Approximate Reasoning using Fuzzy Set Theory

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Lecture - 24
Composition of Fuzzy Relations

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Hello and welcome to the 2nd of the lectures, in this week 5 of the course titled Approximate Reasoning using Fuzzy Set Theory, a course offered over the NPTEL platform.

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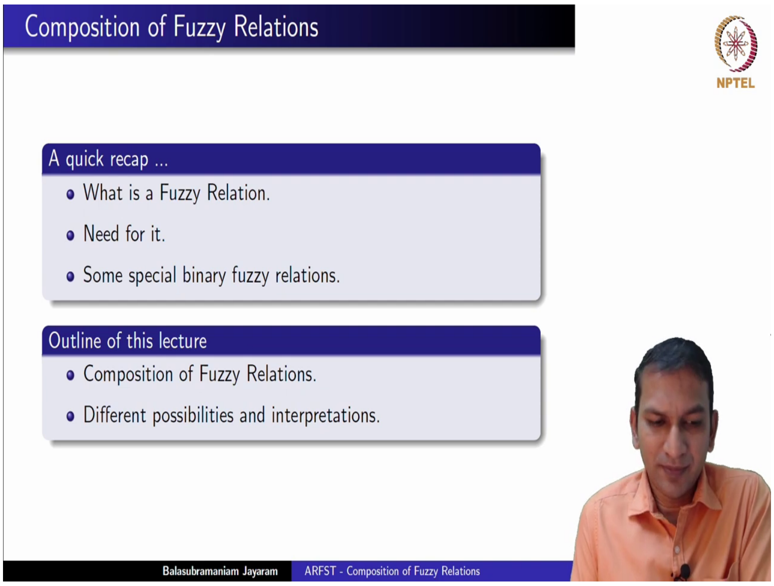

In the last lecture, we looked at and had a gentle introduction to fuzzy relations. We looked at the need for it. And also saw, some special binary fuzzy relations. In this lecture, we will look at compositions of fuzzy relations. Specifically, we will see that when we move from the classical relations to the fuzzy relations, we could interpret there are different possibilities and we could interpret them differently when it comes to composing fuzzy relations.

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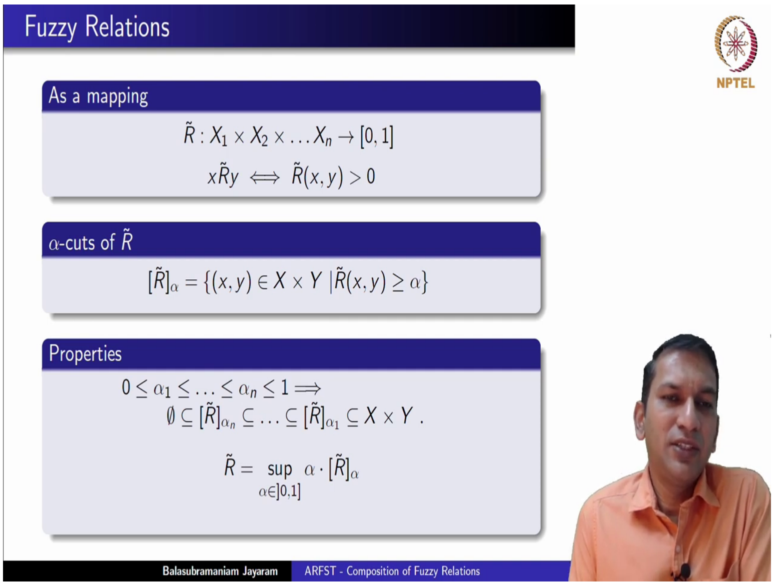

A quick recap, of what we saw in the last lecture. So, a fuzzy relation as a mapping from Cartesian product of spaces to the unit interval [0, 1] and we say x is related to y; in under the fuzzy relation R for a binary fuzzy relation. If x, y belongs to the support of the corresponding fuzzy set or the fuzzy relation of the Cartesian products. We could again take the alpha cuts just like we did in the case of fuzzy sets and consider the subset that we get of the Cartesian product.

And, once again for increasing alphas the subsets would be telescopically decreasing. We could also resolve the fuzzy relation this way, as we do for fuzzy sets.

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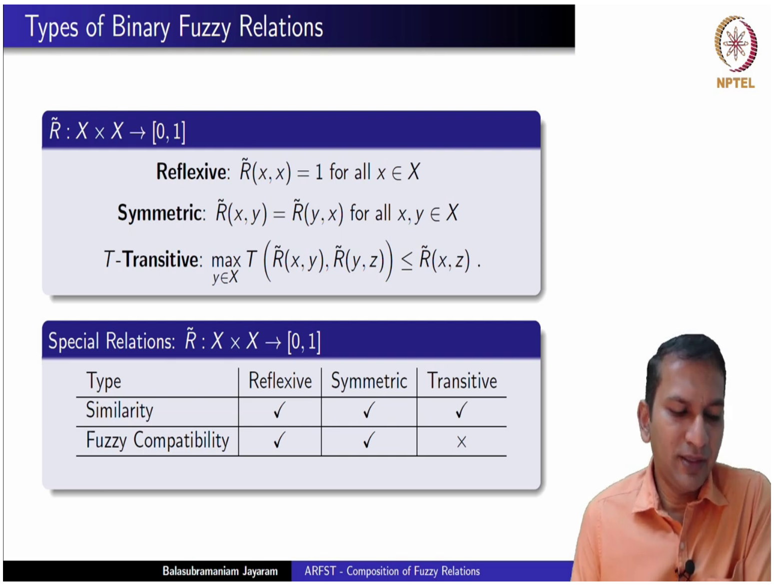

What is interesting is, the way we have defined reflexivity and transitivity. Reflexivity it is defined as R(x, x) is 1 for all x; symmetry is clear. Transitivity we always talk about T transitivity where, T is a t-norm. We have seen these two special fuzzy relations. We have defined them; we are yet to see which we will take up in the next lecture. If it is if a fuzzy relation is reflexive, symmetric and transitive we call it a similarity relation or a fuzzy equivalence relation.

These are two terms that you will see in the literature. And, if it is only reflexive and symmetric and not transitive, we call it a fuzzy compatibility relation.

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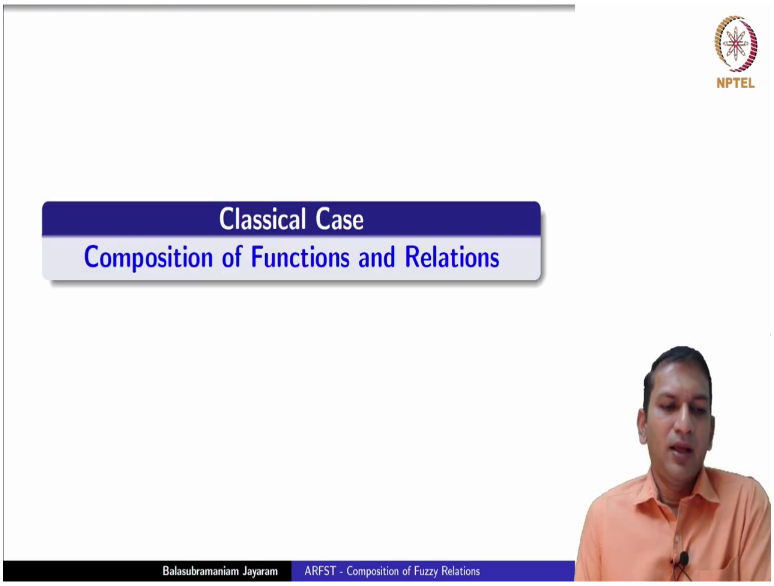

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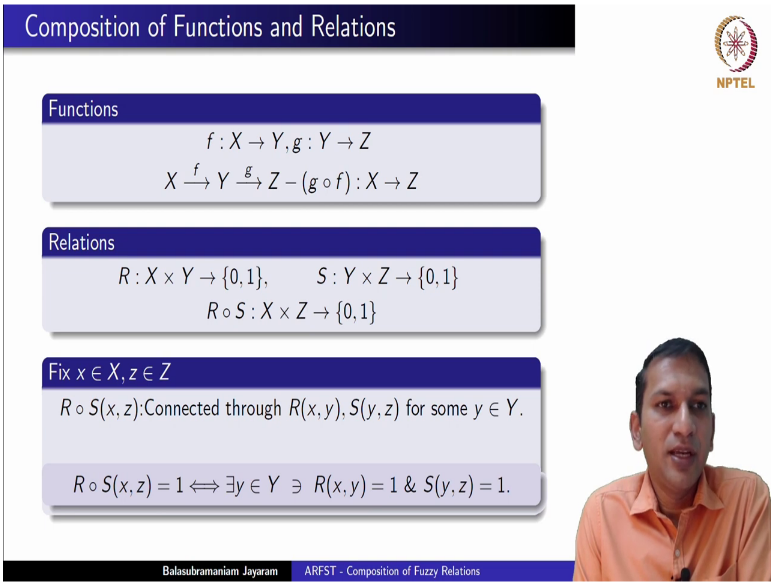

Now, let us look at composition of functions and relations in the classical case. Let us have a function f from X to Y and g from Y to Z. We typically write the composition of functions as g circle f f acts first to take X to Y and then g acts on it to take Y to Z. So, typically we write it as g circle f, as a function going from X to Z.

However, if we look at relations when the classical relation case, is if R is a relation on X cross Y; and S is a binary relation on Y cross Z, we typically write it as R circled S; which is a relation from X cross Z to the set [0, 1]. We will follow this convention when we are discussing relations. Now, the question is if for a fixed x and z, how should R circle S of x, z be? What value should it take?

It is clear that, this relation which expresses whether an x and z are connected or not related or not is. In fact, we expect it to be connected through some y in Y and the relations R and S. So that means, we decide whether x and z are related under R composed with S, only through some y, and if that y were to be related to x under R and related to z under S. So, perhaps we could look at it like this.

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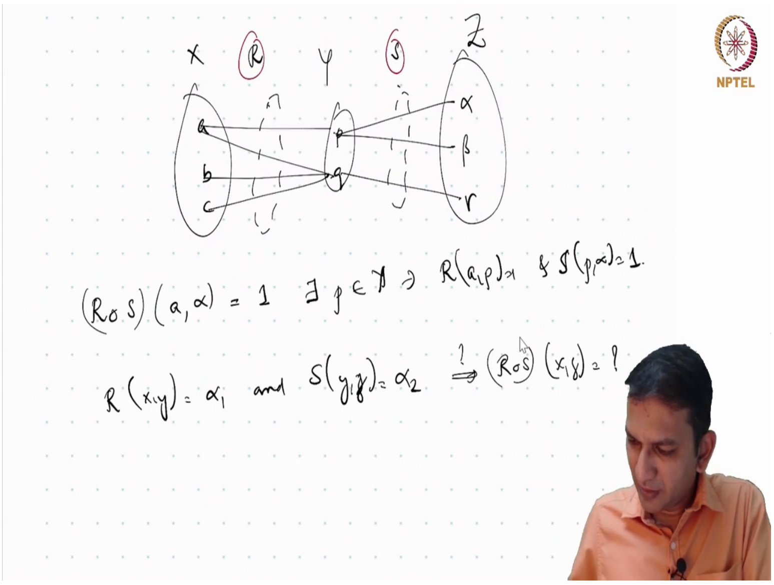

So, let us take an example. So, I have a set X, set Y and set Z. a b c here; just p q here and alpha beta gamma. So, let us say that I indicate. So, this is my relation R on X plus Y. So, b and c are related to q; let us say p is related to alpha; p is also related to beta; then q is related to gamma. So, this is the relation S. So, now, if you were to ask, R circle S of a, alpha, what will it be? It is clear that a is related to p and p relates to alpha.

So, this is 1; because there exists a p in p in Y, such that R of a, p is equal to 1 and S of p, alpha is equal to 1, ok. So, now, then, it is easy to write it like this; that R circle S of x, z is 1; if and only if there exist some Y such that R of x, y is 1 and S of y, z is 1; that means, x and z are related to each other under the relations R and S by some Y at least 1 Y exists.

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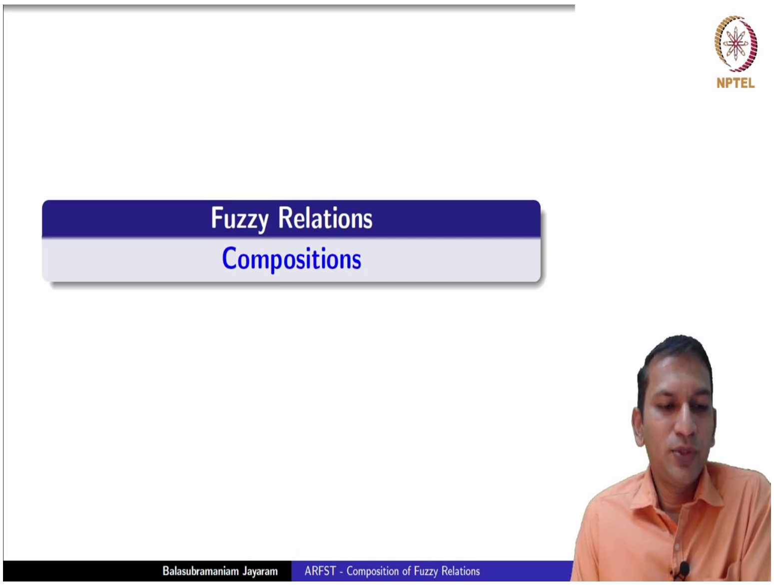

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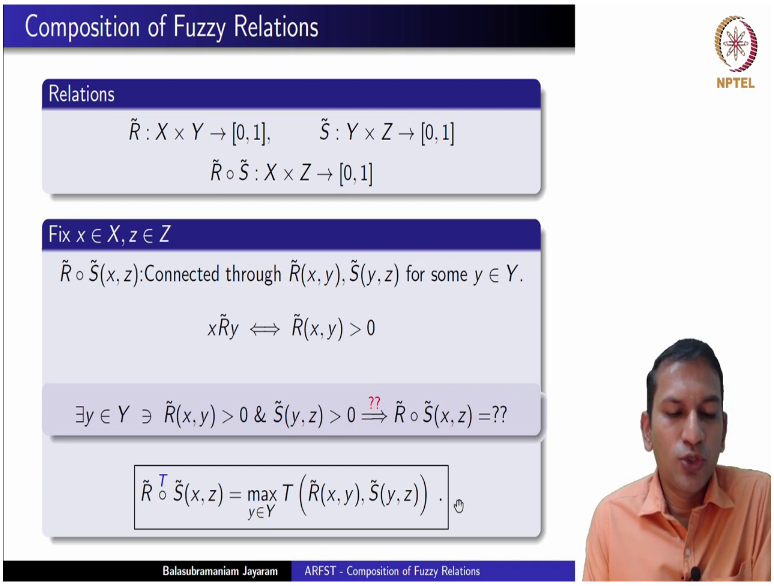

Now, let us look at the compositions with fuzzy relations. So, once again, these are mappings on the Cartesian product of X, Y and Y, Z going to 0, 1. And now we write it as R circle S, as mapping from X cross Z to 0. The question again is how do we decide the value of x, z under the relation R circle S. clearly, it is again should be connected through some y and Y. Note that, when we write in the case of fuzzy relations, x is related to y binary fuzzy relation.

We only say that R of x, y is greater than 0. Even though, we write it as R tilde just to distinguish that it is a fuzzy relation, we will read it as R and slowly we will drop this tilde notation just to avoid the cumbersomeness of the notation. I think the context will make it clear whether we are discussing classical relations or fuzzy relations.

So, under a fuzzy binary fuzzy relation, we say x and y are related; if x, y belongs to the support of the fuzzy relation or the fuzzy set of this Cartesian product; that means, we only know R of x, y is greater than 0. So, now, what we want is, given that there exists a y such that R of x, y is greater than 0 and S of y, z is greater than 0, we want to say whether R circle S of x, z is greater than 0.

And if it is greater than 0, we want to be able to determine. So, remember, it is not only about saying whether it should be greater than 0 or not, what should the value be? For instance, it is perhaps we are given that R of x, y is some alpha 1 and S of y, z is equal to alpha 2 and from here we are asking the question, what is your R circle S of x, y? What value should it take?

So, now let us look at this diagram that we have here. And perhaps if (Refer Time: 08:03) to remove this, we are asking the question, what should R circled S of x, z be for any given pair of x measure?

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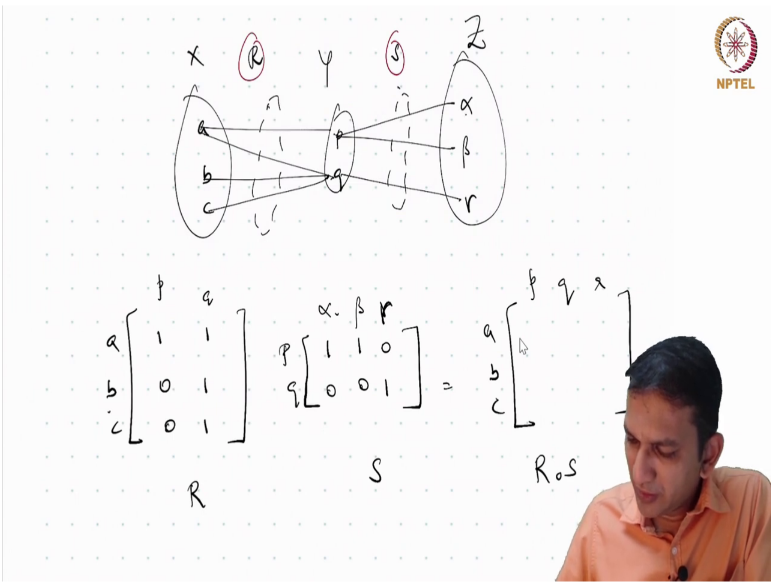

To answer that, let us write these relations in the corresponding matrix form. So that means, R you would write it like this, a b c p q and S we would write it like this. p q alpha beta gamma. So, this is your R and this is your S and what you want is, R circle S which will be from a b c p q and R. Now, a is related to p and also q; b only to q; c also only to q and p is related to both alpha beta, but q is related only to gamma. Now, let us write what is R circle S from the given diagram. We know that, a is related to alpha through p.

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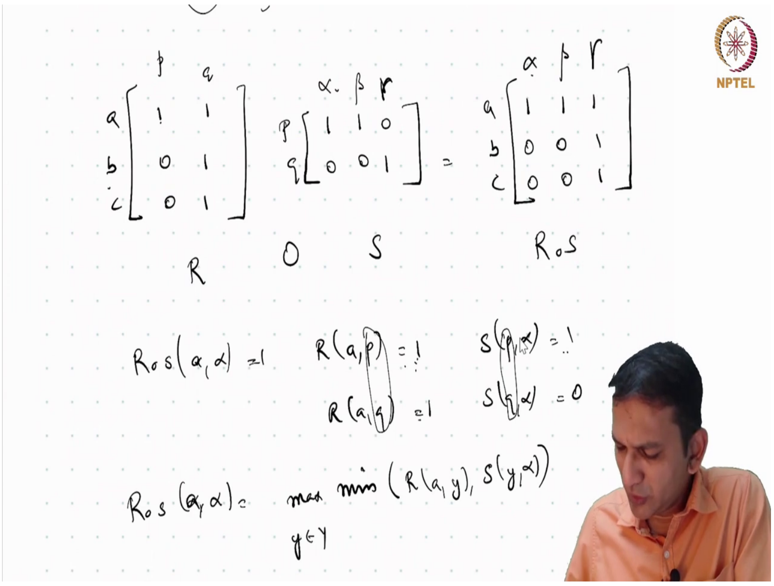

So, a alpha beta gamma. So, a is related to alpha through p, 1; a is related to beta also through p so, it is 1 and a is related to gamma through q so this is 1. If you look at b, b is related to q and it is related only to gamma. So, b is not related to alpha or beta. And, so is the case with c. So, now, for the moment let us assume that we are only given these two matrices which represent the relation.

And, from there we need to come up with this matrix R circle S. how would we handle this? Is the question. Now, we are given two matrices and we are actually obtaining another matrix. And, all we are doing is using this composition operation. So, now in fact, it appears that what we are doing is some kind of a matrix multiplication. Look at this 1 1 1 0. So, in matrix multiplication, essentially row into column all we do is inner product of this row with the column, since the orders will match.

So, essentially, we are doing some of products, but here instead of some and product we seem to be using some other operation. What exactly is it? As you have seen composition is somehow related to transitivity. So, what we are doing is some kind of finding the transitive nature of these relationships between a and alpha through p. So, essentially you are writing R circle S of a, alpha, we say this is equal to 1, if there exists. If 1 some pair is 1. So, some pair is 1.

So, some pair is 1 means, over as you vary over y, there exists a y, such that for some y both of them are 1 and you do not really worry about if for some other y both of them are 1 or not it can very well be, but otherwise also you do not bother. So, essentially you can write this as a, alpha has minimum, because you want both of them to be 1 only then it should be 1 minimum of R of a, y, S of y, alpha max y Y because, both of them have to be 1 for it to be 1. And, for some y it should happen; not for everywhere and that is where maximum is coming to picture.

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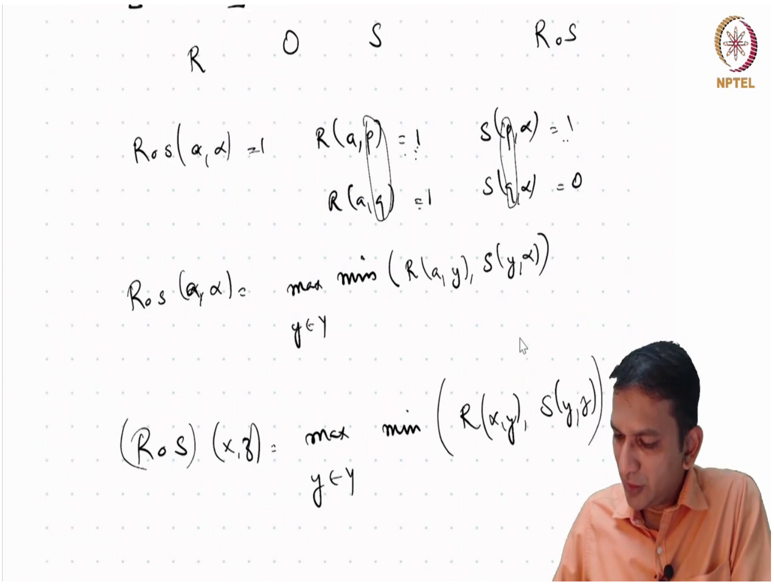

So, now you see here, essentially all we are saying is R circled S of x, z is max y belong to Y min of R of x, y, S of y, z. So, essentially what we are doing is, what we call a max min composition, but we generalize it to max T composition. Of course, for any t-nom as we have seen in the case of transitivity, even here for any t-nom it would work in the case of classical relations, that is where the relations are either 0 or 1.

Now, let us apply this here and see whether it is true. Consider the first row in the first column 1 1 is 1; 1 0 is 0, max of that is 1. Similarly, 1 1 is, if you take these two 1s and 1 0 then it is 1 minimum of 1 1 is 1; minimum 1 0 is 0 and maximum of that is 1. So, that is how you got this one. So, essentially if you take these two matrices and if you do a max-min composition or max T composition, that is how you get the composition of relations.

And simply, we could use this in the case of fuzzy relations too. Because, now, these are not just 0 or 1 values; they are these values come from the entire interval 0 1. And, this formula is valid in the classical case two. So, it is a very nice way of capturing, what happens in the case of transitivity.

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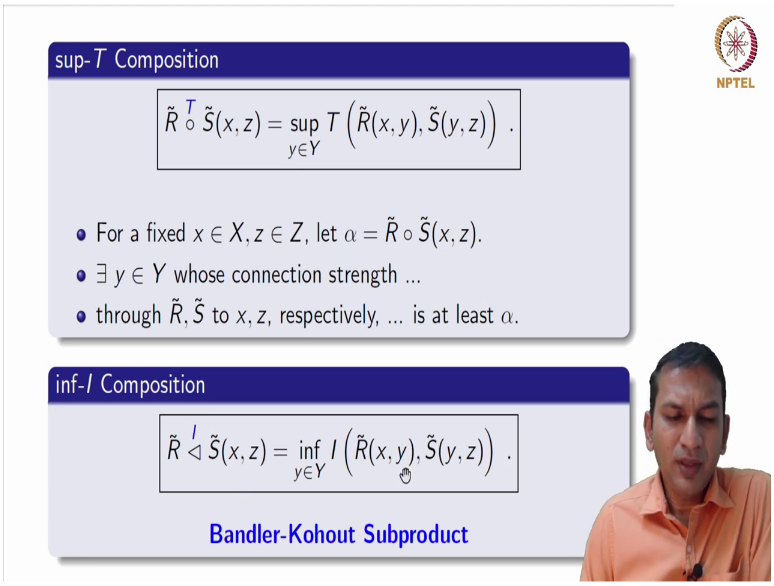

And, this is what is called sup- T composition, max T composition, in the case that y is finite; obviously, sup becomes max but, otherwise, in general instead of talking about maximum we talk about sup- T composition. Now, if you want to interpret this, if you fix an x and z and if R circle S of x, z is alpha what does it mean from this formula?

It means there exists a y in Y, whose connection strength through R and S to x and z respectively, through R to x and through S to z is at least alpha. Remember, supremum of all of them. So, now, the maximum of what is happening inside this bracket is what you are taking. And T of this, just imagine T is min. So, what would come out is actually something smaller than both of them or at least the small the smaller among the two.

Or in the best case, that is when use T is min. otherwise, typically we know that t-norm is smaller than min for example, if you take product then it will be a product of these two and these are numbers coming from [0, 1] which means it will neither be R(x, y) or S(y, z) will be smaller than that. So, essentially, we are saying that if you use min, then we know that there exists some Y, whose connection strength is at least a to either x or z at least alpha.

Now, once we have interpreted this sup- T composition matrix, let us go back to our matrix here. And, let us try to generalize this slightly further. So, what we seem to have is when we are talking about composition is this.

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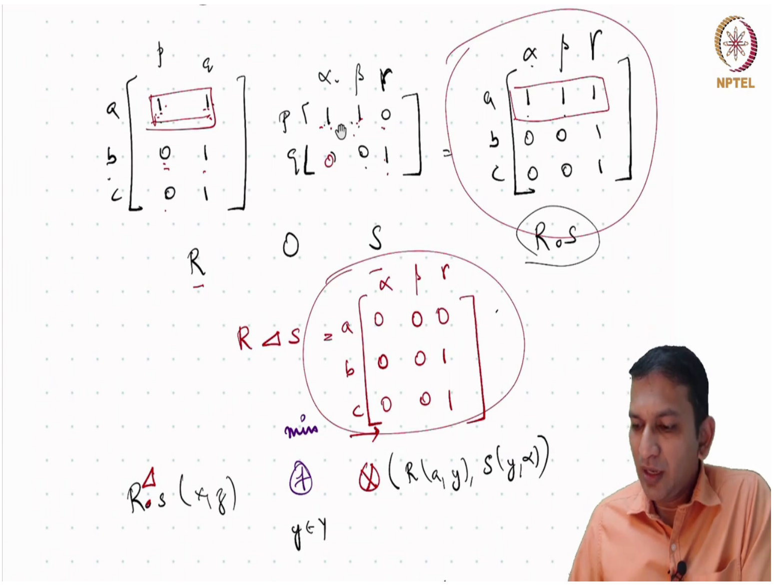

So, if you have R circle S x, z, we seem to have 1 binary operation here. I will denote it by circuit forms. And, another binary operation. So, we took this as max and min, but why not some other operation? In the case of fuzzy relations, we have a lot of fuzzy logic operations. So, why not try something else?

So, for the moment let us assume that we use minimum here and the implication here so; that means, instead of doing max-min composition, we are doing min implication composition. So, I think it will be worthwhile to see, what happens when you apply this composition in the classical cases. So, let us keep R and S same. And, now we are going to apply this composition.

So, for the moment I will indicate this by this. So, we want to see what is R delta S? Now, we are taking this into this. So, 1 implies 1 is 1; 1 implies 0 is 0; minimum of these two will be 0. So, if you take a b c alpha beta gamma and a alpha is 0. So, look at a beta is a same values. So, 1 into is 1 is 1; 1 implies is, now we are looking at this and this. So, we will be looking at here in this. Once again it will be the same. 1 implies 1 is 1; 1 implies 0 is 0. So, minimum of these two will be 0.

Now, if you take the first row into last column, 1 implies 0 is 0; 1 implies 1 is 1, but minimum of these two is 0. Now, let us look at what happens for the rest of the cases? How this b related to alpha beta gamma? 0 1 is 1; 1 0 is 0 in terms of implication. So that means, again minimum when we take it is 0. Now, 0 1 is the same 0, something interesting happens in this case, 0 implies 0 is 1 and 1 implies 1 is 1.

So, minimum of these two is 1. Now, since c has the same thing (Refer Time: 18:53). Now, if you compare these two compositions, they are quite different. Not just that, they are perhaps revealing also in some sense. Now, a had at least 1 of them, coinciding with alpha beta gamma; that means, there exists a p such that a p is 1 and p alpha is 1; a p is 1 and p beta is 1 and a q is 1 and q gamma is 1. And that is how we got 1s there.

But, now, when you changed the composition to that of minimum implication composition, we see that was not enough and are we insisting them? So, what does this capture?

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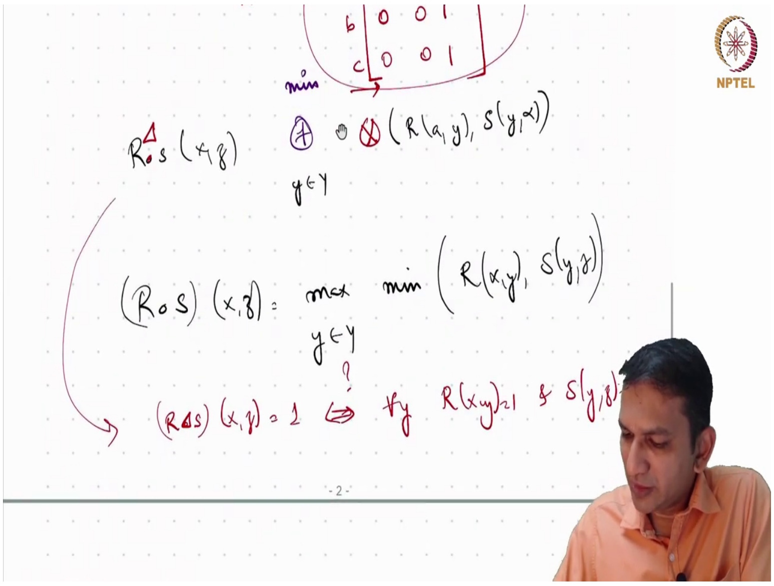

So, when you are using min implication, what are we capturing? Are we saying that, R circle R of x, z is equal to 1; if and only if for all y R of x, y is 1 and S of y, z is 1. This is what we are capturing. Now, if this is what we are capturing, then look at b and gamma. This is 1 1. Yes, but we have we have 0 0. So, perhaps then we need to modify this say, ok.

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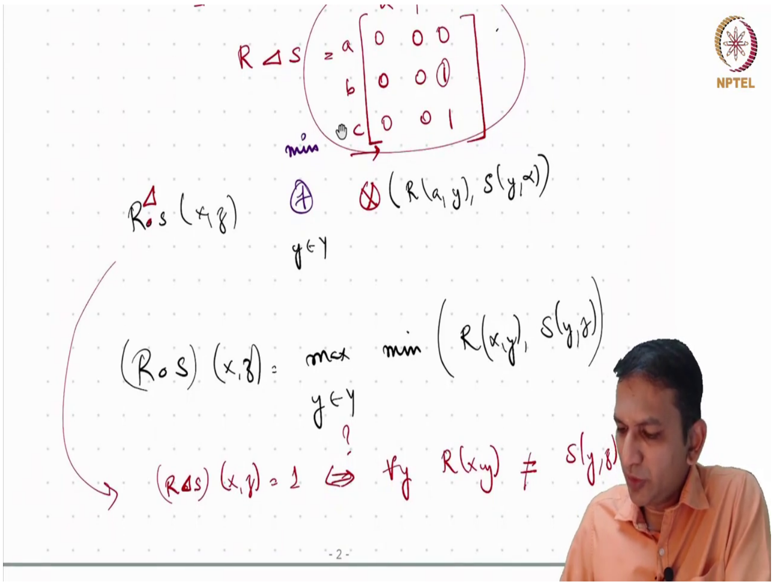

Perhaps, what we are capturing is really not they should be equal to 1 both of them. But maybe that these are actually equal; that means, x is related to z under this composition, means they become 1; if and only if through every y R(x, y) and S(y, z) should be actually identical, but let us put this conjecture also to test.

Instead, let us take here, instead of 0 let us take 1. Now, what happens to b gamma 0 implies 1; 0 implies 1 is 1; 1 implies 1 is 1. So, what we will get is still 1. And, now you see here that R of x, z x, y is actually not equal to S(y, z) when y is equal to when y is equal to p.

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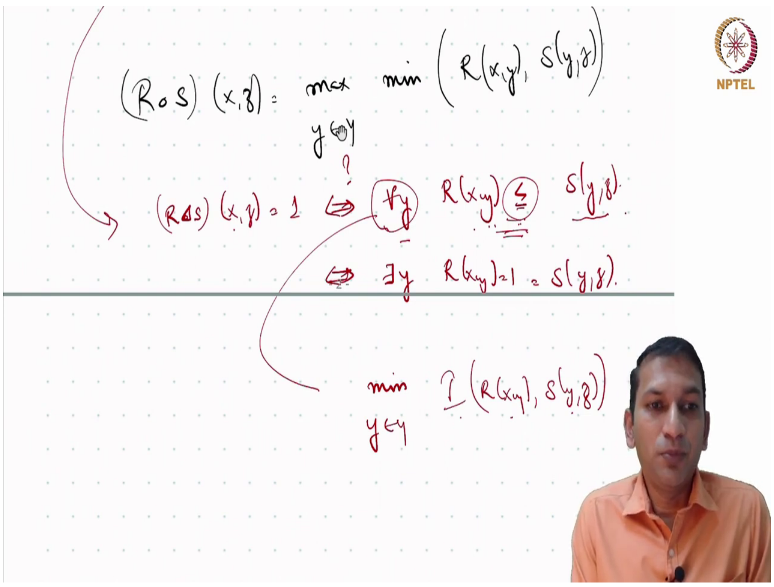

However, what it says is, perhaps this could be a better way to capture it. But what we want seems to be that if through this composition if x and z has have to be related. Then for every y the connection strength of y to x should be smaller than the connection strength of y to z. Put in a more positive way, if x and z has have to be connected under this composition then, for every y the connection strength of y to z under S should be at least as big as the connection strength of y to x under R.

So, now you see here, earlier what we asked was, there exists y such that R x, y is equal to 1 is equal to S y, z. So, here we insisted on R of x, y being equal to 1; in the we are still talking about in the case of classical relations and composition. We insisted that through some y R of y should be related to x to degree 1 and also to z to degree 1. But it should happen just for one way.

Whereas, in this case we have made this inequality make this an inequality and, in that sense, less tangent, but what we are asking is this should happen for all (Refer Time: 22:50) So, there seems to be a nice trade-off. So, when you when you want to actually capture this, for all y turns out to be minimum or infimum and this you could consider it as an implication R of x, y and S y, z that is what we have captured.

So, we have interpreted it like this, but initially we are actually used implication. But, if you want to come back, assume that you are using an implication which has ordering property; that means, whenever this is smaller than this that is going to be equal to 1. And that is when you will get 1. So, essentially, you could think of it as using an implication with an ordering property here. So, now, this gives us another way of looking at composition. And, interpretation and through the interpretation we are getting another composition operation itself.

This is called the inf-I composition I stands for implication of course; it is given like this. And, we will use this symbol the left triangle symbol. And, which is defined like this. Infimum over Y, I of R x, y, S y, z. Note that, we could use any implication here, and any t-norm here, only this operation of supremum or infimum they are fixed. To denote this, we are having this T above the circle and I above this left triangle.

In fact, in the fuzzy set theoretic literature, this composition is also known as the Bandler-Kohout subproduct as it was proposed by these two people Bandler and Kohout. Both of these compositions are extremely important and they play an important role especially considering the focus of this course, which is inferencing and clearly in fuzzy relational inferencing, ok.

So, now we have come up with two different types of compositions. Now, are there others can we use instead of a t-norm and a multiplication can use some other fuzzy logic operation. Yes, you can. But you need to be careful about interpreting them. And, where you can use them. For our purposes these two compositions would serve more than quite well.

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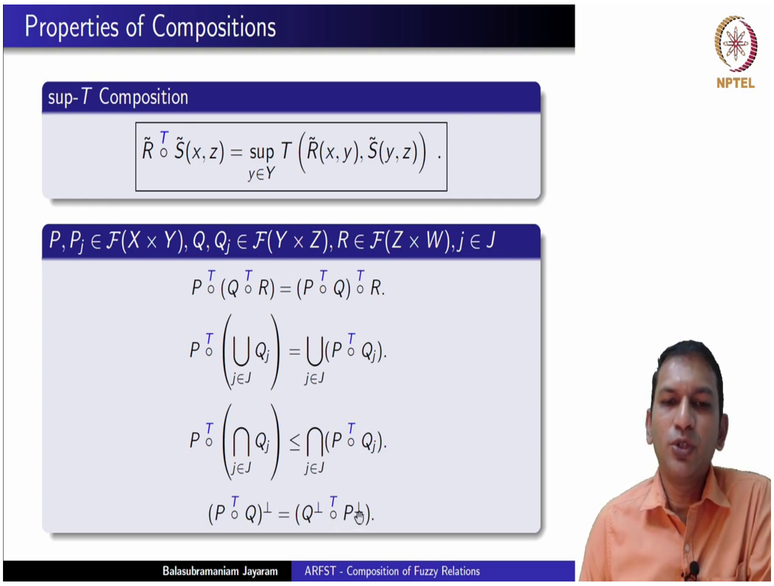

Now, let us look at properties of this composition. So, let us take P P j indexed with j as fuzzy relations on X cross Y; Q and Q j are fuzzy relations on Y cross Z and R is a fuzzy relation of Z cross W and j is coming from this set j. Well, if you consider this equality, it says P circle Q circle R is equal to P circle R circle R. Essentially, it says, that the circle T composition, the sup-T composition is in fact, associated.

This should hardly be surprising because we are using a p-norm which is anyway associated and it can be actually inherited from there. Similarly, is the case here. We see that, the composition distributes our union. And in the case of intersection, we actually get some kind of subadditivity here. This final equality should also be interesting, where this symbol denotes transposition.

So, if you have a relation P from X cross Y to 0, 1. The transpose operator from Y cross X to 0, 1. Essentially, we are transposing the corresponding relational matrix. So, it says that with respect to the sup-T composition P circle Q transpose is nothing but, Q transpose composed with P transpose.

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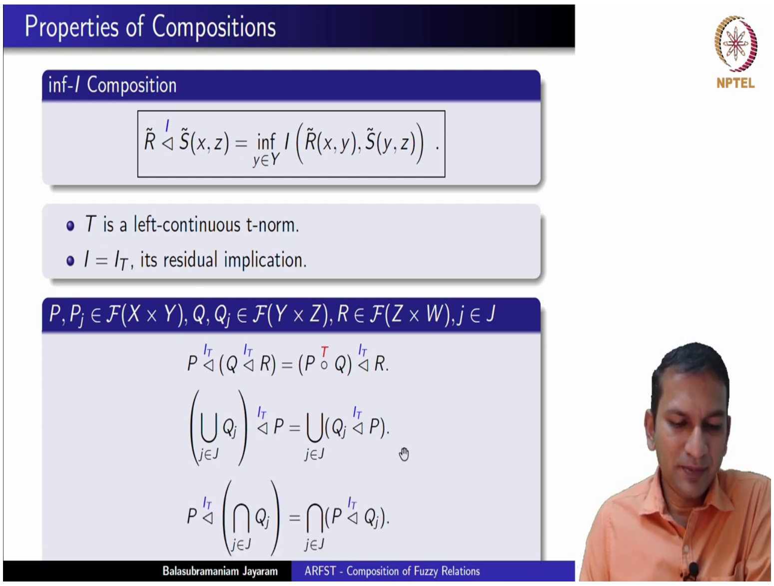

Now, if similar kind of properties can also be expressed or explored and detailed, for the inf-I composition, but what is interesting is, if you pick a left continuous t-norm and the corresponding residual implication I T. remember, if T is left-continuous, we can call the R implication as the residual implication of T or residuum of T.

So, if you fix T to be left-continuous t-norm and I to be its corresponding R implication. And, with the same relations coming from these sets, we have some interesting properties. Look at this, P with respect to the inf-I or the Bandler-Kohout subproduct composition. P composed with Q composed with R, is actually equal to P sup- T composition with Q and BKS composition with respect to R.

So, you see here, it does not have associativity as such. But it has some kind of an associativity and it is with respect to the sup- T composition. Remember, T is left-continuous and I T is actually its corresponding residual. Of course; it also has distributive as we have seen earlier.

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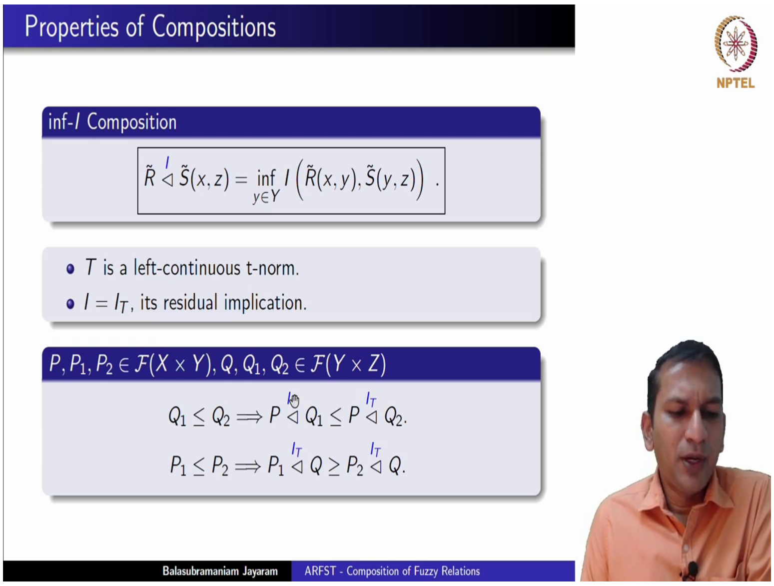

But what is interesting is, because of this distributive we also have, the corresponding monotonicity; that means, if Q 1 is less than or equal to Q 2; that means, point wise in terms of the relational matrix; if the numbers on Q 1 are smaller than the numbers of on Q 2 at the same location, we can show that this is the kind of monotonicity that we have in the composed relations.

And, if P 1 is less than equal to P 2, then the monotonicity reverses of course. This is also clear because, I T has an implication is decreasing in the first variable and increasing in the second variable.

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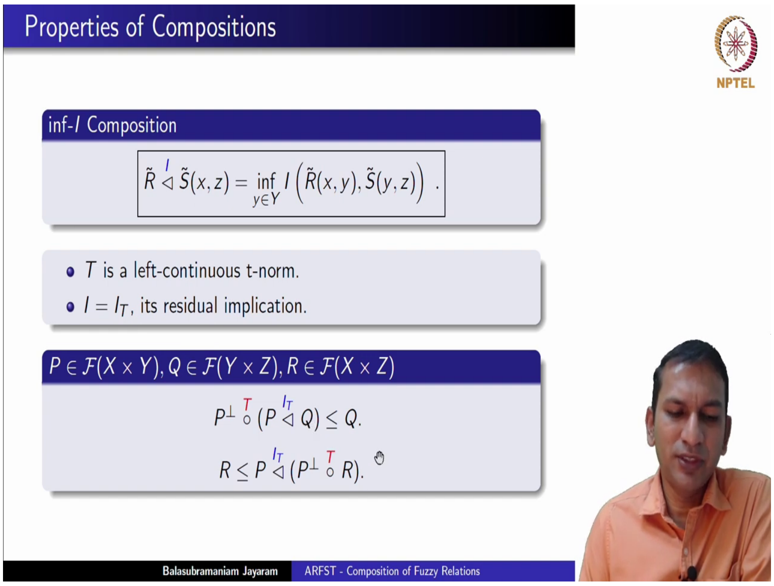

But what is very very interesting are these two properties. For this let P be from X cross Y; Q from Y cross Z and R from X cross Z, look at this inequality. What does it say? P transpose sup- T composition with P composed with Q under the BKS product is actually less than or equal to Q. Now, let us look at it in terms of dimensionality. So, P transpose P is from X cross Y.

So, P transpose is from Y cross X is X cross Y, Q is Y cross Z. So, finally, what you get is a relation on Y cross Z. So, it is comparable, dimensional it is correct. Look at this inequality R is from X cross Z; P is from X cross Y; P transpose is from Y cross X. So, this will give you something from X cross X only and R is again from X cross. So, these are two inequalities which are of course, dimensionally correct.

But they see they see that, they show you the interplay between sup- T and the BKS sub product composition when T is left-continuous t-norm and I is the corresponding residual these are two inequalities which play an extremely important role, when we want to discuss fuzzy relational equation. And, we will discuss them when we are discussing interpolativity of fuzzy relation inference going forward.

And at that time, perhaps we will recall that we have seen this inequalities involving fuzzy compositions, relational compositions.

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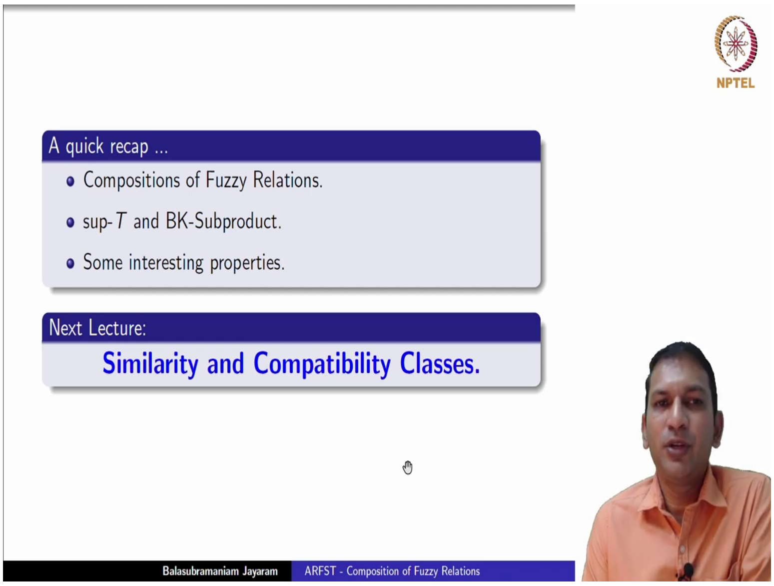

Well, A quick recap we have discussed compositions of fuzzy relations. And we have seen in the case of fuzzy relations, we could have different interpretations and hence different possibilities, which led us to talk about sup-T and inf- I composition of the BK-Sub product composition. And we have also listed out some interesting properties which will come up again when we discuss inference systems, especially fuzzy relational inferences.

What next? We had introduced fuzzy equivalence relation or similarity relation and compatibility relation. In the next lecture, we will look at how these special relations are going to be useful for us.

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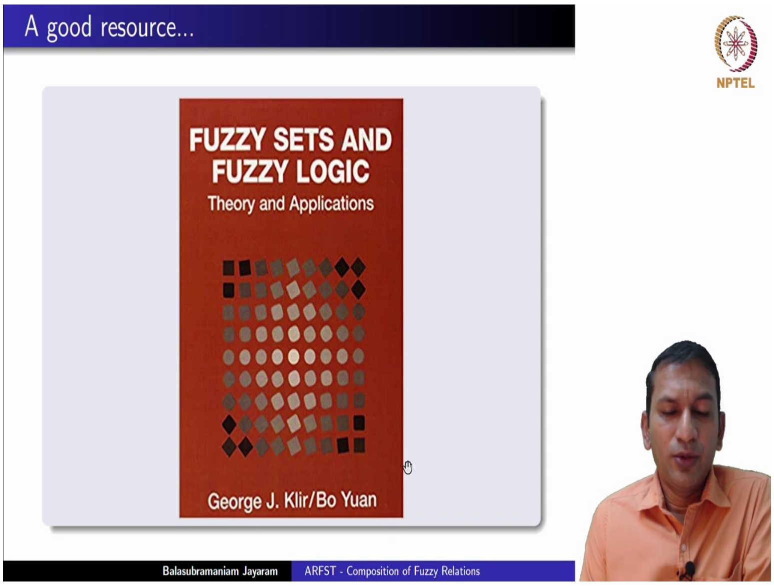

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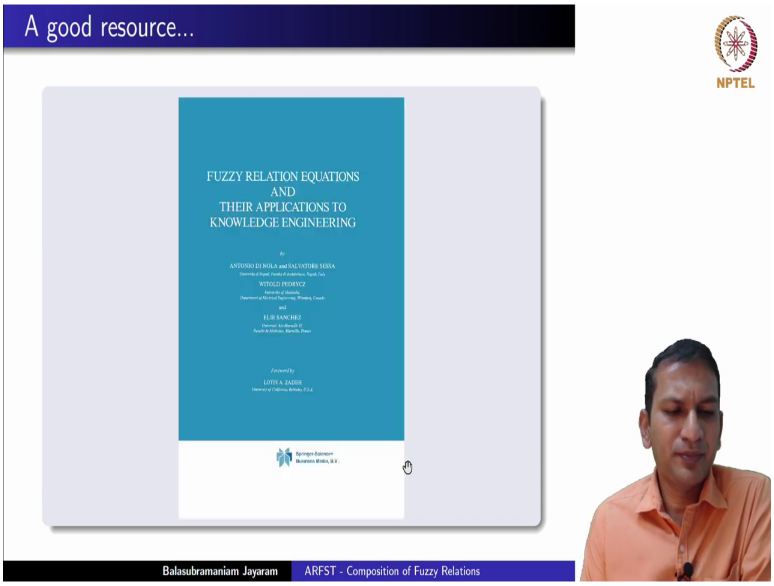

A good source for the topics discussed in this lecture, is once again the book by Klir and Yuan. You could also look into this book by Di Nola and Sessa which deals with Fuzzy Relational Equations And Their Applications To Knowledge. Glad you could join us today for this lecture. I am looking forward to meeting you in the next lecture.

Thank you.