Approximate Reasoning using Fuzzy Set Theory
Prof. Balasubramaniam Jayaram
Department of Mathematics
Indian Institute of Technology, Hyderabad

Lecture - 21
Construction of Fuzzy Implications – IV

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Hello and welcome to the last of these lectures, in this fourth week under the course titled Approximate Reasoning using Fuzzy Set Theory, a course offered over the NPTEL platform. So, far this week we have exclusively concentrated on the operation of fuzzy implication.

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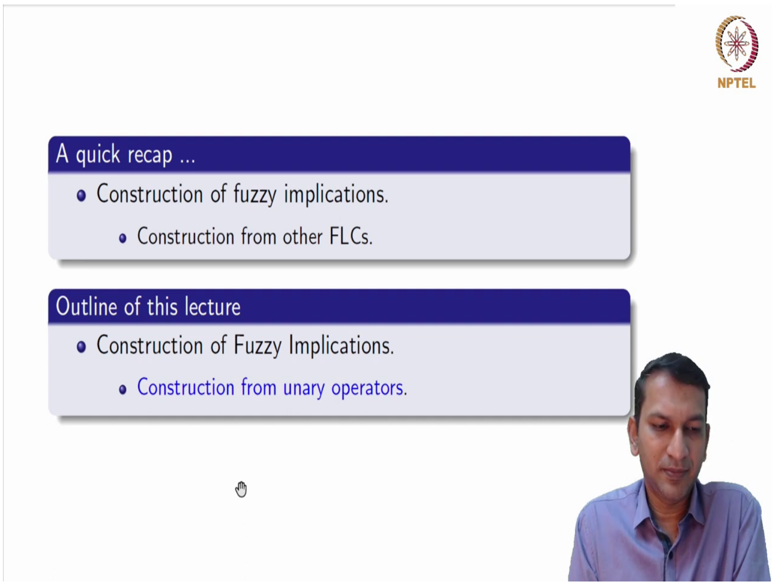

In the last few lectures, we have seen how to construct fuzzy implications from other fuzzy logic connectives. In this lecture, we will look at constructing fuzzy implications from unary operators.

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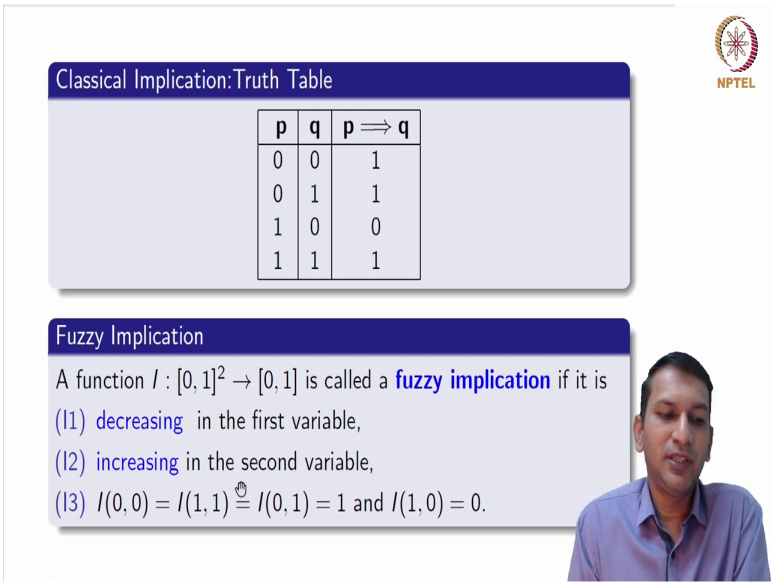

A quick recap. We have seen that this is the truth table for the classical implication. Based on this we have defined a fuzzy implication which is a mixed monotonic binary operation on unit interval [0, 1], decreasing in the first variable, increasing in the second variable with these four boundary conditions.

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We have seen how to construct fuzzy implications from other fuzzy logic connectives.

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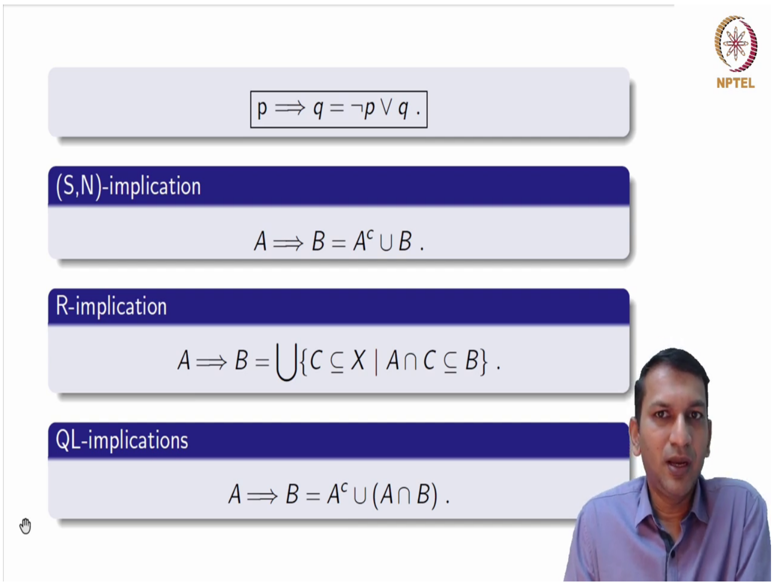

In this quest of ours what really helped us is the equivalent formulation of p implies q in terms of this logical formula negation p or q. We were quick to translate this into the language of set theory and saw this as A complement union B, which allowed us to come up with the family of (S, N)-implications by simply substituting negation for complementation and a t-conorm for the union.

And using another equivalent formulation that of finding out the largest subset C of X such that A intersection C is contained in B we came up with another family of fuzzy implications namely the R-implications. Once again taking Q from the first formula and writing it in yet another equivalent form we came up with a family of QL-implications.

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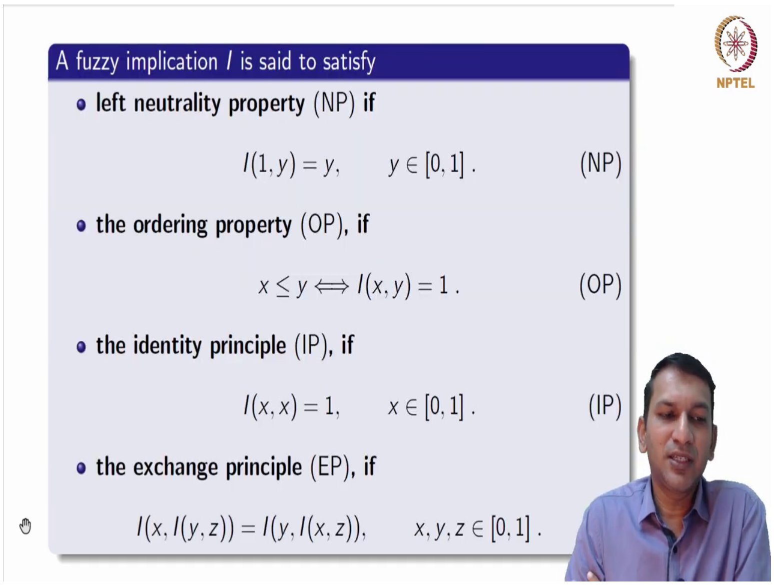

And all of these families we discussed when if and when they satisfy these four desirable properties namely the left neutrality property the ordering property the identity principle and the exchange principle.

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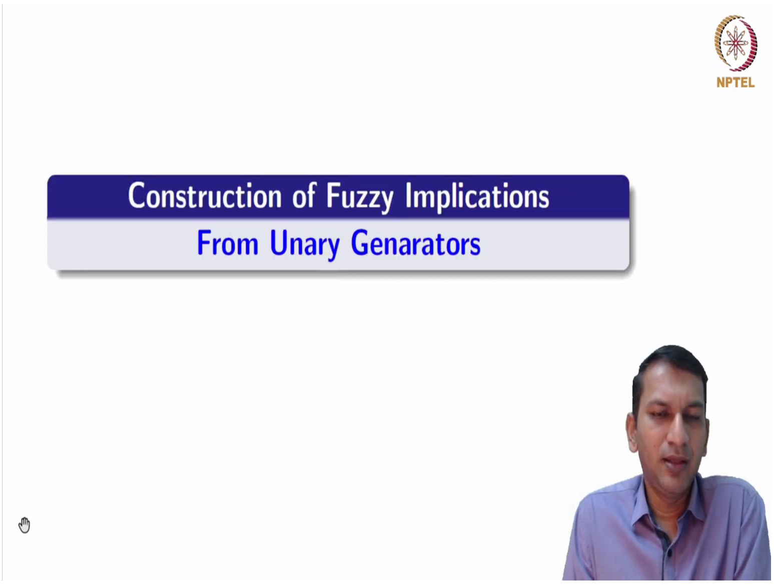

In this lecture we will look at constructing fuzzy implications from unary generators or unary functions.

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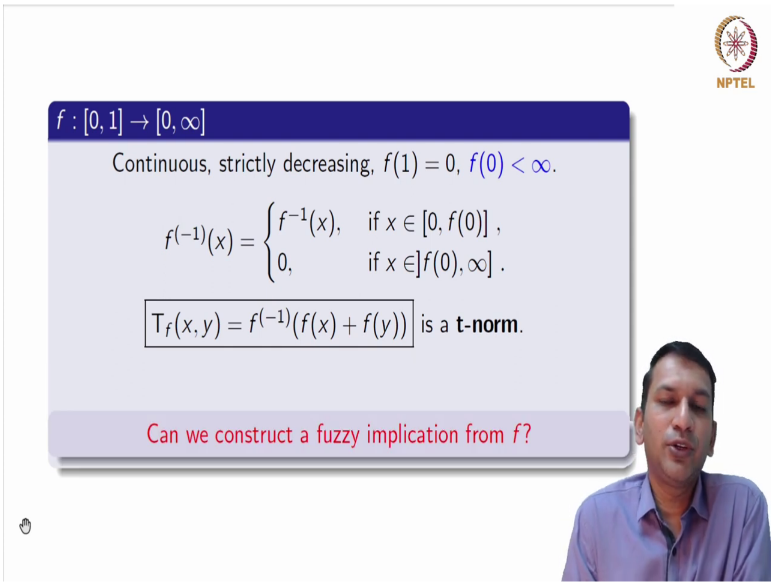

Let us look at a function f from [0, 1]to [0 infinity), which is continuous strictly decreasing, such that f(1) is 0 and f(0) is less than infinity. We have seen this kind of generators when we discuss the additive generators of triangular norms, we have seen that you could write the pseudo inverse in this way.

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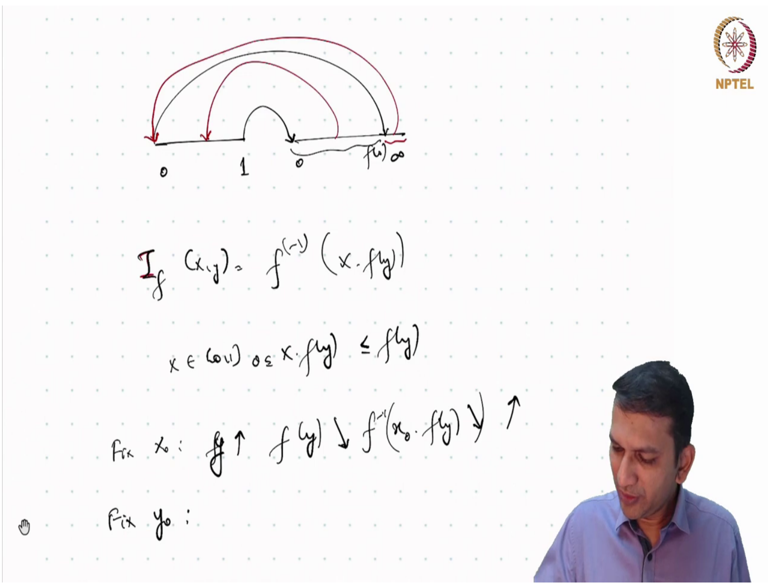

Quickly you might recall what we have is a function strictly decreasing function f from [0, 1] to [0, infinity) such that 1 maps to 0 and f(0) can map to either infinity or some value less than infinity and we have seen that if you want to talk about the inverse for anybody within 0 and f(0) then it is the actual inverse.

But anybody here between f(0) and infinity what we do is if pseudo inverse maps it to 0 that is the definition we have given here. We have also seen that if you define a function T_f that is function T from this f as f inverse of f pseudo inverse of f of x plus f of y it actually becomes a t-norm its commutative, associative, monotonic in both variables and 1 becomes a neutral element. The question is can we construct a fuzzy implication from such unary operations f.

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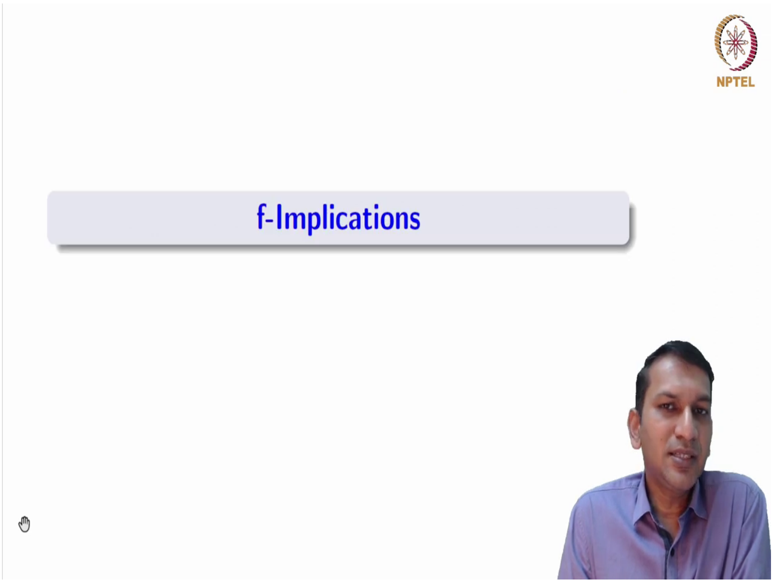

This is less lead to the introduction of the family of f-implications or often also called as f-generated implications.

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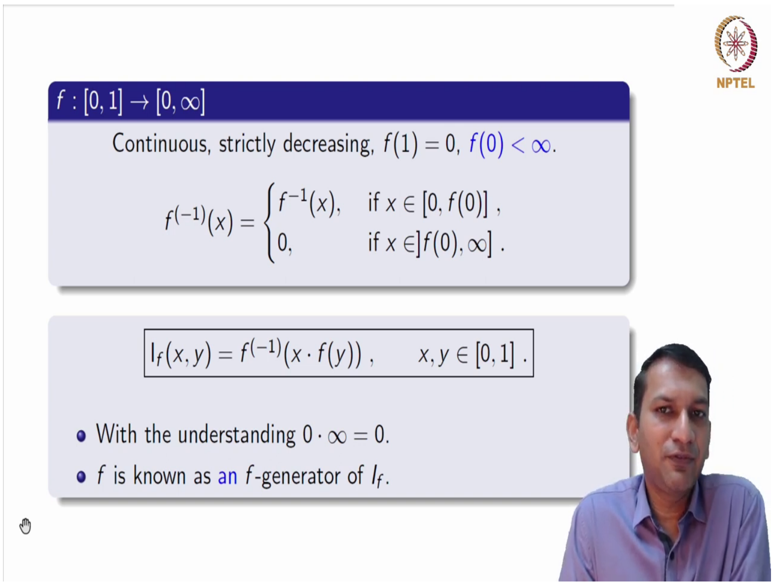

Take this function f from [0, 1] to [0 infinity) is continuous strictly decreasing f(1) is 0 f of 0 is less than infinity whose pseudo inverse is given like this. Let us look at the formula which is given like this. An I obtained from f as f pseudo inverse of x into f of y where x and y are coming from [0, 1].

With the understanding that 0 dot infinity is actually 0 we call such an I_f; we call f which gives us this I_f as an f generator of I_f not the word and because there can be many such generator functions here for the same function if ok.

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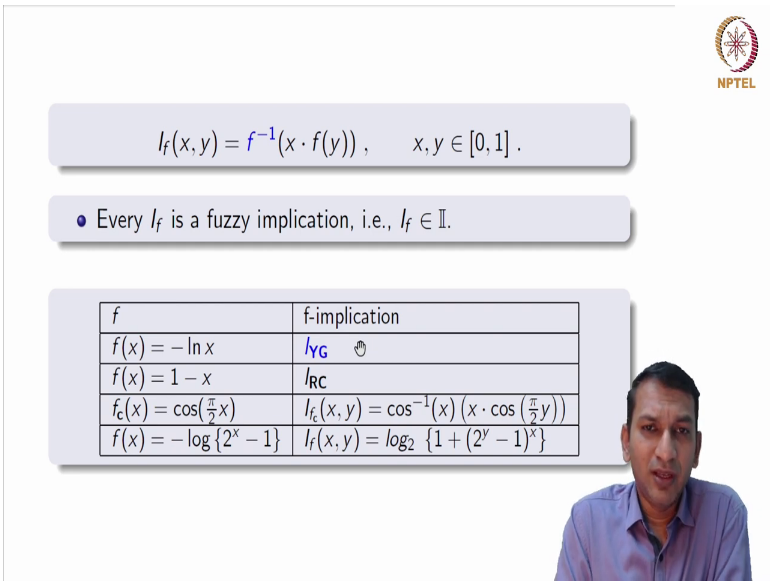

So, now if you look at this function once again note that we have not written f pseudo inverse, but only f inverse this can be easily explained. So, what we have is that I_f of x y is given as f pseudo inverse of x dot f y. Note that x comes from [0, 1]and so, x dot f y will always be less than or equal to f of y which means f pseudo inverse of f y on this value will always be the inverse because it is going to lie between 0 and f of y and f of y is the image of y which means it lies between 0 to f (0).

Thus, it is also sufficient to write this as f inverse and not use the pseudo inverse. Well, now that we have defined such a binary function the question is it really a fuzzy implication. Yes, it is a fuzzy implication.

To see this once again let us consider this. Let us fix an x_0. Now as y increases since f is a decreasing function f(y) decreases which means x_0*f(y) decreases. But when you apply f inverse which is also decreasing function, this entire thing increases which means this function is increasing in the second variable.

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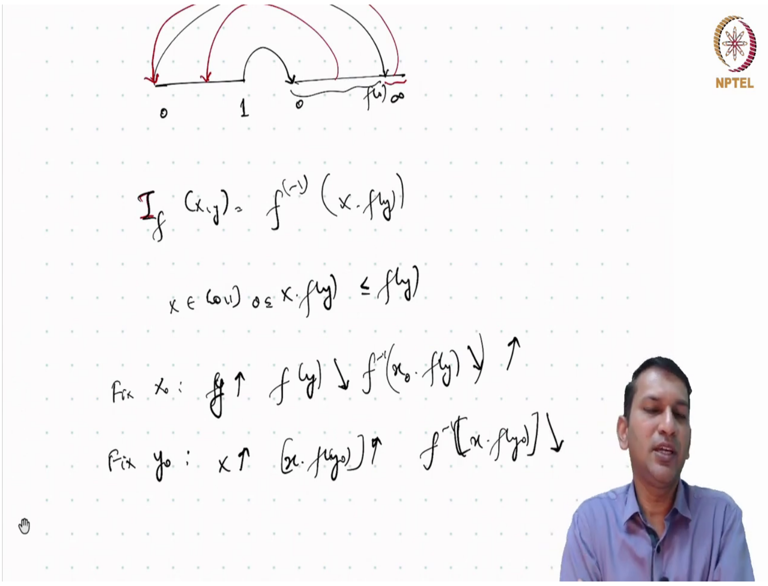

Instead if you fix a y_0 and look at this formula as x increases x dot f of y naught this increases, but if you apply f inverse on this f of y_0 clearly this decreases because f inverse is also a decreasing function. Well, now let us look at some examples.

If you take the function minus long x as the generator f we get the Yager implication. After long Yager's implication which is one of the basic fuzzy implications as we have expressed it as makes an appearance now and when you take f of x as 1 minus x you get the Reichenbach implication.

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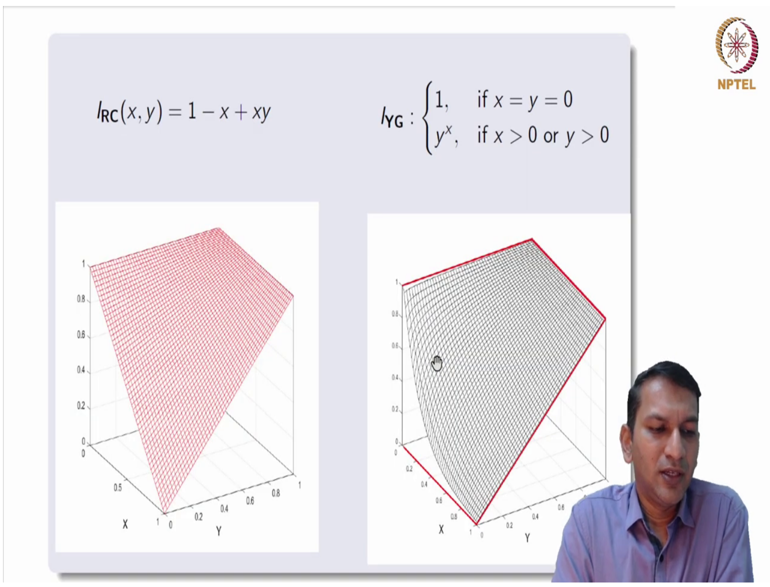

So, these are the 3D plots of these two fuzzy implications. By now these must be familiar to you. Now, let us look at the properties that these fuzzy implications have. Clearly both of them have the neutrality property while the natural negation of Reichenbach is 1 minus x in this case of Yager we find that it is actually the smallest fuzzy negation.

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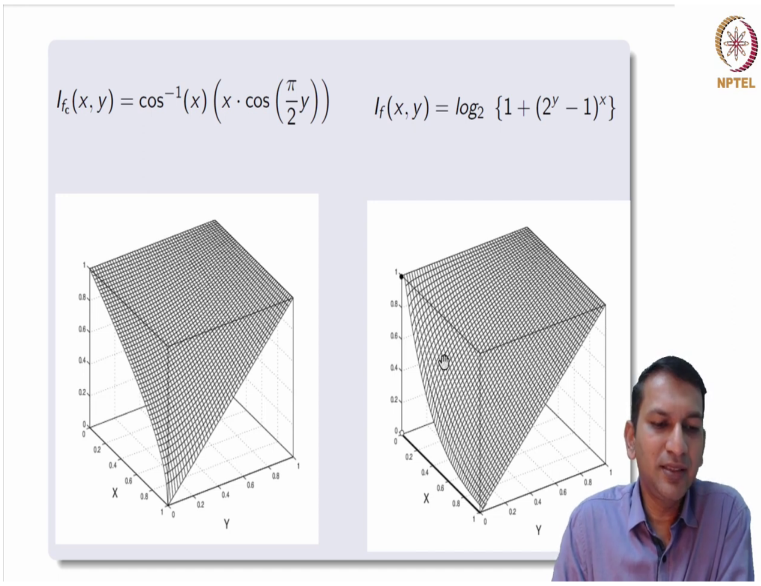

When you consider the other two generators these are the 3D plots of the corresponding fuzzy implications. You see here once again that both of them have the neutrality property, none of these has got the identity principle or the ordering property and the natural negation here is not 1 minus x, but it is a continuous strictly decreasing negation whereas, in this case its once again the smallest fuzzy negation.

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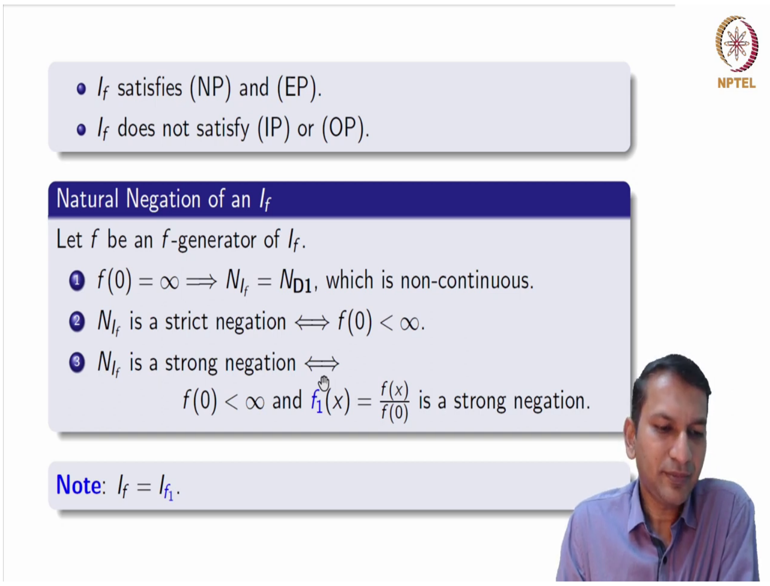

So, if you look at what properties they satisfy, it can be shown that any f implication satisfies both the neutrality property and the exchange principle. Now, how do we show this?

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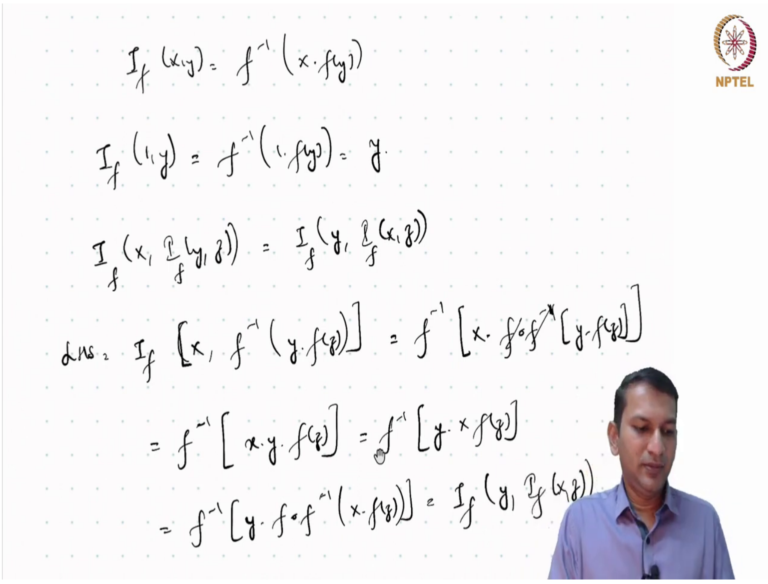

Once again, I_f of x y is equal to f inverse of x dot of f y. For neutrality what we need to show is I of 1, y should be equal to 1, but this is f inverse of 1 dot f of y, clearly this f inverse of f of y, which is equal to y. Now what about the exchange principle? We want to show that I of x, I of y z is actually equal to this and of course, we need to show this for the f inverse. So, let us look at LHS. Once again we will unfold from the inside.

So, I_f is f inverse of y dot f of z, this can be written as f inverse of x dot f circle f inverse of y dot f of z. Now, clearly f composed of f inverse vanishes and what we get is f inverse of x dot y dot f of z and now it is clear that all we need to do is by the commutativity of product you could write it like this and reinsert f circle f inverse. What we get is I_f of y in I_f of x, z.

Thus, any f implication satisfies both the neutrality property and the exchange principle. Unfortunately, none of the I_f implications satisfies either the identity principle or the ordering property. Once again this can be seen.

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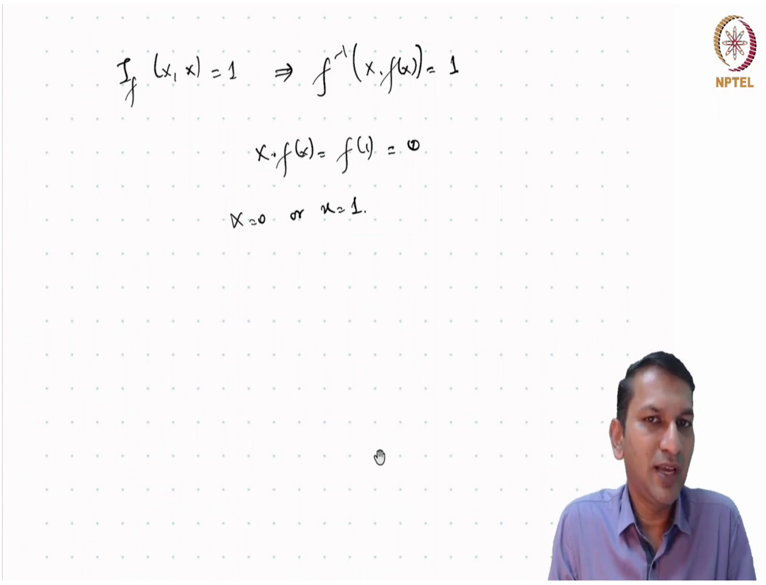

If you want I of x x to be 1 this implies f inverse of x dot f of x to be equal to 1. Now that means, x of f x is equal to f of 1, but by our assumption f of 1 is 0 such kind of unary functions as what we are considering. Which means x dot f of x is 0 clearly shows that either x is 0 or x is equal to 1.

So, for no other x this is going to be valid. We know that if an implication has OP it implies IP, but the contra positive since it does not have IP clearly it cannot satisfy OP. What about the natural negation of an I_f? So, f is an f generator for I_f. If f of 0 is infinity. So, clearly there are two possibilities for f of 0 either it is infinity or is finite.

If it is infinity, then it can be shown that what you would get is the smallest negation which is non continuous. On the other hand if f of 0 is less than infinity, then it is always a strict negation further if f of 0 is less than infinity and if you define this generators f n as f of x by f of 0. If this is a strong negation then we can show that the natural negation obtained from such an f implication is in fact, strong.

Well, if you recall we say that f is an f generator for the function f that is because you can have many different generators for the same function in f. Note here that if you consider this f 1 in the case of f of 0 being finite, then what you would have obtained with f is essentially the same as what f generated implication you would obtain with this f 1 also. In that sense this f is not unique, it is only unique up to a particular generator.

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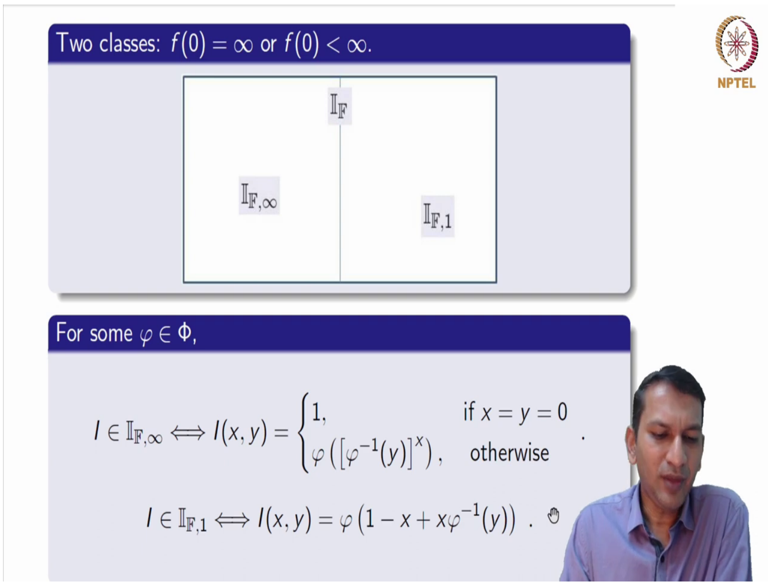

Well, now if you look at the family of F implications itself we know that these are being generated by these numeric functions f and we can broadly partition this into two based on whether f of 0 is infinity or f of 0 is less than infinity. Now, it has a very terse representation. Quite a beautiful work, which has led to this.

If you consider this capital phi which is set of all increasing bijections on the unit interval and if you take one such bijection, phi belongs to this set. If I belongs to this set; that means, if it is generated from an f such that f of 0 is infinity, it can be shown that I will actually that I can be represented this way for some increasing bijection on the unit interval 0, 1.

And if the f implication is generated from an f such that f of 0 is finite then such an implication can be represented for some phi an increasing bijection of [0, 1]in this form.

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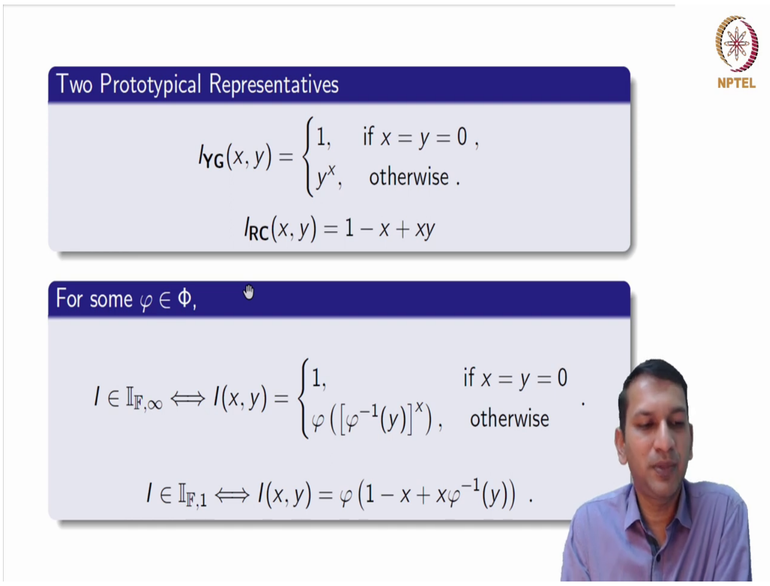

Now, what is interesting is look at these two implications, Yager's and Reichenbach. We know that both of them are in fact, f implications, but not just that they seem to be the prototypical representatives of f implication. Why do we say so? Look at this representation. If I belongs to I F infinity it means F of 0 is infinity or I belongs to F of I F 1; that means, F of 0 is finite we see that these are the representation that are given to us.

Now, take phi to be the identity function which is an increase in bijection on 0 and what you get is this I x, y is in fact, the Yager's implications and if you substitute identity for phi here what you would get is actually Reichenbach implication. So, every f implication in a sense is some transformation of either Yager's implication or the Reichenbach implication.

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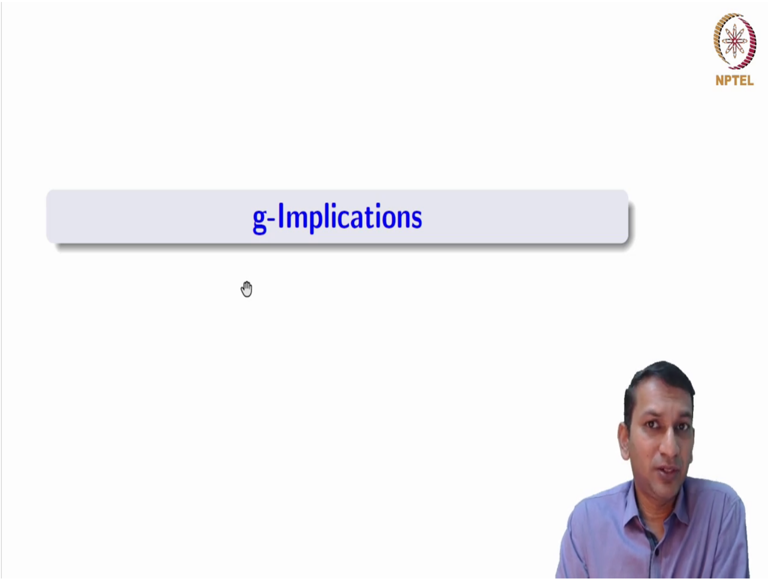

Well, this is one family of implications or the f implication that we have generated from decreasing continuous functions.

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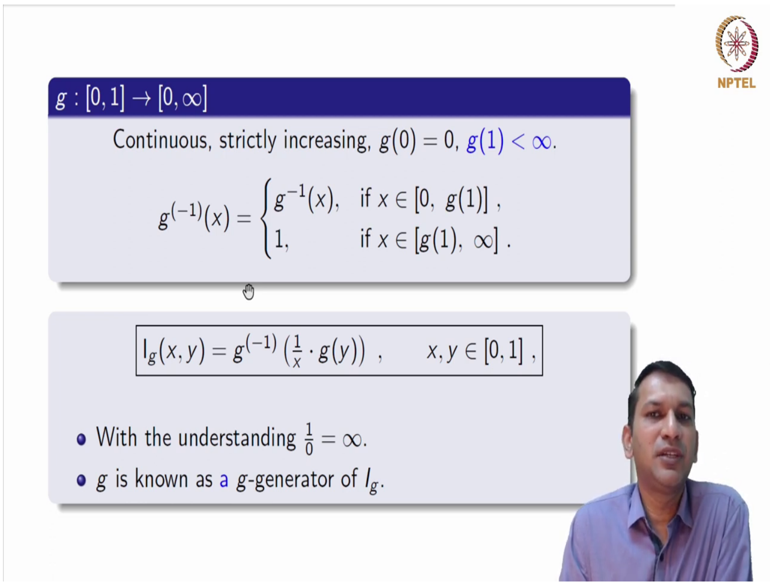

The obvious question is what we have continuous strictly increasing functions from [0, 1]to 0 infinity, so obviously, g of 0 is 0 and it is g of 1 that could either be infinite or finite. Once again we could discuss the pseudo inverse in this case if g of 1 is finite and look at this formula I g. With the understanding that 1 by 0 is infinity we call such a g as a g generator of I g, clearly it means that this generator is not unique.

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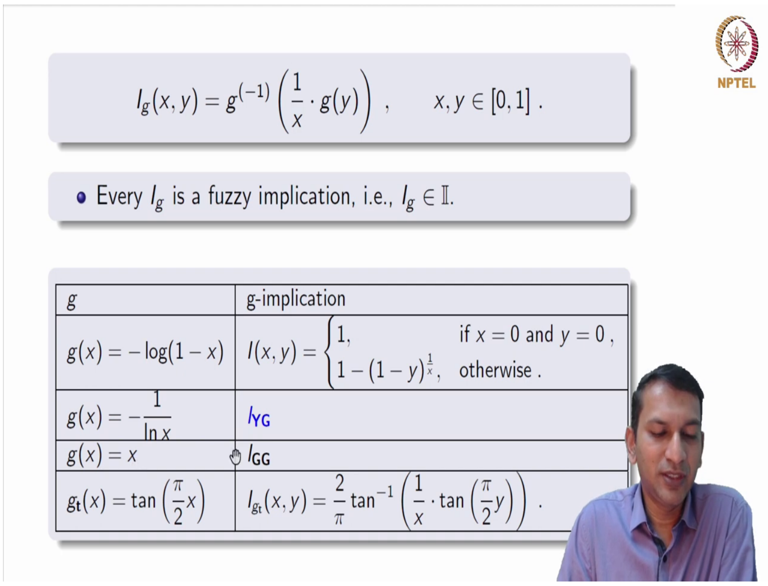

What can be shown quite quickly is that every such I g is a fuzzy implication. The proof is similar to what we have seen in the case of f implications. And if you look at some examples you see that if g of x is minus 1 by long x, we actually get the Yager's implications and g of x is the identity function we get the Goguen implication.

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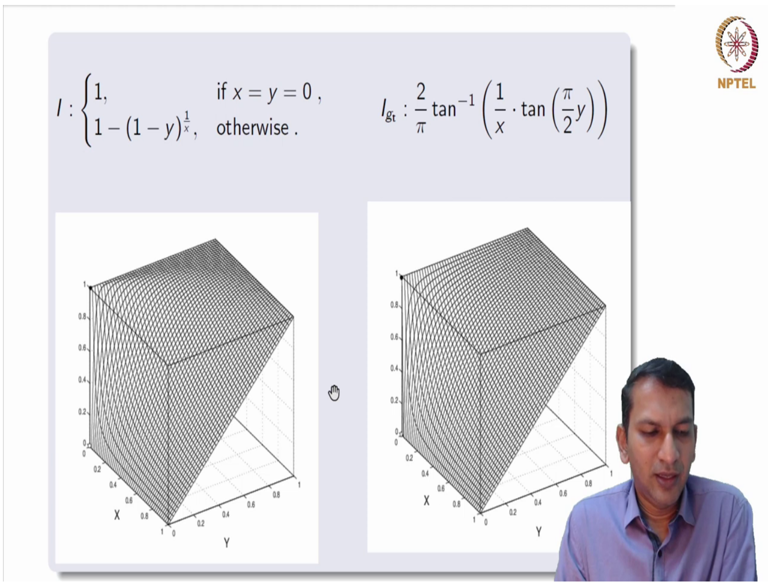

The plots of Yager's and goguen we know. So, these are the plots of the other two fuzzy implications that we have given here which are obtained from these 2 g generators minus log of 1 minus x and tan by 2 of x. You see once again here that we actually get neutrality property for both these fuzzy implications and interestingly both of them seem to have their natural negation as the smallest fuzzy negation.

And none of these two has OP or IP of course, we know that Goguen implication does satisfy OP and hence IP whereas, Yager's does not satisfy IP or OP.

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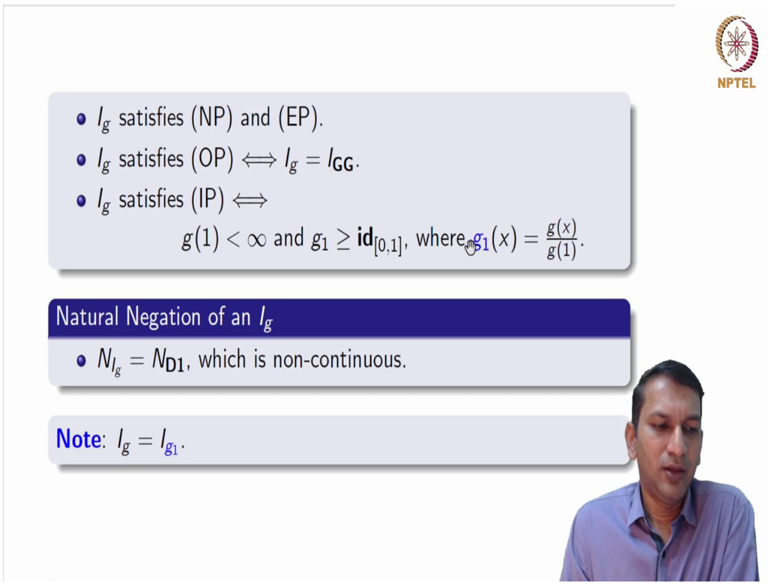

If we look at the property they satisfy definitely every g implication satisfies both the neutrality property and the exchange principle, it can be shown as we have done it in the case of f implications. What about OP? It can be shown that the only g implication which satisfies the ordering property is the Goguen implication.

If you are looking at identity property, yes, there are many g implications which do satisfy the identity property, but under this condition; that means, g of 1 has to be finite and this function g of g 1 which is defined like this it has to be greater than the identity function on the unit number.

What about the natural negation of g implication? As we have seen it is always the smallest fuzzy negation in that sense it is always non continuous. What is interesting is given a g, g of 1 is finite, you could construct I g which is the g implication, but you could also construct the same g implication using this function g 1 which is essentially a scaled version of the original g.

In that sense it is the generator is unique up to a constant multiple of any other generator.

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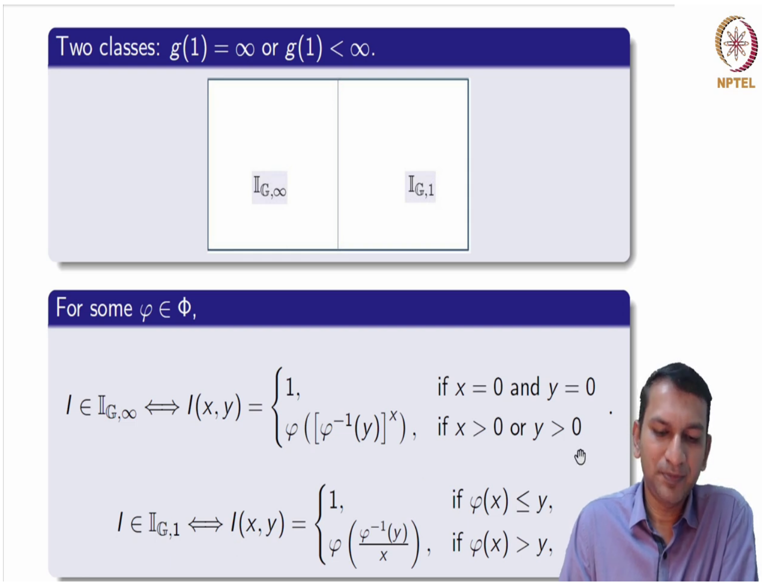

Well, once again we could partition the set of all g implications into two based on whether g of 1 is finite or infinite and what we have are these two interesting representations. Once again if I belongs to I G infinity; that means, it is generated from generator g such that g of 1 is infinity then such an I can be represented like this for some increasing bijection on [0, 1]interval.

And similarly any g implication obtained from the generator G such that G of 1 is finite such an implication can be represented like this for some increasing bijection on the unit interval.

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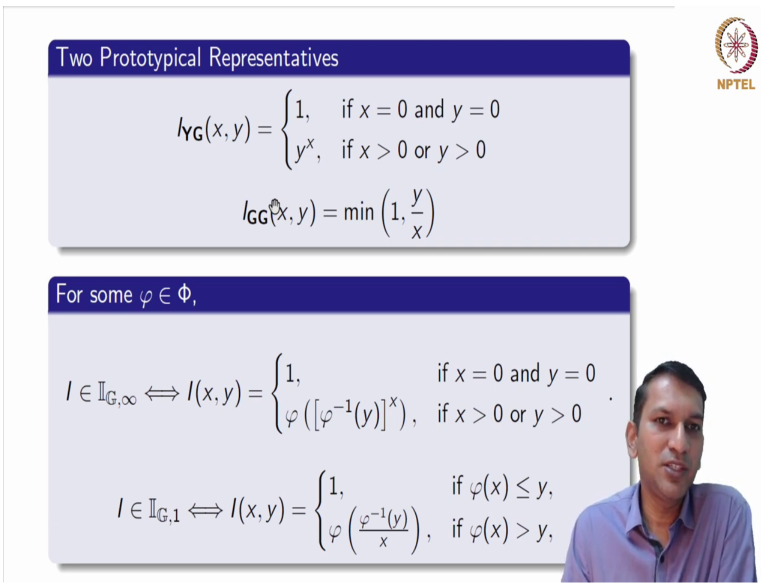

We know that the Yager implication or the Goguen implication, both are G implications not just that they turn out to be prototypical representatives of this family. Why do we see so? Look at this. If we take this increasing bijection on [0, 1]to be just the identity we see that this turns out to be the Yager's implication and in this case it turns out to be the Goguen implication.

So, the class of all G in generated implications where G of 1 is infinity is somehow just a transformed Yager's implication and the other part that is G generated implications from g which are such that G of 1 is finite they are nothing but some kind of a transformed version of the Goguen implication.

So, these are the two important families of fuzzy implications that can be obtained from unary operators. Of course, there are many more such families that have been constructed from unary functions, but for our purposes these two families would serve as well.

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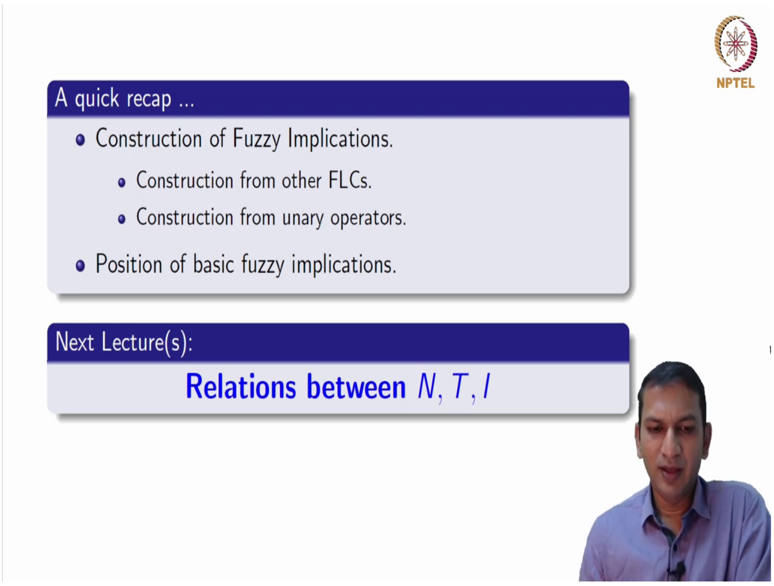

A quick recap of what we have been discussing in the last few lectures. We have discussed construction of fuzzy implications. Firstly, from other fuzzy logic connectives and in this lecture from unary operations. What should be very heartening is the role played by the nine basic implications, how they have actually turned out to be coming from these five families that we have seen so far.

Not only do they belong to these five families that we have discussed that is not the only reason why we discussed these five families, you will see that both these families and these basic fuzzy implications themselves will play a very interesting role in the application that we will discuss especially from relational or fuzzy inference systems point of view.

What next? So, far we have seen three basic fuzzy logic connectives that of negations T norms and implications. In the last lecture of this week, we will see some relationships that exist among these three.

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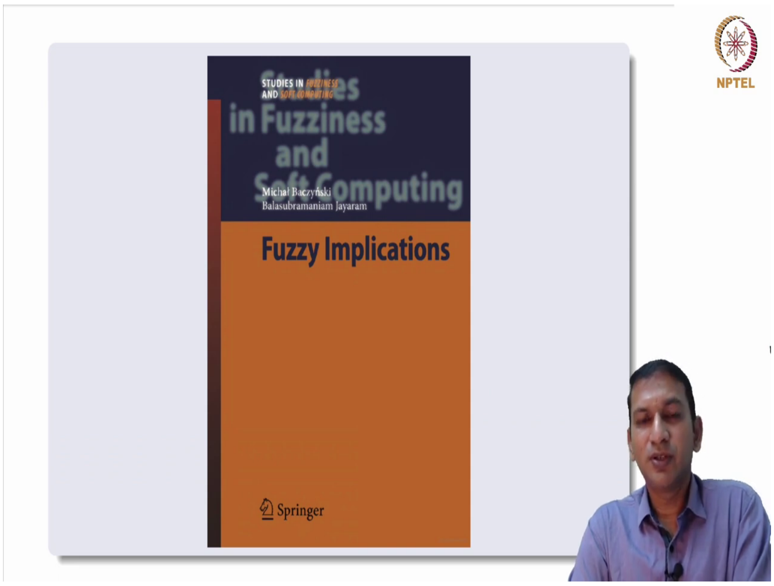

Once again a good source for the topics covered in this lecture is the book on Fuzzy Implications. Glad that you could join us today for this lecture. Looking forward to meeting you again in the next lecture.

Thank you.