case it is a binary relation we say an x is related to y under the fuzzy relation. If x, y belongs to the support of the relation means R of x y should be greater than 0.

Since they can also be viewed as multi dimensional fuzzy sets we could apply many of the concepts that we had on fuzzy sets for instance we could talk about the alpha cuts and clearly when you look at the corresponding level set and for each one of those alphas set are from the level set if you consider the alpha cuts then what you are going to pick up R elements which belong to the relation to degree greater than or equal to alpha.

And this essentially will give you subset of the Cartesian product of the spaces and in the case of binary relation what we pick are actually ordered pairs from x cross y and we also know that as you keep increasing the alpha then the alpha cuts start to become smaller and smaller and also that their fuzzy relation itself can be resolved or composed using the corresponding alpha cuts. Now these are some concepts that will play a major role in this lecture here.

(Refer Slide Time: 03:27)

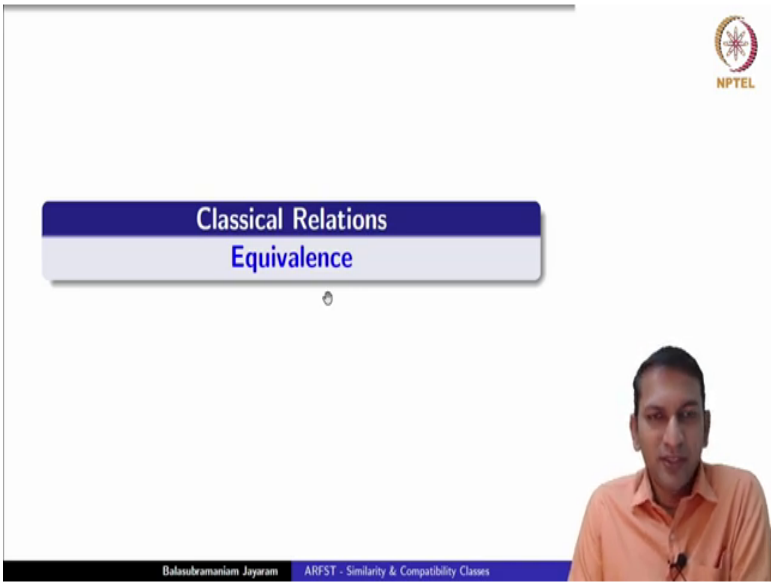


We have also seen different types of binary fuzzy relations they were course we have generalized them from the classical binary relations exact concepts. So, we say it is a fuzzy relation is reflexive if R(x, x) is 1 for every x it is symmetric if R(x, y) is equal to R(y, x) we do not talk about transitivity instead we talk about T-transitivity where T is a t-norm and it is defined like this. In the previous lecture we have seen how actually we have generalized this from the classical transitivity of relations.

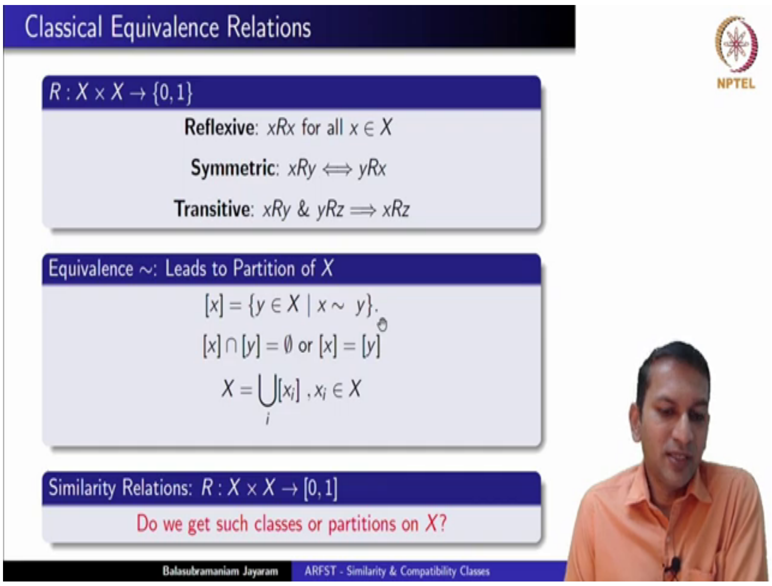
So, it is given as follows that max over y element of X such that T of R x y, T of R x y, R y z should be less than or equal to R x z. So, essentially we are saying that the connection strength between x and z should be at least as much as the connection strength of x and z through some y that is there in the support of the relationship ok.

So, in this lecture we are going to look a little deeper into two special binary fuzzy relations the similarity relation which is essentially the fuzzy equivalence relation which means it is reflexive symmetric and transitive and also the fuzzy compatibility relation which is only reflexive and symmetric, but not transitive.

(Refer Slide Time: 05:05)



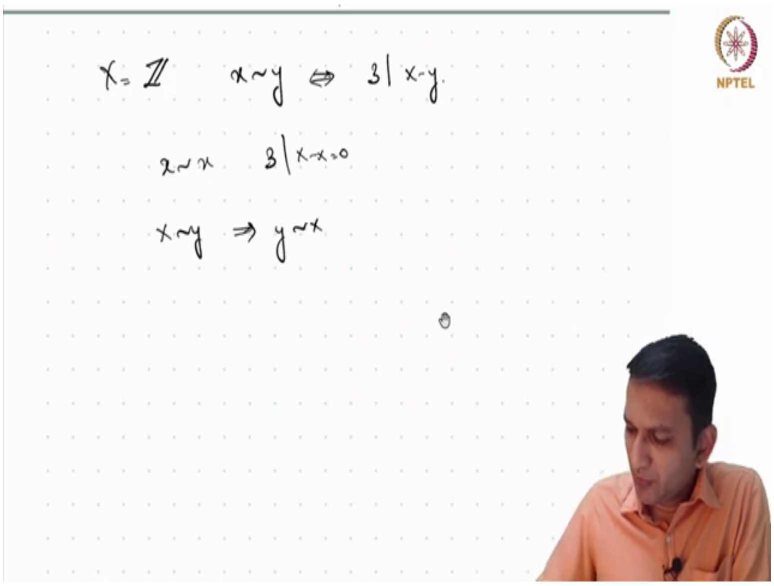
(Refer Slide Time: 05:17)



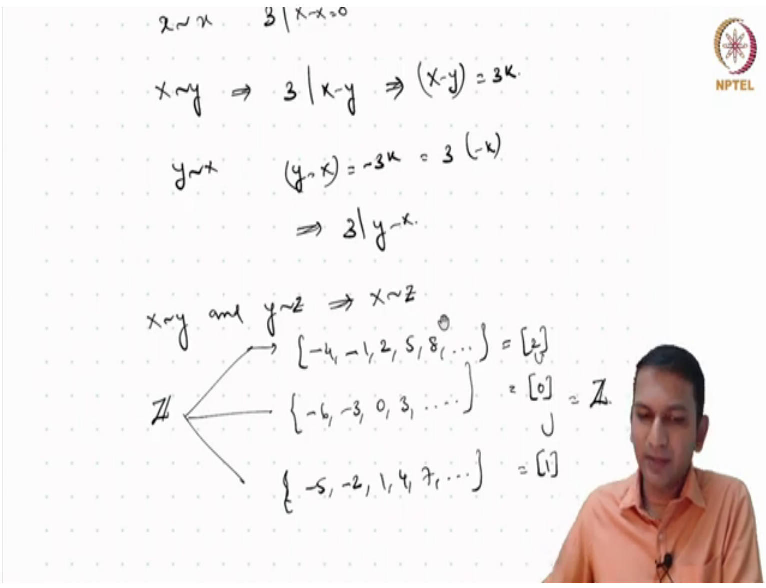
Now, to study similarity relations means, we need to know what usually are the properties of the classical equivalence relation. Why do we study equivalence relations? So, this is something that is familiar to us all by norm. A classical equivalence relation is something which is symmetric reflexive and transitive. Now what is the need for an equivalence relation? Well, an equivalence relation always leads to a partition of the underlying set. How?

Let us denote by the symbol x in square brackets the equivalence class of an element x. It is set of all those elements y such that x is related to y under this equivalence relation. Now we know that if we pick up two equivalence classes either they are disjoint or they are actually equal. Not just this the x itself can be written as a union of these disjoint equivalence classes; that means, these equivalence classes form a partition of x.

(Refer Slide Time: 06:20)



(Refer Slide Time: 06:54)



Let us look at a simple relation let X be equal to Z and let us relate x to y if and only if 3 divides x - y this is something that we have seen yesterday clearly it is reflexive x is related to x because 3 divides x - x which is 0. If x is related to y then this implies y is related to x why because if x is related to y this means 3 divides x - y means x - y is actually some 3 times k.

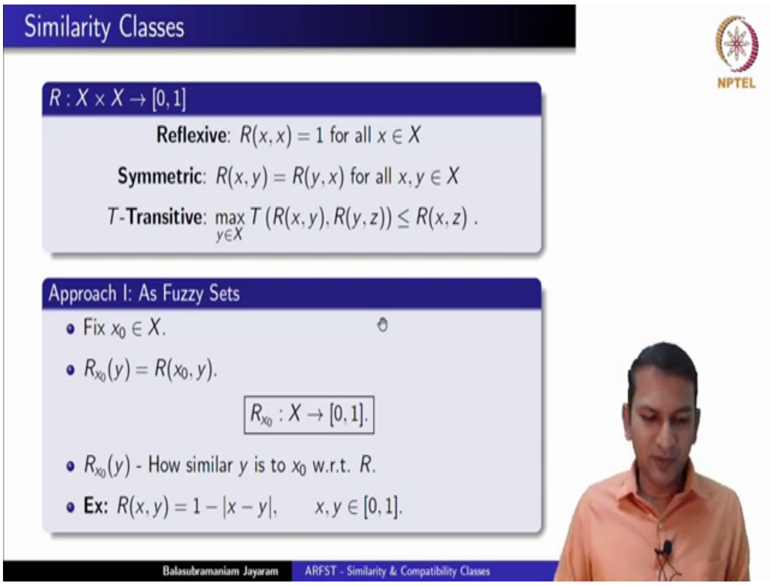
Now, this also implies y - x y is related to x because we can say that y - x is equal to - 3 times k or 3 times - k which implies that 3 divides y - x. Similarly, you can extend that if x is related to y and y is related to z then this implies x is related to z. What this does is it is familiar to us now the entire Z it partitions into 3 pieces essentially the set the multiples of 3, then shifted copies of them - 5, - 2, 1, 4, 7,… and - 4, - 1, 2, 5, 8,….

And you see that this is essentially the equivalence class of 2, equivalence class of let us say 0, equivalence class of 1. And union of them will actually give us Z. Now this is perhaps one of the most important properties that favor having a classical equivalence relation or an equivalence relation on the set X itself. The question immediately arises if we have fuzzy equivalence relation a similarity relation do we have this (Refer Time: 08:51)? Do we get such classes or partitions on X?

(Refer Slide Time: 08:55)

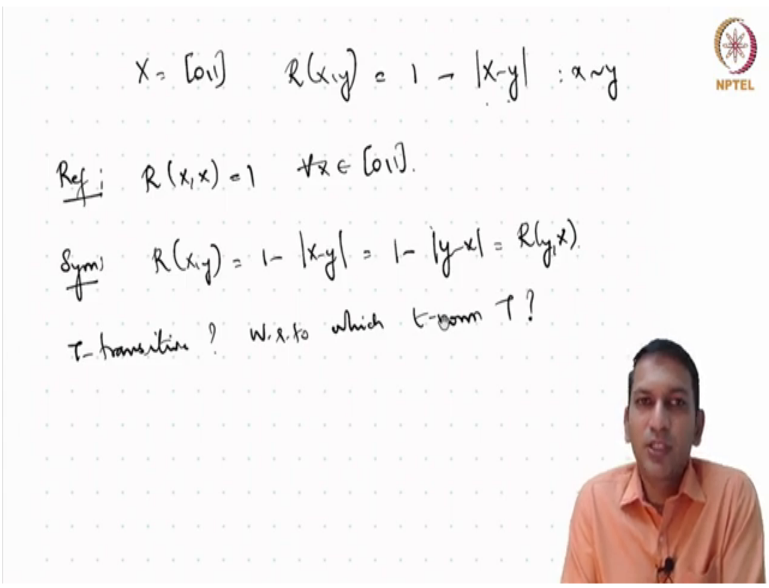
Let us look at the similarity classes the classes that we get from fuzzy similarity relations in slightly deeper detail.

(Refer Slide Time: 09:08)



Similarity classes themselves can be approached in two different ways recall relation is fuzzy equivalence relation if it has these three properties. You could look at similarity classes as fuzzy sets themselves, how? Let us fix an x naught element of X remember we are talking about binary fuzzy relations. So, fix and x naught from X and define this function R subscript x_0 of y as R(x_0, y). Clearly, then this R_x0 is nothing but a fuzzy set on X itself. Now what does it tell us? R x naught of y tells us how similar y is to x naught with respect to R.

(Refer Slide Time: 10:05)

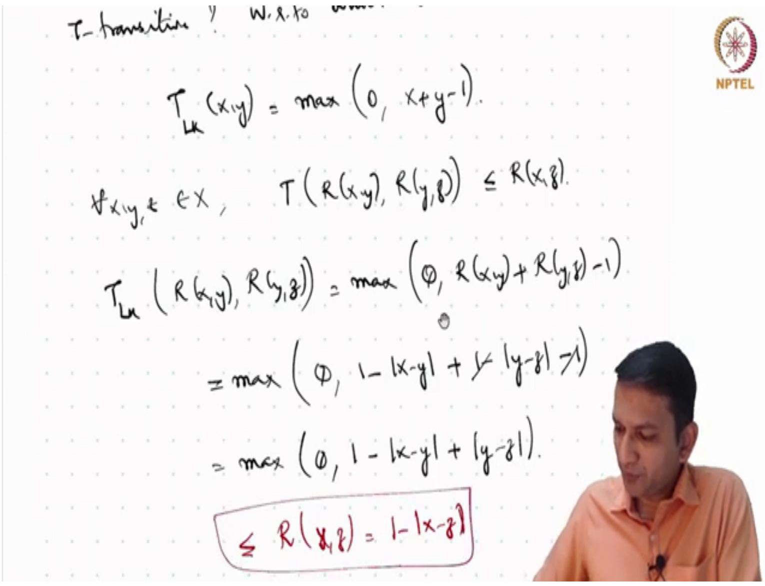


Let us look at one particular example. Consider this relation we have X to be [0,1] and R of x, y is given as 1 -|x – y| you could also think of this as x related to y. So, now, we are asking the question is this a similarity relation on the set x which is [0,1] then we need to show this reflexive by reflexivity we mean R of x, x should be equal to 1 for all x in x in this case it is [0,1].

Note that this is true when x is equal to x, x - x is 0. So, it is one. Now is it symmetric? Well, R(x, y) clearly is 1 -|x – y|, but this you can also write it as 1 -|y – x| which is R(y, x). So, it is also symmetric now the question is this relation T-transitive, if so, with respect to which t-norm T? Now this is not an easy question to answer if you are given a relation as was told from the matrix itself you can say whether relation is reflexive or symmetric, but transitivity is not very obvious.

Now, as it happens, due to some interesting theoretical results that are available we can show that this particular relation that we have is in fact, T-transitive with respect to the Lukasiewicz t-norm.

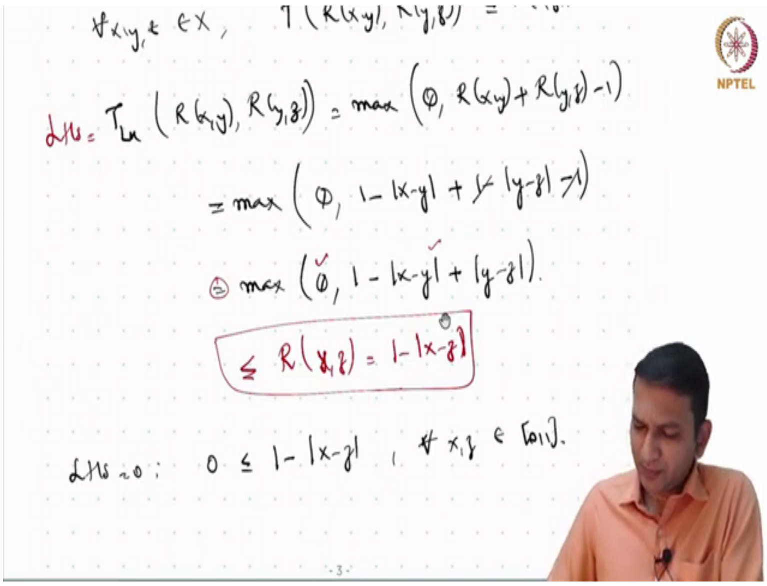
(Refer Slide Time: 11:47)



Recall what is the Lukasiewicz t norms this max of 0, x + y - 1 for T transitivity what should we show we need to show for every x, y, z element of X we need to show that T of R x y, R y z is in fact, less than or equal to R of x, z and in this case we want to show for this relation R that we have here and the T LK t norm; Lukasiewicz t norm.

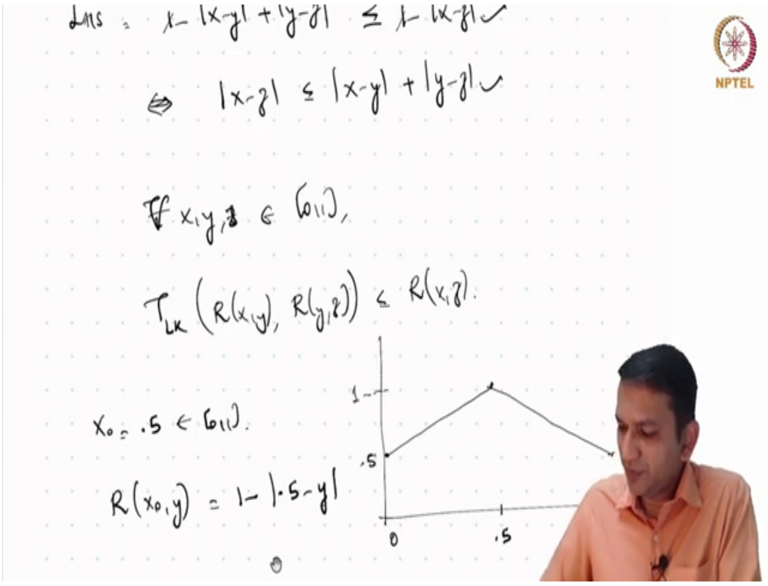
So, let us start with this. So, what is T_LK of R(x, y), R(y z)? Is nothing, but maximum of 0, R of x, y + R of y, z - 1. Now what is R? Maximum of 0, R of x, y is 1 - | x - y |+ 1 - |y – z| - 1 this is nothing but max of 0, we cancel out this one this one here. So, 1 -|x – y| + |y – z|. Now our task is to show that this is actually less than or equal to R of x, z which is 1 - |x – z|. So, this is what we need to show.

(Refer Slide Time: 13:48)



Now, look at this from here we know that either it is 0 or this if it is 0, I will call this LHS. If LHS is a 0 then we know that 0 is always less than or equal to 1 - |x – z| for any xz coming from 0, why is this so? Note that R of x, y R is a function from [0,1] square to [0,1]. So, it is always greater than or equal to 0. So, when LHS is 0 then it is not a problem.

(Refer Slide Time: 14:20)

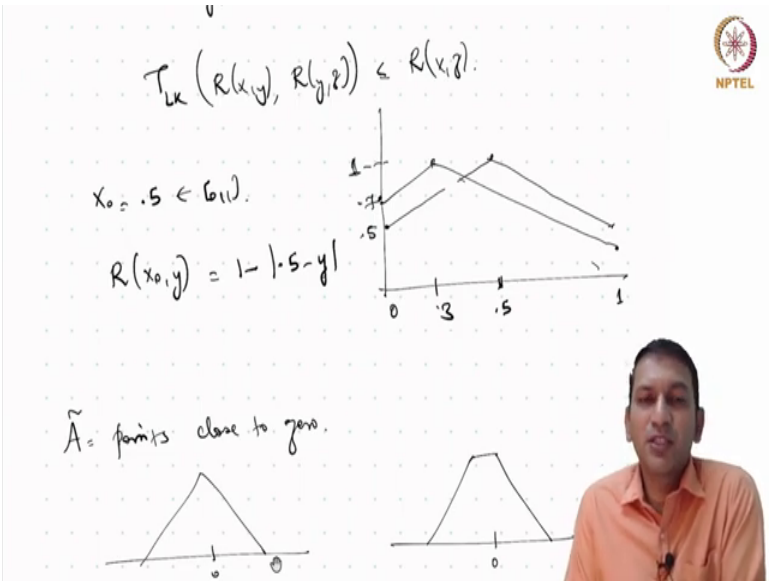


But what if LHS is actually 1 - |x – y| + |y – z|? The question is, is this less than or equal to 1 -|x – z|. Well if this were true this will be true if and only we cancel this and you can write it as |x – z| is less than or equal to |x – y| + |y – z|, but this is true we know this is true because |x – y| is nothing but a metric on the set x and metric satisfies triangle inequality.

In this case mod satisfies triangle inequality which means this is true and hence this is true. And hence what we have is for this relation we have that for any for all x, y, z element of [0,1] we have that T LK of R x, y, R y, z is less than or equal to R of x, z; that means, this relation that we are looking at is in fact, T LK transitive. Now why did we go to such lens? To show that this is in fact, a similarity relation. Now think of this let us fix x naught to be 0.5 which belongs to 0 and look at what is R of x naught, y this is nothing, but 1 - 0.5 - y.

Now, how would it look like? So, this is 0 1, this is 0.5. At 0.5 it is 1 and at 0 it will be 0.5 and at 1 again it will be 0.5. So, if you draw this, this is a fuzzy set that you get when you fix x naught to be 0.5, what does it tell you? Remember this is the value 1, this is the value 0.5.

(Refer Slide Time: 16:43)



Now, if you recall in one of the earlier lectures perhaps in the first week when we are defining fuzzy set as a concept and given a concept when we are coming up with mathematical expressions that could actually represent that concept using a fuzzy set one of the examples that we have taken up was concept A tilde points close to say 0. And what did we come up with?

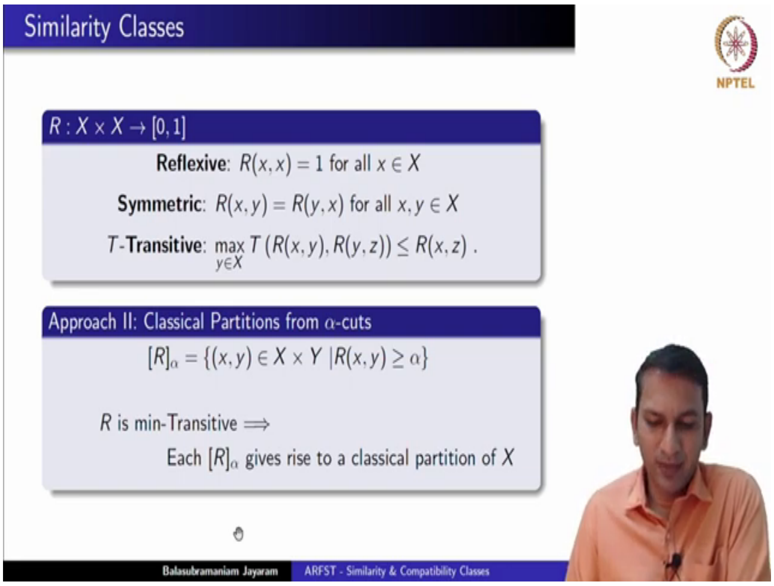
We came up with such kind of a triangular function perhaps in some cases 0 even may have had a trapezoidal kind of a fuzzy set. Now what this represents is in some sense similar at 0.5 if you move away from 0.5 on either side the membership the similarity of those points those values to 0.5 starts to dip. So, you could look at this as expressing the fact how similar a point or how close a point y is to 0.5.

Now, instead of 0.5 if you take it to be 0.3 all we are doing is we are actually just displacing this fuzzy set. So, at 0.3 it is 1, at 0 it will be 0.7 and at 1 it will be. So, you see here from this relation if you fix one of the elements because it is symmetric it does not matter whether you fix the first component or the second component what you get is a fuzzy set which represents how similar that point is with respect to the concept as captured by the relation.

So, in this case where we are dealing with 1 - |x – y| perhaps, it tells you how far away a point is to that particular x naught. Now this should not come as a surprise because what we have actually is |x – y| which measures distance and 1 - of that can be looked at as how similar the these two points are. In fact, we will see this in little more detail in the next two lectures to follow, distance and similarity are in some sense dual of each other well coming back here.

So, we see a similarity class; that means, each row or column in a similarity fuzzy similarity relation the matrix of the fuzzy similarity relation can be looked at as a fuzzy set expressing how close or how similar an element is to the pivotal element of that row under the relationship R. What is the second way to look at similarity classes?

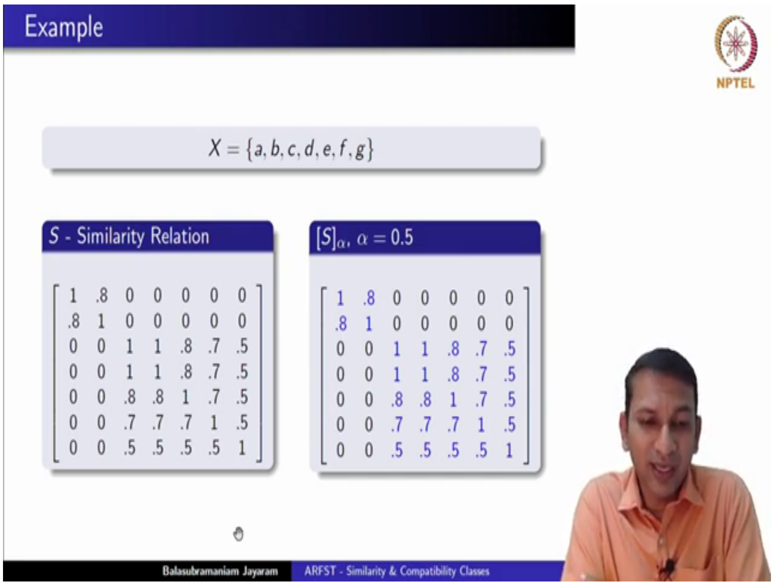
(Refer Slide Time: 19:47)



We would like to see if in some sense it also can do what classical equivalence relations were able to do that is can we get classical partitions? Remember we say that from given this fuzzy relation the alpha cuts will essentially give you subset of the Cartesian product. So, do they form? So, using that can we actually come up with partitions of the underlying space X.

Recall this is your alpha cut what can be shown is if your similarity relation is actually min transitive, then it is possible to come up with classical partitions for almost every alpha that you have there. That means, from the corresponding alpha cuts it is possible to come up with a partition of the set X let us see how this can be done with an example.

(Refer Slide Time: 21:00)

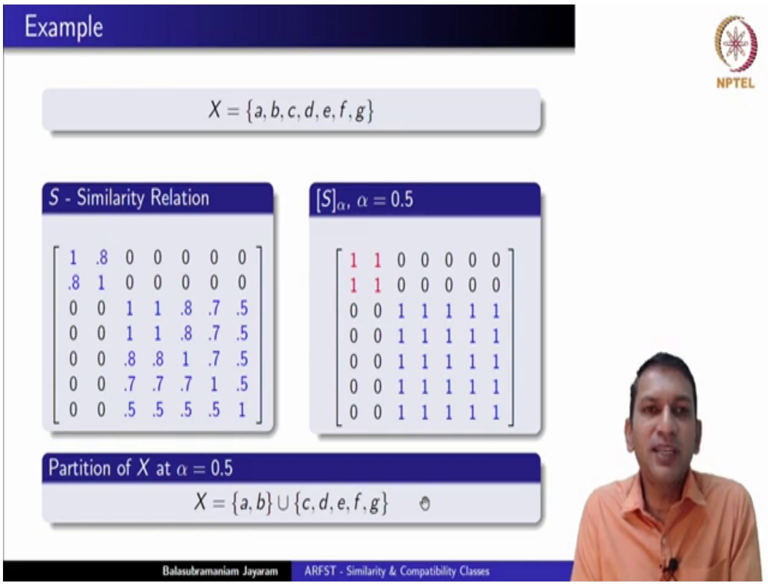


For the moment consider X to have these seven elements a, b, c, d, e, f, g let us take a similarity relation. So, that this is a similarity relation can be verified, but not without much difficulty because it is a 7 cross 7 matrix it is not easy to look at all possible combinations now. So, let us for the moment believe and agree that this relation is in fact, min transitive and it will be validated as we go along because of the partition that we have. So, let us take this similarity relation and let us start to look at the corresponding alpha cuts.

So, we know that for this similar relation, the level set will consist of the unique alphas which are 0.5, 0.7, 0.8 and 1. So, we could have four such alpha cuts on this fuzzy relation which is a similarity relation or if you only highlight those values which are above or greater than or equal to 0.5. So, your alpha cut would look like this from just highlighting the values which are greater than or equal to 0.5.

So, essentially we are saying that these are the elements pairs of elements which are related to each other to at least a degree of 0.5. Now, it is not apparent how we can get a partition of on hence from this matrix.

(Refer Slide Time: 22:25)



So, now let us look at this matrix in a slightly different form; that means, let us consider the corresponding alpha cut which is written in the form of an indicator matrix; that means, wherever the value is greater than or equal to 0.5 we will put a 1. So, the corresponding alpha cut would look like this. It is clear from this matrix which is actually a classical relation on x where it is reflexive symmetric that it is transitive can also be seen by the block nature of this matrix.

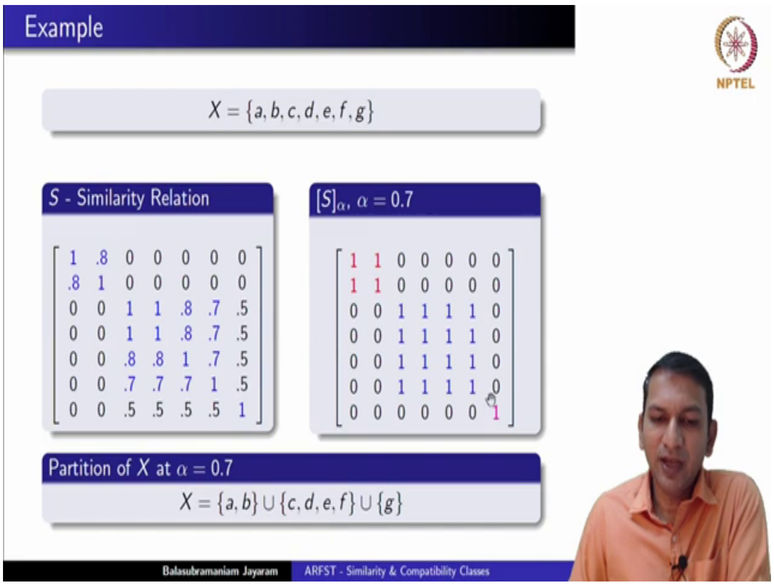
For instance, if you take this element it is related. So, this is a this is b a is related to b and not related to anybody else then b is also related only to a. If you look at this it says c is related to everybody else d c is related d e f and g and what is interesting is d e f and g in turn are related only to these five elements and nobody else that is what we mean by an equivalence class. What you mean by an equivalence class? x is related to y means x is related to every element that y is related to in turn y is also related to every element that x is related to.

So, clearly you see a block matrix kind of a structure here and this allows us to write the following. So, we can obtain a partition of x at the level of alpha equal to 0.5 what would this be? So, look at this block structure means we have X is a, b. So, these two elements are related to each other to a degree greater than or equal to 0.5 and all of these elements c, d, e, f, g they are related to each other any pair of element that you take from here will be related to the degree at least 0.5.

Now, X consists of only these seven elements and we already have a partition at this resolution and look at this, this is a partition that you have caught on X when you do not when all you insist is that the similarity between any two elements in thin the class should at least be 0.5. You do not care whether it is above 0.5, 0.7, 0.8 you only care that it is greater than or equal to 0.5 you do not insist on anything and you actually cut off anybody who is not related to another person another element to a degree less than 0.5.

Of course, this is the smallest alpha here in the level set. So, we would have almost everybody here, but now what happens if you increase it and move on to the next alpha let us look at 0.7; that means, all of these are not going to come into picture.

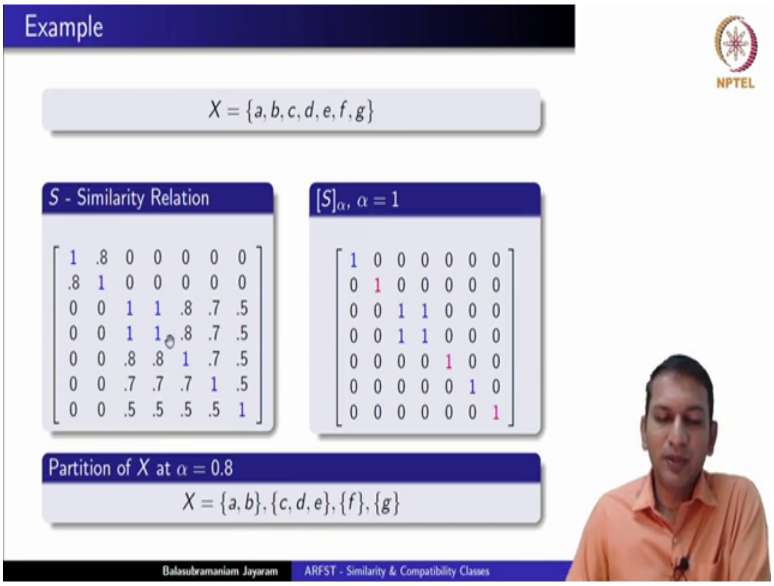
(Refer Slide Time: 25:25)



And if you draw the corresponding matrix of the alpha cut this is what you see all of these are 0; however, this one will remain because it is reflexive. Now once again you see because of transitivity we get a nice block matrix infrastructure and for this and this resolution of alpha the partition that we get is a, b union c, d, e, f and g. Now, nicely it is in fact, a resolution of an earlier partition; that means, this is one partition now we are actually making it much more granular finely granular.

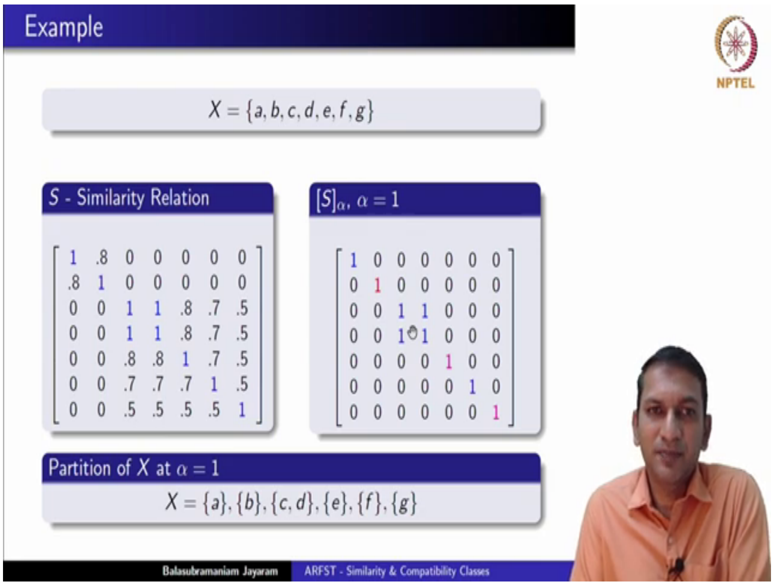
So, at 0.7 this is how the partition of X could be; that means, if you insist that in the classes any two elements should be related to at least the degree 0.7 then these are the ones that are sticking together g separates out a and b are still sticking together now let us up the ante and go to the next alpha level resolution level. So, all of these elements do not play a role and accordingly that is seen here you see the corresponding indicator matrix is what you get. Now, you see here that the partition is further refined.

(Refer Slide Time: 26:34)



Now, f also separates out from the previous set, f is not related to at least one or more of them in this set to a degree greater than equal to 0.8 and so, it separates out, but a and b still stick together because they are related to each other at least to degree 0.8. Now if you go to 1 then these are the elements that will remain and the corresponding indicator matrix is this.

(Refer Slide Time: 27:04)

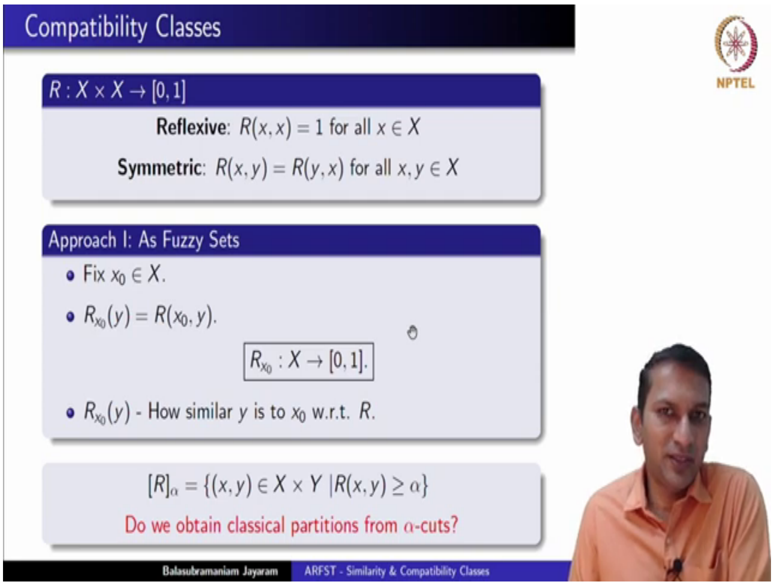


And from here you see that even a and b start to separate, but c and d they actually stay together; that means, c and d they are related to each other under this relationship to a degree 1. In essence they are saying that there is not much difference between c and d well. So, given as fuzzy equivalence relation a similarity relation, we may not be able to get a single partition, but at every resolution we are able to partition the set X and when you threshold it.

So, essentially what we are doing is we are thresholding at particular alpha and because of transitivity min transitivity of similarity relation what we can ensure is, this the matrix that we get the relation the classical relation that you get after thresholding is in fact, transitive and it leads to equivalence classes or partition of the set X.

(Refer Slide Time: 28:03)

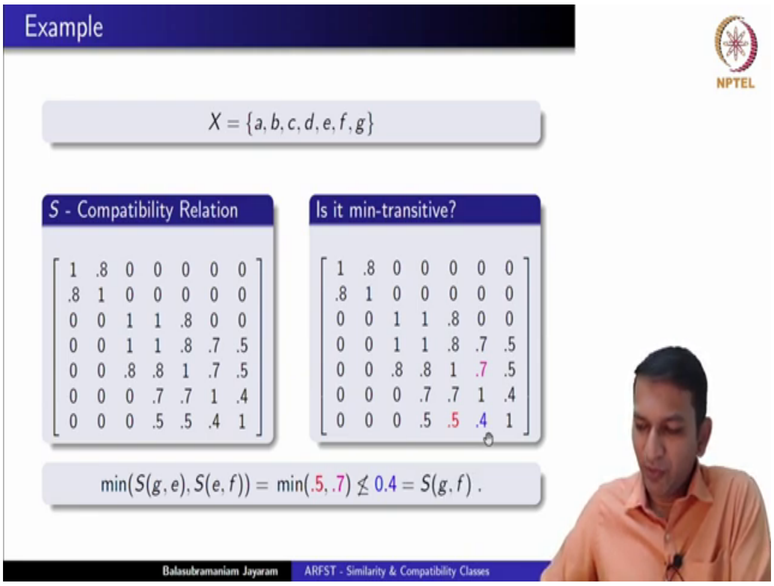
(Refer Slide Time: 28:11)



Now, let us look at fuzzy compatibility relations which are only symmetric and reflexive they are not transitive. Can we obtains a similar concept of compatibility classes here? Yes, we can. For instance once again if you fix an element x naught, you see that it essentially gives you a fuzzy set on x and you could still say that it measures the similarity between any element y and x naught under the relationship R.

This is clear because it is only a fuzzy set we do not have any further restrictions. Now, the question is what about partitions from the corresponding alpha cuts? Do we obtain classical partitions from alpha cuts?

(Refer Slide Time: 28:54)

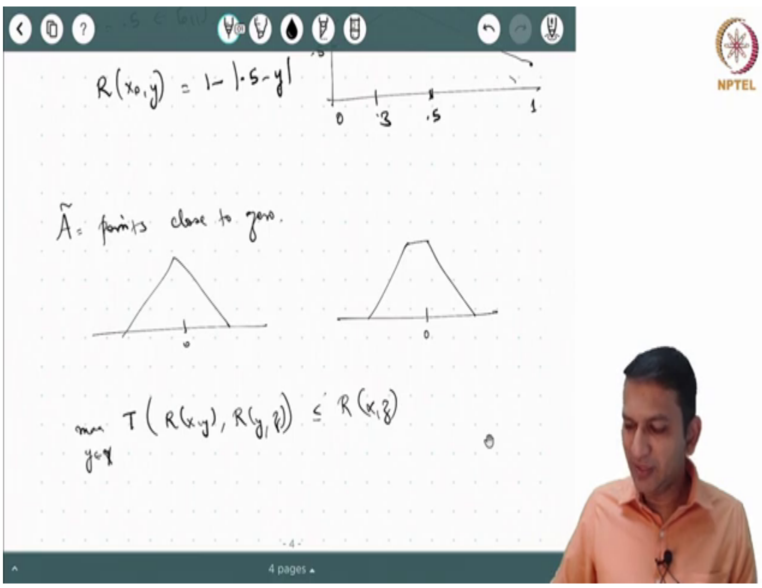


1	.8	0	0	0	0	0
.8	1	0	0	0	0	0
0	0	1	1	.8	0	0
0	0	1	1	.8	.7	.5
0	0	.8	.8	1	.7	.5
0	0	0	.7	.7	1	.4
0	0	0	.5	.5	.4	1

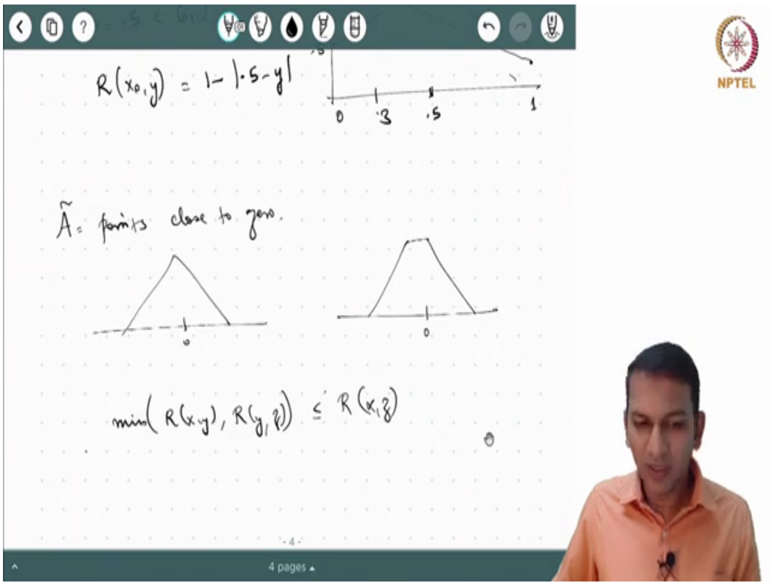
Well, sadly know we will see why with an example. Once again we consider the set x to be a set of these seven elements a, b, c, d, e, f, g and consider this relation S which is a compatibility relation. Clearly, it is reflexive and also symmetric, but the question is it min transitive? Well to prove transitivity is difficult to prove it is not transitive perhaps it is easier because we need to produce just one triple where the transitivity does not hold.

Luckily enough we are able to find such a triple here look at these values. So, you have a, b, c, d, e, f, g here and a, b, c, d, e, f, g here and look at these three numbers. So, 0.5 is actually S of g, e, 0.4 is S of g, f and 0.7 is S of e, f. Now if you look at minimum of S of g, e, S of e, f is equal to minimum of 0.5, 0.7. Clearly, this is not less than or equal to point four which is actually S of gf.

(Refer Slide Time: 30:10)



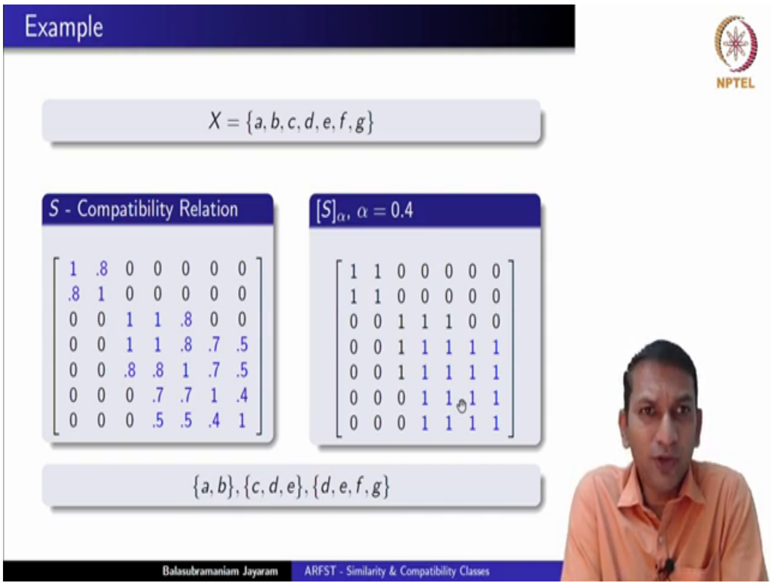
(Refer Slide Time: 30:42)



Recall what we want is for transitivity for T transitivity we want that t of R of x, y, R of y, z should be less than or equal to R of x, z; that means, R of x, z should be at least as big as T of R x, y, R y, z. In fact, for every y that is what we want we also put a max here, but even for one y if this is not valid then we know that it is not transitive. In this case we are looking at min transitivity which means this T is min and what we have shown is there exist at least one triple for which this inequality is not held which means this is not min transitive ok.

So, now as was stated earlier you could look at the first row third row and say ok this is giving me a fuzzy set which tells me how much similar other elements of the set x are to me which is c. So, if you take the last row it tells you how similar other elements of the set of x are to the element g that remains the approach 1 is valid here, but the question now is from the alpha cuts can we get partitions of the set X? And we have shown now this is only a compatibility relation at least it is not min transitive.

(Refer Slide Time: 31:33)



Now, once again let us look at the alpha cuts. Now there are five distinct alphas here 0.4, 0.5, 0.7, 0.8 and 1. So, let us look at the indicator matrix for the value of alpha being equal to 0.4 now this is what we get here. Now this does not seem to have very clear block matrix kind of a structure of course, you see here a, b they are related to each other to degree 0.4 and nobody else.

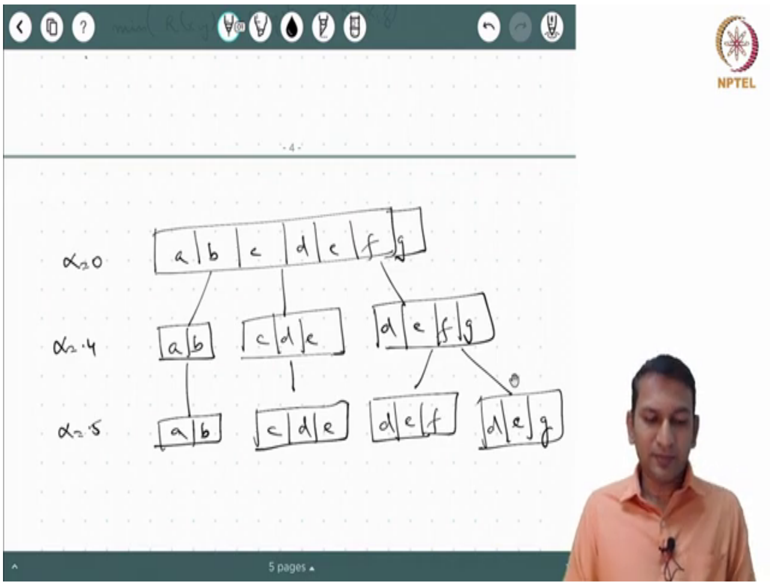
So, in that sense they appear like an equivalence class because remember equivalence class means that x is related to y, then it should be related to every element that y is related to and in turn y should be related to every element that x is related to. It appears that that happens with this these two elements a b; however, if you look closely we also have this kind of a block now what does this say?

How are we highlighting some blocks among this huge among this sub matrix? All of these elements c, d and e they are related to each other to a degree of at least 0.4. Now remember if d is related to e and e is related to f then we want that f also should be related to c; c is related to d, d is related to e then we want that c is also related to e. Now all of them have from this set any pair of elements that you take they will all be related to each other at least of the degree of 0.4 so; that means, this again forms a similarity class or a compatibility class in this case.

Now, we also have other kinds of blocks look at this all of these elements share the same relationship amongst themselves; that means, if you consider d, e, f, g any pair of elements in this d of d, e, f, g will be related to each other at least to degree 0.4. Thus, we have these three compatibility classes. Clearly compatibility these compatibility classes they form a cover of x; that means, their union is x; however, they are not necessarily disjoint.

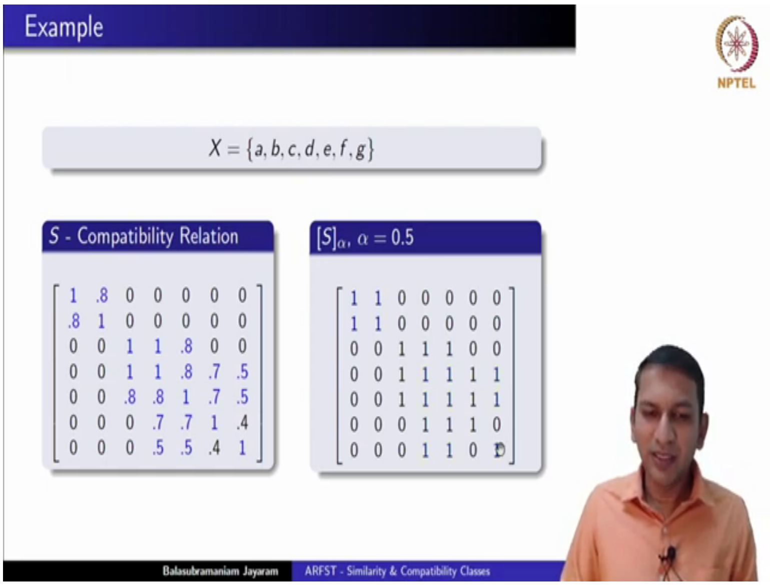
And hence they do not form a partition they only form a cover now allow me to record them differently because when you go one level up it shows something else also.

(Refer Slide Time: 34:04)



So, what we have is we have this set x with seven elements. Now, if alpha is equal to 0 or any value less than 0.4 all of them will form the compatibility class. However, at alpha is equal to 0.4 what we see is these are resolving like this we have a b branching out, we have c d e forming another compatibility class and we have also d e f g which forms a compatibility class at the resolution level alpha is equal to 0.4.

(Refer Slide Time: 35:00)



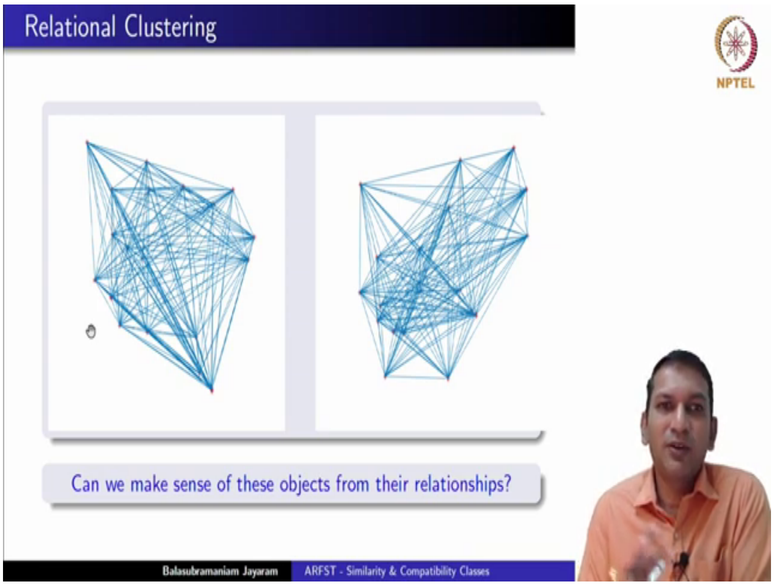
Now, let us move one step further let us take alpha is equal to 0.5 which means these two elements do not come into picture and the corresponding indicator matrix looks like this.

Now, are there some blocks that we can find here? That means, self-contained set of elements which are related to each other and every element is related to everybody else within the set up to a degree 0.5. Once again we have these two these 3 also remain that is c, d, e now you see that d, e, f, g we have only d, e and f here, but interestingly we also have d, e and g. So; that means, what has happened is a, b they have continued to remain together c, d, e also have continued to remain together.

Now, interestingly from d, e, f, g, d, e, f have split like this and we also have another compatibility class which is d, e and g. So, while increasing the alpha cuts resolution level alpha, we have more compatibility classes we see that they are not disjoint they are overlapping. Now this is happening because this relation is only reflexive and symmetric, but not transitive. So, transitivity min transitivity is very essential if you want to actually have a partition on X now the question will arise.

So, what if we have only compatibility classes which anyway form a cover of X? Well, the question is how much information are you able to gain from these compatibility classes? Typically, it is well known that if you have partitions then you are essentially able to group the elements of X. Now with this overlapping sets of compatibility classes if you are able to intelligently interpret them yes fine but otherwise we are at a loss on how to unearth some information from this compatibility classes? So, typically what we want is transitivity.

(Refer Slide Time: 37:47)



Now, let us also look at a particular practical application. These days it is common to talk about social network analysis wherein you have lots of nodes representing perhaps people or objects which are related to each other. And what you have is essentially a graph perhaps a weighted graph the moment you have weights, you could always normalize them and bring them down to say between 0 and 1 the unit interval.

So, essentially what you have is weighted graph whose adjacency matrix looks like a fuzzy relation. Now if you have only this we know that typically these relations are symmetric. That means, if x is related to y to some extent y is related to x same extent for instance this graph if you take two of these nodes they could indicate on a given set of attributes what is their preference to what extent they agree or disagree on the preference?

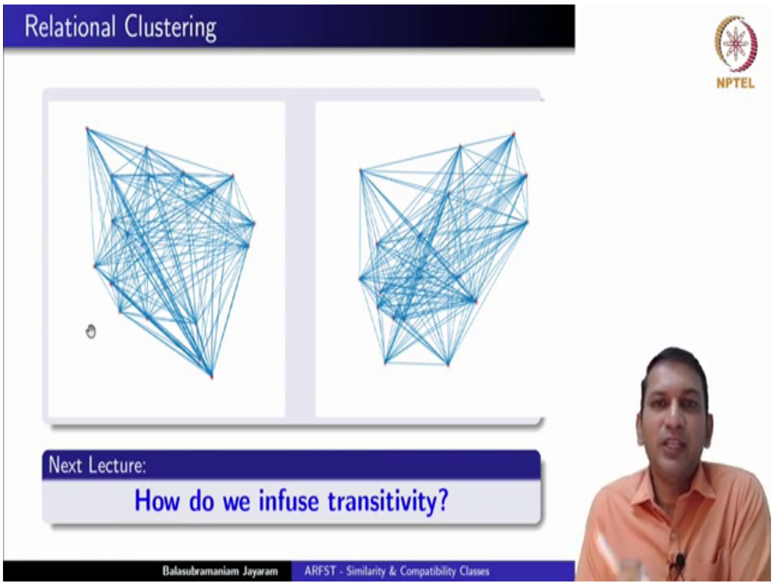
So, that this graph could actually represent them and if you have another graph like this for the same set of people or a different set of objects that you are considering it could represent some other preference relation. Now, how do we make sense of these objects from these relationships remember these relationships may only be symmetric and we could also put a self loop here and then assume it is reflexive.

So, essentially you might only have a compatibility relation not a similarity relation. So, now, if you want to obtain some information among them from this one of the first thing that you would do is to actually group these objects which is essentially clustering them. And since you are going to do clustering based on the relations then it falls under the large class of clustering algorithms called relational clustering.

But remember if you want to look at it from the lens of fuzzy relations, then what you would perhaps want is actually a fuzzy equivalence relation a similarity relation. However, often these are data that are given to you we do not have a nice mathematical form which generated this and often they end up being only compatibility relation; that means, they are only reflexive or they could be made reflexive they are symmetric typically.

But they may not have transitivity and even if they have transitivity perhaps, but as I said you may not be able to find out what is the transitivity and it may not be min transitive.

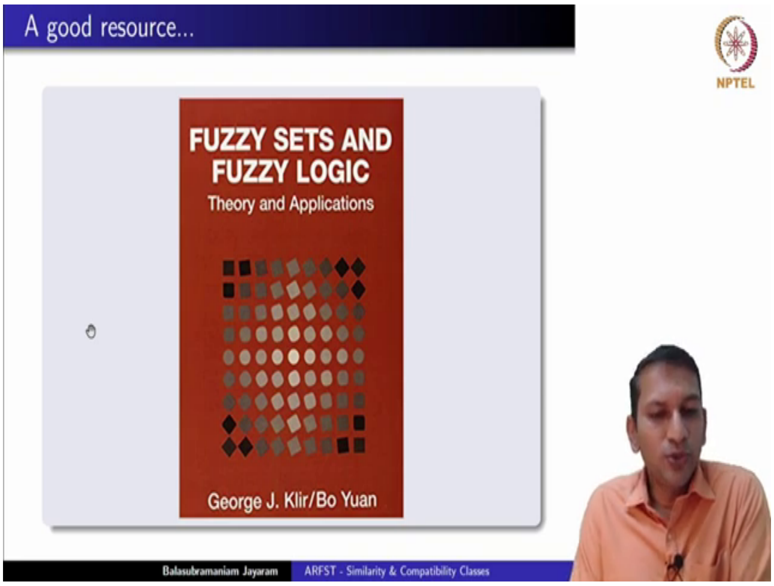
(Refer Slide Time: 40:37)



So, the question is how do we infuse transitivity into fuzzy relation? So, the need for transitivity is clear if you have compatibility relation by infusing transitivity you make it a similarity relation. And once you have a similarity relation you are able to partition this underlying space s X and you know that it is beneficial in many applications. In fact, in the next lecture we will touch upon this how hierarchical clustering can be looked at as relational clustering using fuzzy equivalence relations or similarity relations.

Hence transitivity plays a vital role when you want to do some basic data analysis given a set x of objects. So, in the next lecture we will discuss how to infuse transitivity into given fuzzy relation.

(Refer Slide Time: 41:33)



A good source for the topics that we have seen in this lecture is the book of Klir and yuan almost whatever we have spoken about on the compatibility and similarity classes are available in this book. Glad that you could join us for this lecture and hope to see you in the next lecture soon.

Thank you.