Approximate Reasoning using Fuzzy Set Theory

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Lecture - 23
Fuzzy Relations

Hello and welcome to the first of the lectures in this week 5 of the course titled Approximate Reasoning using Fuzzy Set Theory, a course offered over the NPTEL platform. In the last 2 weeks we have seen, the 3 basic fuzzy logic connectives that of fuzzy negations, triangular norms and fuzzy implications. In this week, we will deal with Fuzzy Relations.

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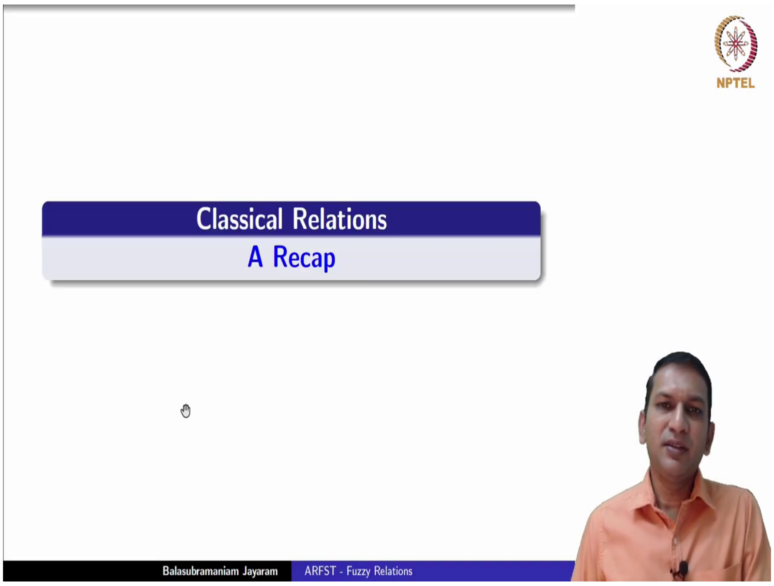

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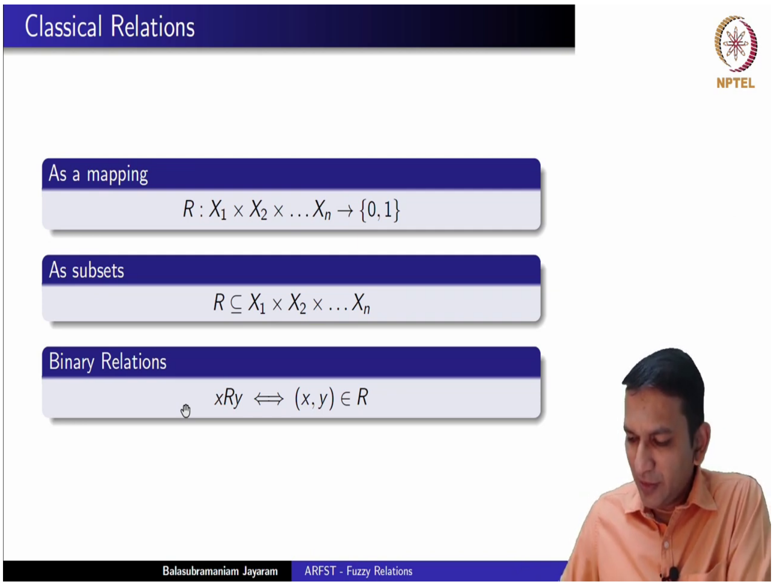

Let us begin with a brief recap of classical relations. A classical relation can be seen as a mapping from the Cartesian product of some sets to the set {0, 1}.

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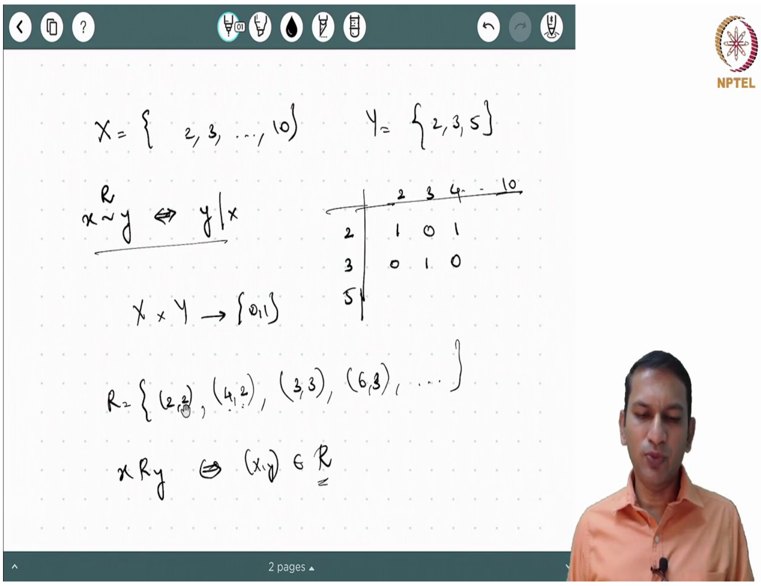

For instance, let us take the set A to be the numbers 1 to 10 and Y simply to be the set {2, 3, 5}. Now, we could define the relation as follows x is related to y if and only if y divides x. Now, this will allow us to build a relation on this set. So, you could indicate this by a matrix, in fact a binary matrix. Let us for convenience just only start with the set from 2 to 10. So, 2 divides 2, that means, we put 1. 2 does not divide 3, so 0. It divides 4, if we put a 1. 3 does not divide 2, so put 0. 3 divides 3, put a 1. Does not divide 4, put a 0.

So, similarly from the way we have defined the relation where we have specified when an x is related to y, we could indicate it by a binary matrix. And this essentially can be looked at as a mapping from X x Y to the set {0, 1}. We could also look at a relation, classical relation as the subset of the Cartesian space. For instance, all we need to do, if I call this relation R, then R can be written as a collection of ordered pairs from X x Y. For instance, 2, 2 will remain R because 2 divides 2, 4, 2 will remain in this collection because 2 divides 4.

Now, this is the second component is Y, the first component is from X. Similarly, 3, 3 will remain, 6, 3 will be in this set. So, it is a collection of elements from the Cartesian product of X and Y. So, in that sense a relation can be seen as a subset of the Cartesian product. In the case of binary relations, it is customary to write like this, indicate if R is a relation x R y if and only if x, y belongs to R, where now we are interpreting R as a collection of elements as a subset of X x Y.

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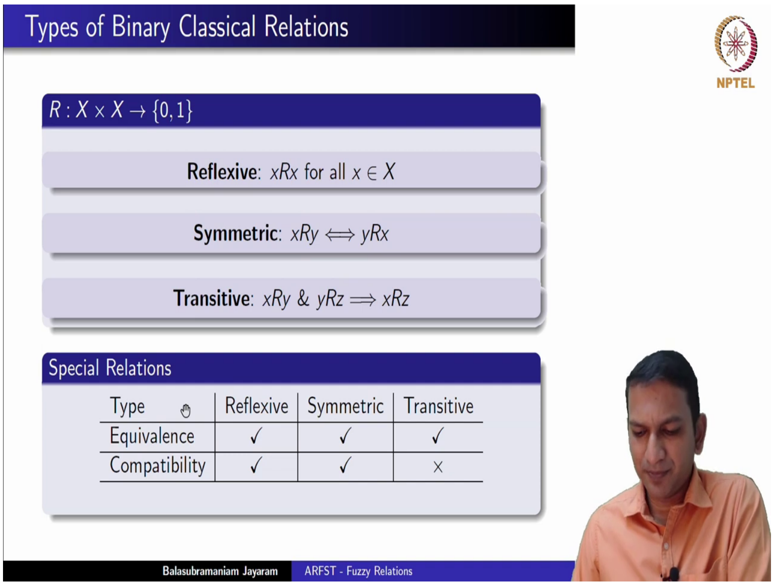

Now, if the set X and Y are same, in the case of binary relations if both X and Y are same then we can come up with some specific or special binary relations. A relation is said to be reflexive, if x is related to x itself under the relation for every x. It is said to be symmetric if x is related to y, then y should also be related to x. It is said to be transitive, if x is related to y and y is related to z, then x also should be related to z.

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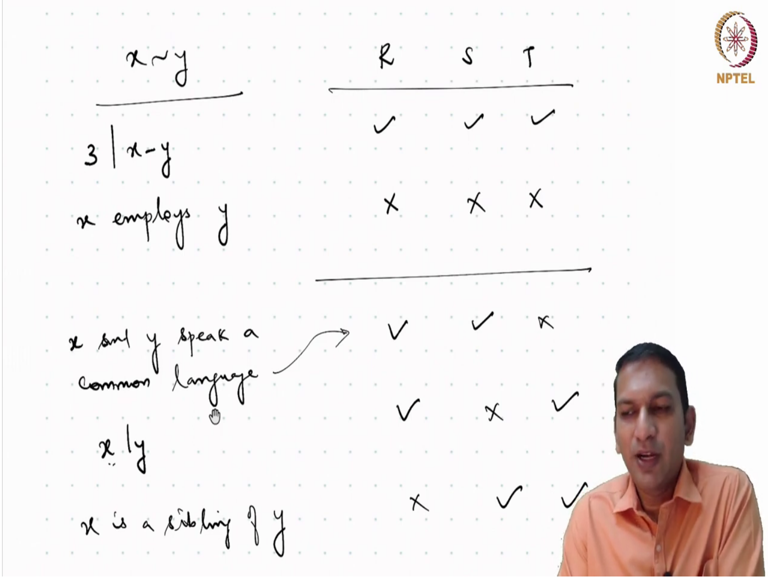

Let us spend a few minutes on this to understand that these 3 properties are in fact, mutually independent. Towards this end let us come up with relations having different properties, final relations having different properties. The domain of the relation x will be clear from the way we are defining it. So, we will not specifically write out or specify it explicitly. Now, also note that if now we are looking at R as of X cross X to {0, 1}, we know that it could be written in the form of a matrix.

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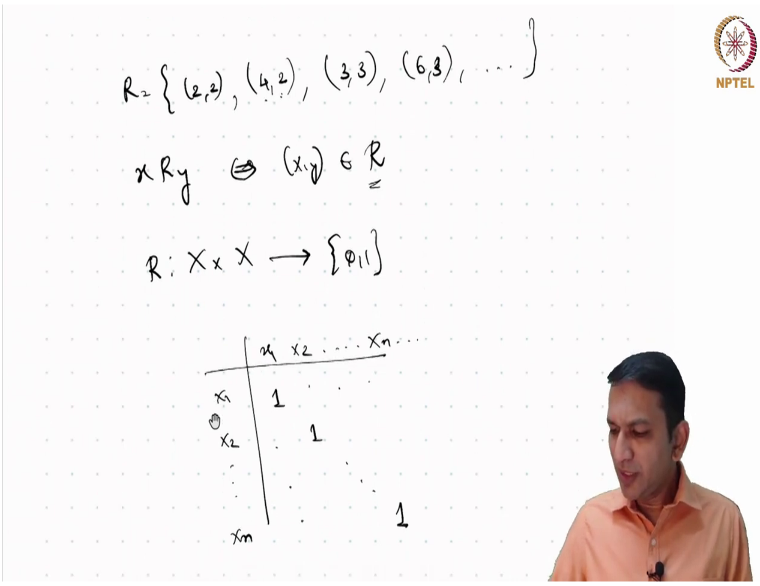

If we could list out the elements of X; note clearly that reflexivity means on in this matrix the diagonal becomes 1. Symmetry implies that this matrix is symmetric. Now, what does transitivity mean? That, it is not immediately obvious or it cannot be seen from the matrix alone the corresponding relation, but we will try to interpret it and see what is inside the definition of transitivity.

So, now let us link begin with defining some functions, some relations having 1 or more of these properties. Let us begin with a relation which has all of these properties. For instance, if you consider x and y from set of natural numbers, you could define a relation like this, x is related to y if 3 divides x- y.

Clearly, you could say 3 divides 0, if you allow that then x is related to itself. It is reflexive. If x- y is divisible by 3, then y- x is also divisible by 3. And if x- y is divisible by 3, and y- z is divisible by 3, then it is clear that x- z is also divisible by 3. Simply you need to add x- y plus y- z, you get x- z. Both of them are multiples of 3, so the addition also will be multiple of 3.

Now, what about a relation which is none of these? Do we have such a relation? Quite interestingly, think of it like this, x is related to y only if x employs y. Note that not all relations have to be actually over the set of numbers or mathematical in nature. So, if you think of this a person may not be employed in his own company. So, x may not be reflexive. If x employs y, does not mean that y will y also employs x, that is not true. So, it is not symmetric.

And clearly it is not transitive, if x employs y and y employs z, it does not mean that x employs z because x could have one company A and y could have another company B and so x may not be employing z. So, here is a simple example of a relation which is neither reflexive, symmetric, nor transitive.

Now, let us look at relations which have two of these properties. So, what would be a nice interesting example for a relation which is only reflexive and symmetric, but not transitive. Think of this, x is related to y if x and y speak a common language.

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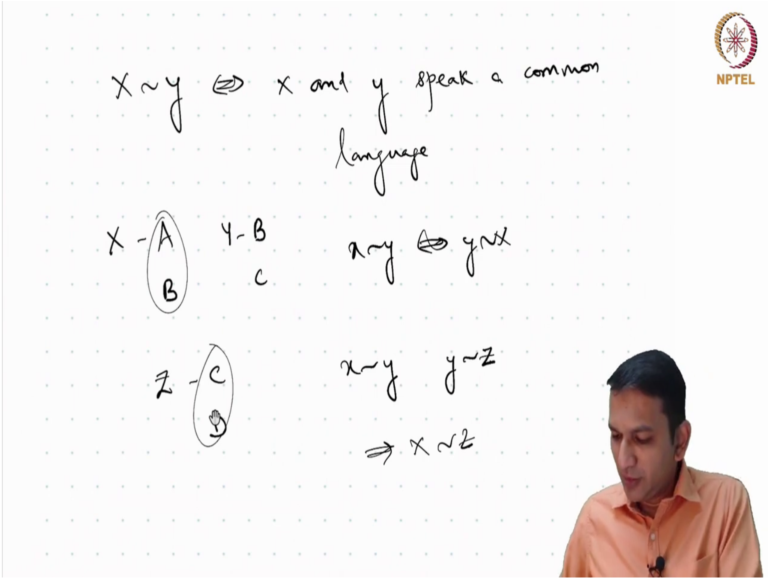

So, clearly if you look at this, so we have said x is related to y if and only if x and y speak a common language. Clearly, it is reflexive because x whatever he speaks is understood by x himself. So, it is common. If let us say that x speaks language A and B, and y speaks language B and C. So, now, there is something common between x and y, if x and y speak a common language then obviously, y and x also speak a common language which means it implies that x is related to y if and only if y is related to x.

But this relation is not transitive. Why? For example, let us see here that x is speaking the language as A and B, y speaks the language as B and C, so there is something common between x and y. Now, bring in z he might speak the language C and D, so that means, y is related to z because of C; however, this does not imply x is related to z because there are no common languages between x and z, assuming that all x, y and z speak only these two languages. So, you have such an example here.

What about a relation which is reflexive and transitive, but not symmetric? An interesting example would be x divides y and x is not equal to y, right. So, in I mean sorry x divides y. So, now, it is clear that x divides himself, so it is reflexive. x divides y does not mean y divides x, so it is not symmetric. If x divides y and y divides z, then obviously, x divides z. So, clearly, it is a reflexive transitive relation which is not symmetric.

Now, what about a relation which is not reflexive, but symmetric and transitive? Perhaps you could say x is a sibling of y. So, clearly, it is not reflexive, it is symmetric and also transitive.

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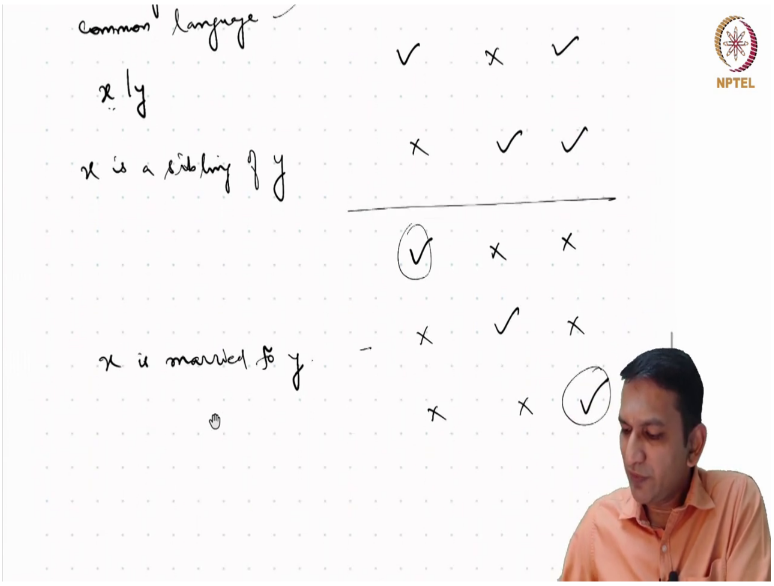

Finally, let us look at relations which are only which satisfy only one of these properties. In this, this case is simpler, you could define a relation which is symmetric, but not reflexive or transitive. For example, you can say x is related y only if x is married to y. So, clearly, x is related to y and y is related to x, but x is not married to himself or herself, similarly if x is married to y and y is married to z, does not mean that x is married to z. So, it is not transitive.

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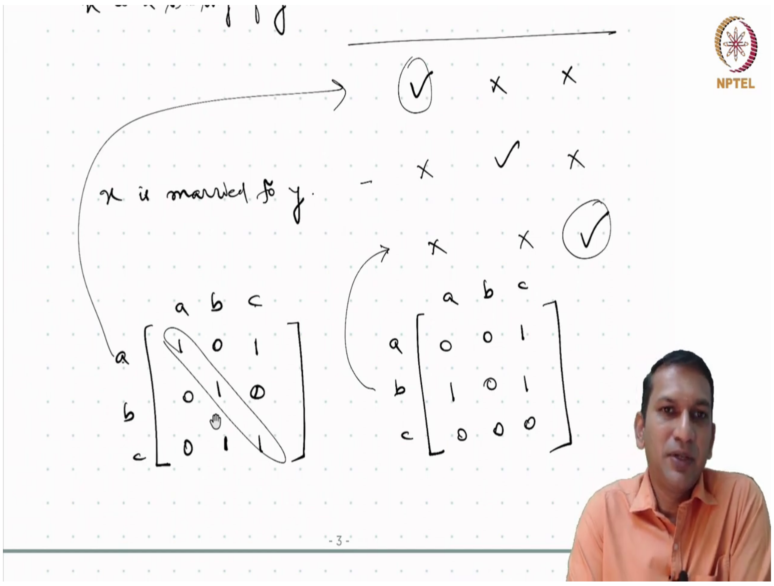

Now, how do we specify a relation which is only purely reflexive or purely transitive, but this is where we relate we resort to writing them as matrices or relations. Let us consider x to be with 3 elements a, b, c. We want it to be reflexive, but not symmetric. So, let us put 0 here and 1 here and 0 here. So, clearly, it is not symmetric.

Now, how do we ensure it is also not transitive? Now, see that a is related to c and let us make c also related to b, a is related to c, c is related to b. So, if it were transitive, then a related to c and c related to b would imply b is related to c, but let us put a 0 here. So, clearly, this relation is only reflexive, not symmetric and not transitive. So, this would be a nice example here.

What about a relation which is neither reflexive nor symmetric, but only transitive? Let us construct one like that know. So, clearly we do not want it to be reflexive. So, let us put 0s throughout on the diagram, you could also of course, have 1 somewhere. Now, let us relate a to c and let us relate b to a. So, b is related to a and a is related to c, so we should ensure b is related to c, and we do not need anything else. So, you see here 0, 0, 1 and 0, 1, 0 which means it is not symmetric, it is not reflective of course. And there are only 3 pairs in this relation and they ensure it is transitive.

So, b is related to a, a is related to c, so b is related to c; a is related to c, c is not related to anybody else, so transitivity does not come into picture; b is b is related to a, a is related to c and so and b c, b is related to c. So, the only possible relations are these and it is transitive. So, this would be a nice example here. So, as you can see often we may not be able to come up with a relation expressible on that not very simple easy to explain relationship, but we would when you actually construct or get some relations they may satisfy one or more of these properties.

Well, based on these 3 properties we can classify binary relations themselves. If a relation is reflexive, symmetric, and transitive, we know it is an equivalence relation. If it is only reflexive and symmetric, it is called a compatibility relation, ok.

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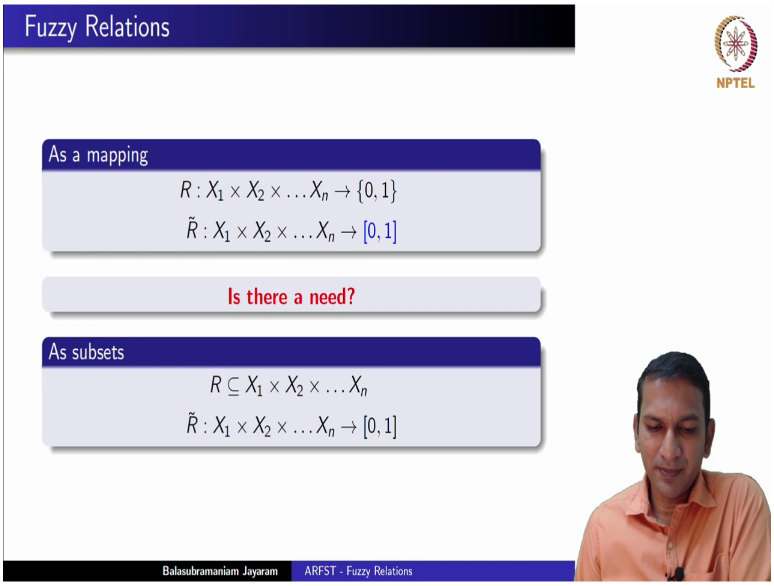

With this let us move to a fuzzy relations. Clearly, given a classical relation, how do we make it a fuzzy relation? By expanding the domain to the entire interval [0, 1]. So, it is a mapping from the Cartesian product of spaces to the unit interval [0, 1]. The immediate question is there a need for it. We can ask this question though in the case of classical relations, but let us look at it.

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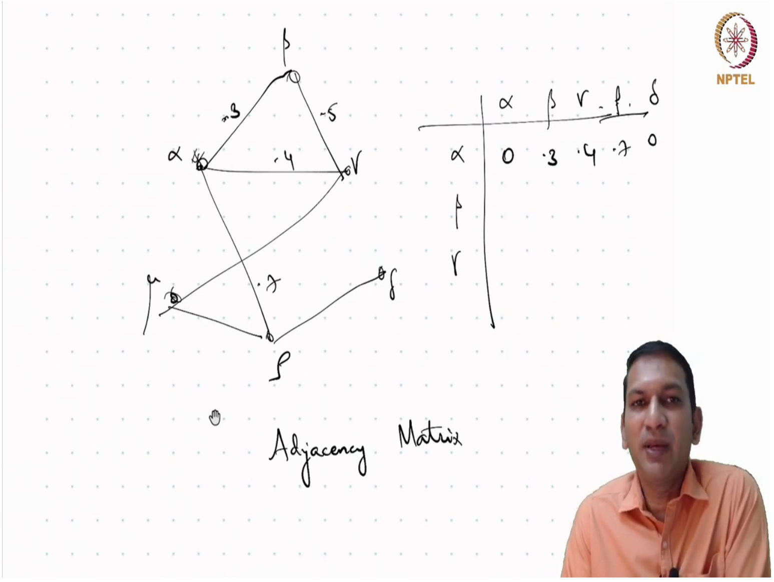

To understand this let us look at a graph. So, what is the graph? Graph consists of two things v and e, v is the set of vertices and e is the set of edges relating the vertices. So, consider this graph where you have 6 nodes and let us say this is how they are linked. So, now, if you label them as alpha, beta, gamma, delta, rho, mu, then if you wrote here like this, since alpha is related to beta you get a 1, alpha is related to gamma you get a 1, it is related to rho you get a 1, alpha is not related to delta, so you get a 0.

So, now, alpha whether it is related to itself or not, if you have a self-loop, yes. Otherwise, you should put a 0 here. So, you see here from the graph if it does not have a if each node does not have a self-loop then it does not reflex it. However, what you have is the graph being represented in terms of what is called an adjacency matrix. This clearly is a binary relation, binary classical relation, binary in the sense of having values in 0 or 1.

Now, it is also possible that these weights have some, these edges have some weights. They could be cost, they could be distance, what have you; they could be any number between 0 and 1 or even any negative or positive real numbers. If these were actually representing weights, then what we would have in this matrix are numbers from 0 to 1, 0, alpha beta is 0.3, alpha gamma is 0.4, 0.7 and 0. So, you see here immediately the adjacency matrix of a weighted graph turns out to be a fuzzy relation, that means, a relation which takes numbers between 0 and 1.

Now, which immediately tells you that some of the graph operations, what the operation that you could do on graphs can also be seen in terms of the operations that are being performed at the adjacency matrix. And now looking at it as fuzzy relations, perhaps we could bring into play some fuzzy logic operators, fuzzy set theoretic connectives and related operations to glean something more about this graph.

Going forward, we will discuss at least one such thing when we discuss a similarity classes soon enough during this week. But for the moment we clearly see that fuzzy relations do arise quite naturally in many settings. So, there is a need to discuss fuzzy relations. If you want to look at them as subsets, clearly this is a classical subset in the case of classical relation, and in the case of fuzzy relations it is essentially a fuzzy subset or fuzzy set on the Cartesian product of sets.

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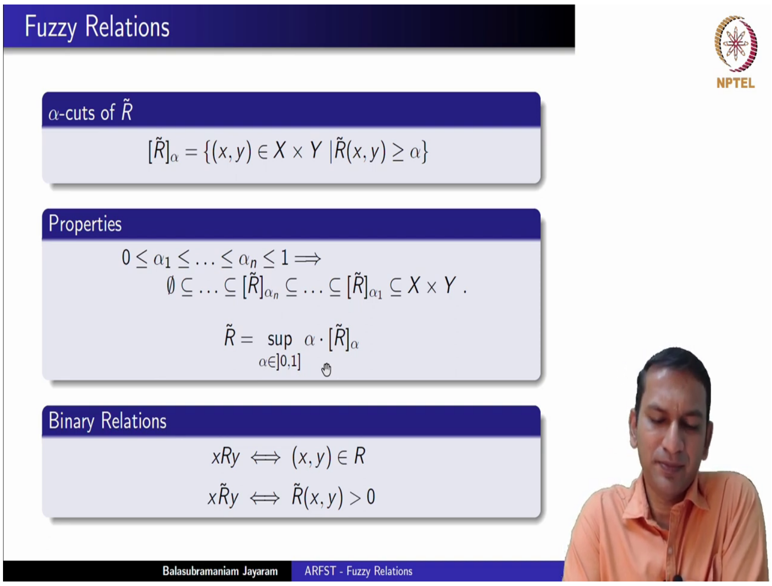

Now, you could also, given a fuzzy relation you could also consider the alpha cuts because fuzzy relations are essentially fuzzy sets on the Cartesian products. So, if it comes to the alpha cuts, what you would get is a binary relation which essentially means then you are looking at the subset from the corresponding Cartesian product.

In this case, what we have indicated is only for the binary relation. And immediately many things coming into play. This we will see soon enough. So, if you have the level set of this fuzzy relation and you pick some alphas from there. If alpha 1 to alpha n is an increasing sequence, then we know that the corresponding alpha cuts will actually be a decreasing in terms of the subset. Means higher the alpha, smaller the number of elements we are going to have in that subset.

And you could reconstruct the fuzzy relation once you know the corresponding alpha cuts and the alpha the level set. This is very much the same in the case, as in the case of fuzzy set because fuzzy relations are fuzzy sets on Cartesian products. Finally, we indicate binary relation, classical binary relation like this x is related to y if and only if x, y belongs to the subset. In the case of fuzzy relations, we simply say that x is related to y if R of x, y is greater than 0. That means, x, y should belong to the support of this fuzzy set.

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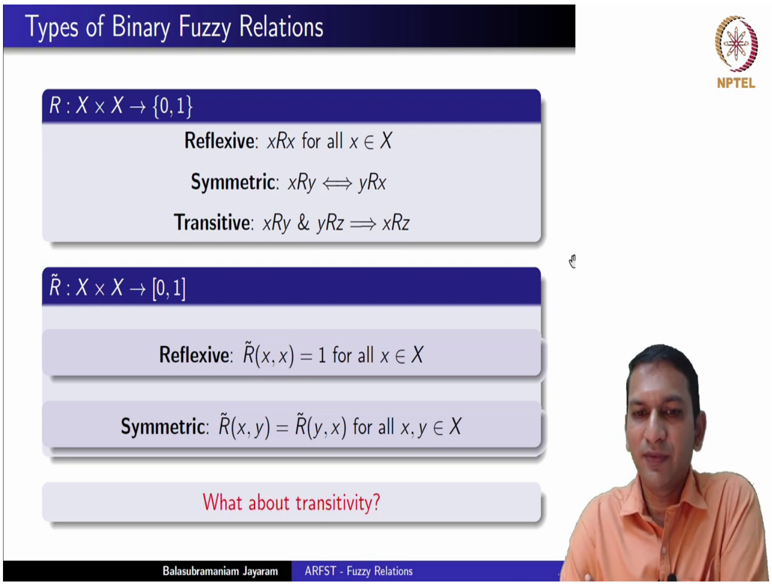

Now, can we talk about these special relations also? Means can we talk about when a fuzzy relation is reflexive, symmetric and transitive. In the case of reflexivity, this is how it is defined. We say a fuzzy relation is reflexive if R of x, x is equal to 1. Just a few minutes earlier, moments earlier we saw that x is related to y in the case of binary fuzzy relation, if R of x, y is greater than 0. So, it should just belong to the support.

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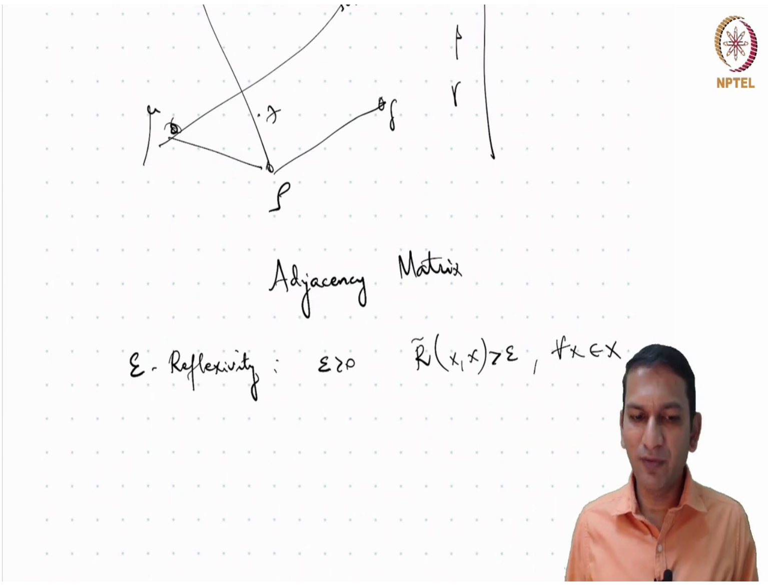

So, we could also have defined that x is the relation is reflexive if R of x, x; x, x is greater than 0. Now, such definitions also exist. So, it is called epsilon reflexivity. So, fix an epsilon greater than 0 orbit in small and say that R tilde is reflexive, it is greater than epsilon for all x in X.

Now, if R of x, x is greater than 0 for every x and assuming that it is finite, you could always find an epsilon which seems to which is the minimum of all such possible pair that you can take and then call that as the epsilon and it becomes epsilon to reflexive. However, in this course when we refer to reflexivity, it will always be under this definition that is R of x, x is equal to 1 for all x.

Now, when do we say it is symmetric? Clear, that if x is related to y, under the relation to some degree, then y should also be related to x under the same degree. So, essentially the matrix of the relation, fuzzy relation should be symmetric. The question is how do we define transitivity.

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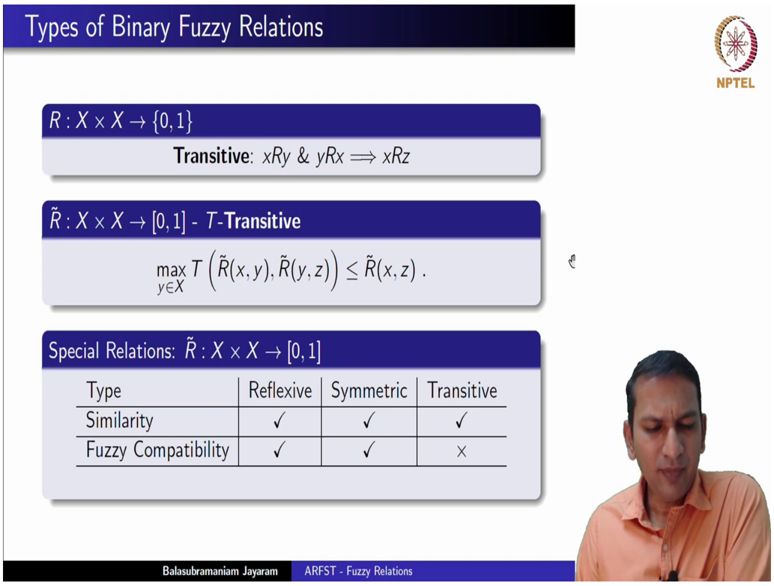

So, you know this is transitivity for classical relation. It is not easy to read off whether a relation is transitive or not just by looking at the matrix of numbers. So, we need to look a little deeper here. So, towards that let us try and understand what we mean by transitivity here. So, transitivity says that if x is related to y and y is related to x, they should imply it should insist that x is related to z.

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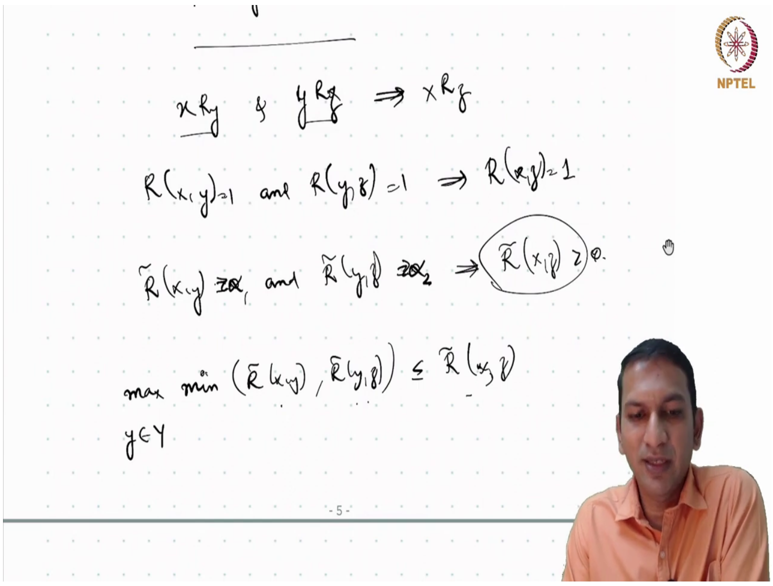

Now, in the classical case what it means is R of x, y is equal to 1 and R of y, z is equal to 1. They should imply R of x, z is equal to 1. But now here we are only saying that x is related to y and y is related to z. In the fuzzy case, this would mean R tilde of x, y is greater than 0 and R tilde of y, z is greater than 0.

Now, what we need to show is that this should imply that R tilde of x, z is greater than 0. But we cannot define a fuzzy relation taking arbitrary values because we know that R of x, y is perhaps taking the value alpha 1 and R of y is taking the value alpha 2. So, somehow this the relation, the strength of the connection between x and z should somehow be dependent on the strength of the relations between x and y, and y and z.

Now, how do we define this? Now, let us look at it. In the classical case, the transitivity is defined only if x is related to y and y is related to z. Now, it can happen that x is related to y and y is not related to z, but still x may be related to z or the other way round. So, essentially what we insist here is if you look at this, we want that minimum of R of x, y, R of y, z should be less than or equal to R of x, z.

So, when both of them are 1, then we want this to be 1. If one of them is 0 or both of them are 0, then we really do not care. It does not affect the transitivity. Now, this is the definition that we could use also in the case of fuzzy relations. However, notice that x and z may be related through many ways. So, that means, we essentially take that R tilde of x, z should be greater than or equal to all possible connection that can exist between x and z through the different ways and all you want is it should be max over min of this.

And you will see that this definition of transitivity will also hold in the case of classical relationship. So, this is typically taken as a relate the transitivity of a fuzzy relation. Why only minimum we could also use any t-norm, so typically it is translated as T-transitivity for a given t-norm. And you will see that in the classical case because we are dealing only with 0s and 1, it is exactly the same as the definition of transitivity that we have seen in here.

Now, once again if you look at fuzzy relations because we have defined these 3 properties, if a fuzzy relation has all these 3 properties, reflexivity, symmetry and transitivity, we call it either a fuzzy equivalence relation or typically it is called a similarity relation. If it is only reflexive and symmetric and not transitive, then we call it fuzzy compatibility relation.

These are two very important types of fuzzy relations. Moving forward in this week, we will see their role in analyzing data itself based on the properties that they allow us to extract from the fuzzy relations.

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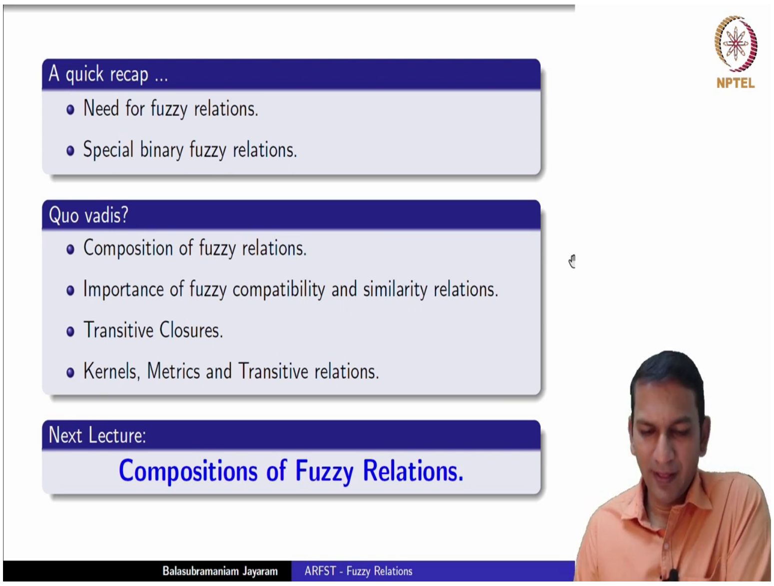

So, in this lecture, we have just seen briefly what are fuzzy relations, the need for them and we have touched upon a couple of special binary fuzzy relations that of similarity and compatibility. Moving forward, we will look at how to compose such fuzzy relations, the importance of fuzzy compatibility and similarity relations that we have just introduced, the importance of transitivity itself.

In fact, a large part of this week, the different lectures that we will have can be seen as elucidating the importance of transitivity of a fuzzy relation and their relationships to many other concepts that readily do not spring to our mind when we are talking about relations. Those are that of kernels and matrix. We will also see how these are related to transitive relations.

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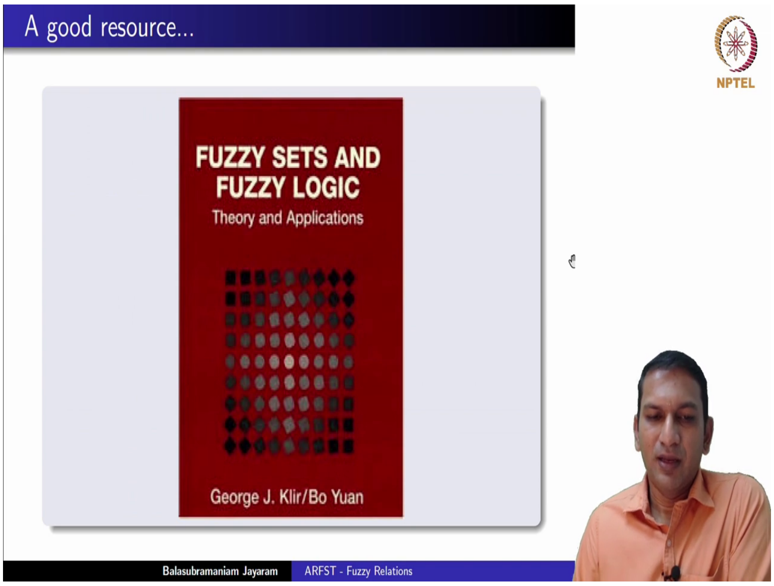

In the next lecture, we will deal with Compositions of Fuzzy Relations. A good source for the topics covered in this lecture and perhaps also in the rest of the week on fuzzy relations is this very good book by George Klir and Bo Yuan titled Fuzzy Sets and Fuzzy Logic.

Glad you could join us for this lecture. And looking forward to meeting you in the next lecture.

Thank you.