Approximate Reasoning using Fuzzy Set Theory
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Lecture – 20
Construction of Fuzzy Implications – III

Hello and welcome to the next of the lectures under the series titled Approximate Reason using Fuzzy Set Theory. A course offered over the NPTEL platform.

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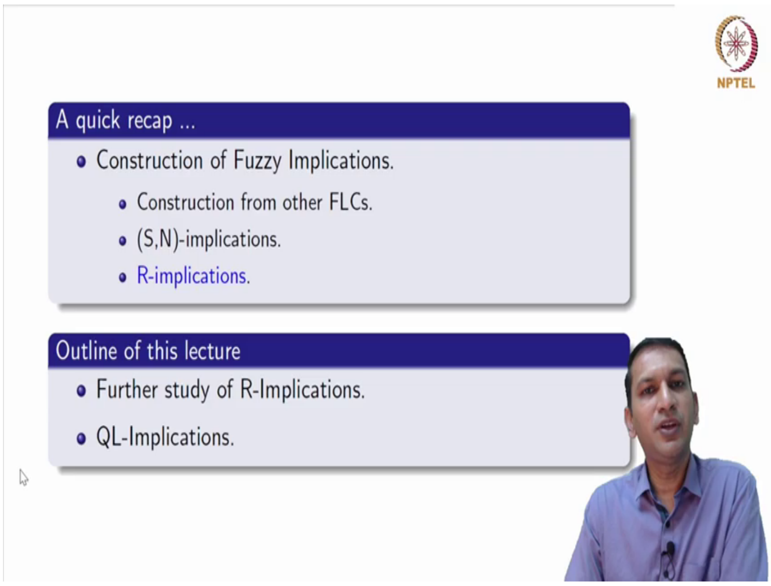

In the last few lectures we have been discussing the Construction of Fuzzy Implications. Specifically their construction from other fuzzy logic connectives. We have already seen the family of S N implications which are obtained from a t-conorm and a negation; we have had a gentle introduction to the family of R implications which are obtained from just t norms.

In this lecture we will further study R implications vis a vis how to obtain their formulae for yet another T norm and also the properties that they satisfy we will also look into the final the third and final family of fuzzy implications using this construction in this lecture series that of QL implications.

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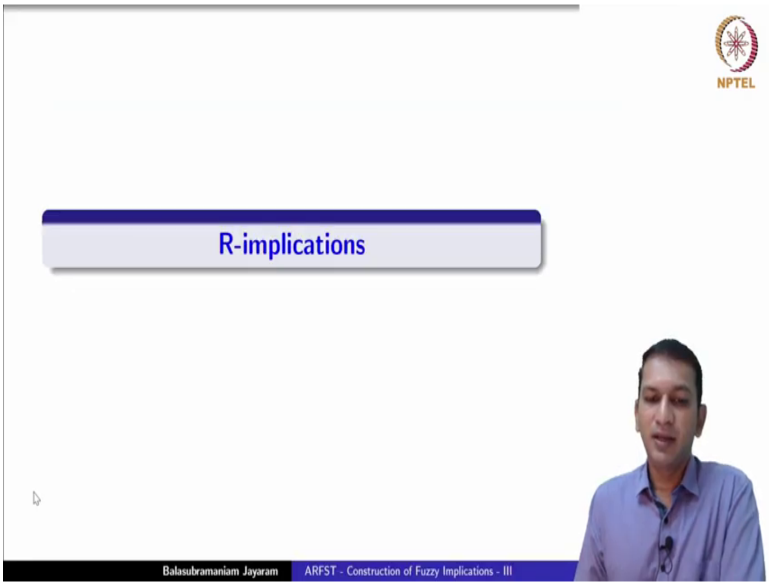

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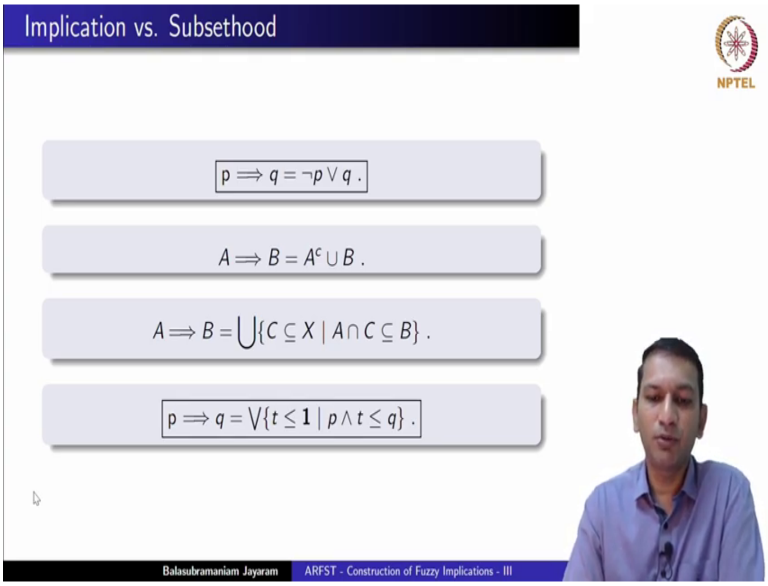

Let us have a quick recap of R implications. We have seen that p implies q in the classical logic setting can be equivalently represented by the formula negation p or q from where we obtained our inspiration for the family of fuzzy implications.

Writing it in terms of its set theoretic equivalence we have seen that A implies B can also be represented as A complement union B. Staying in the set theoretic framework we have seen that this could also be equivalently written as the largest subset C of X such that A intersection C is contained in B. Now moving back to the language of lattices inputting it there we see that p implies q can be written in this equivalent form.

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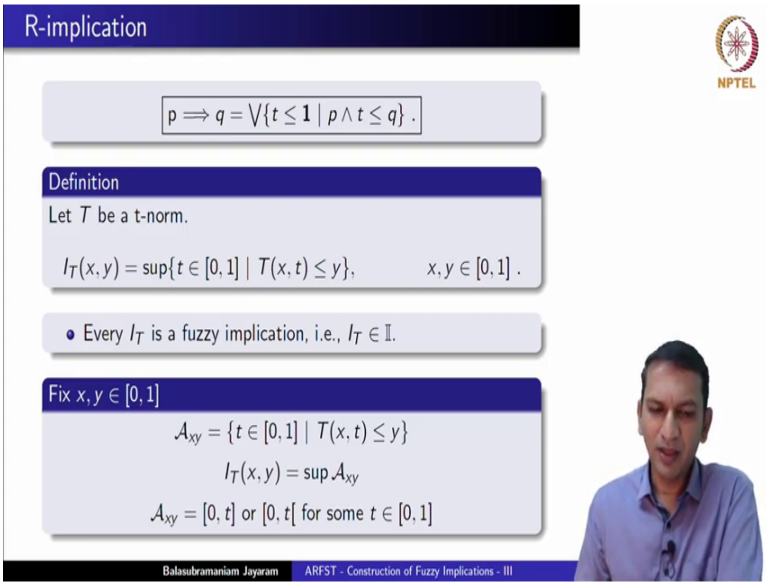

We took our inspiration for the family of R-implications from here and as we saw it is the meek that is playing an important role here the conjunction which at first cut has been generalized to that of a t norm. Hence given a t norm we could define this function which we denote as I_T, I_T(x, y) is equivalent to the supremum of all those t such that T of x t is less than or equal to y.

Now, it can be shown that every such function is in fact, a fuzzy implication. In the last lecture we have seen that, if you fix an x and y from 0 1 then the implication itself can be decoded into the following simpler form. You can look at the set A_{xy}, xy is for this fixed pair x, y from [0, 1] A_{xy} can if you denote by A x y the set of all t such that T of x, t is less than or equal to y clearly I_T(x, y) is actually equal to the supremum of the set.

Using the properties especially the monotonicity of a t norm we have seen that the set A_{xy} for a fixed x, y can either be the closed interval [0, t] or the [0, t) interval for some t belonging to [0, 1].

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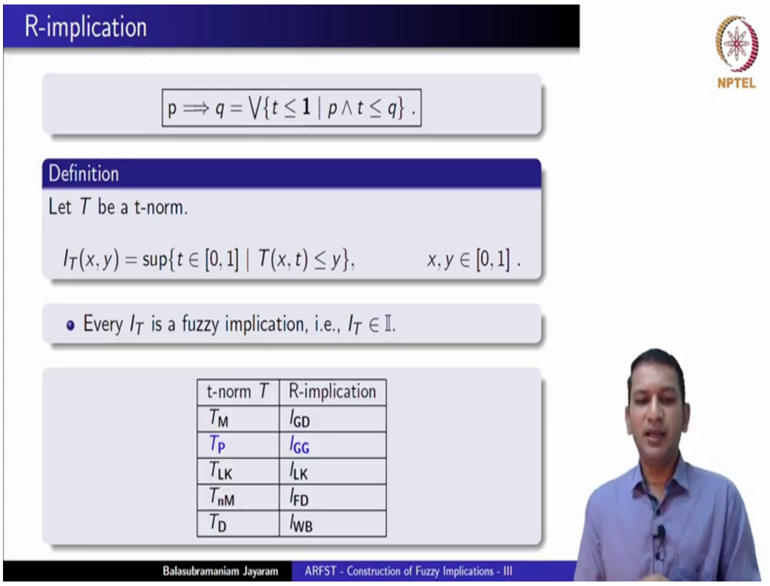

We have also seen that or at least we have noted that these basic implications fuzzy implications of Godel, Goguen, Lukasiewicz Fodor and Weber they actually belong to this family of R-implications. In the last lecture we have seen how to obtain the girdle as an R implication from the minimum t-norm. In this lecture let us take up 1 more such t norm maybe that of the product t-norm and we will go ahead to show that the Goguen implication can be seen as an R-implication obtained from the product t-norm.

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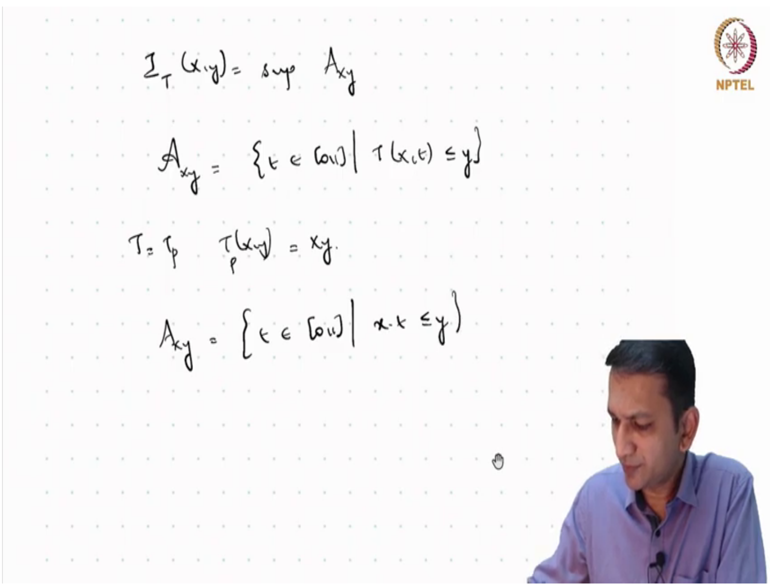

Once again let us recall this is written can be written as A x y note that a x y is set of all t in [0, 1] such that T(x, t) is less than or equal to y. Since our in our case T is T_P that is T(x, y) is equal to T_P(x, y) is equal to x*y. Now for any arbitrary x, y if you want where to write A_{xy} is a set of all t element of [0, 1] such that x into t is less than or equal to y.

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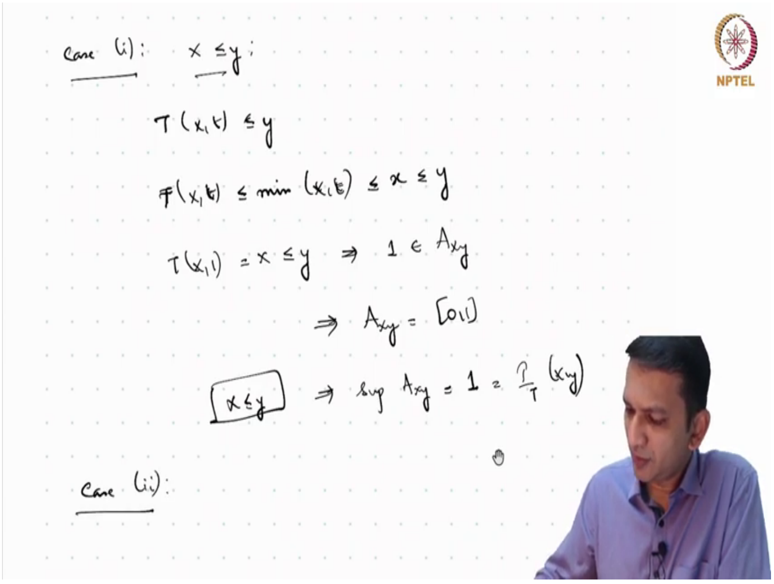

Now, once again given any x y we can consider two cases case 1 x is less than or equal to y. We have seen already that in this case for the Godel implication I of id of xy is actually equal to 1. In fact, this is true for any t norm t why is it so? Let us consider an arbitrary T and what we want is T(x, t) to be less than or equal to 1. Now we know that for any T, T is always less than or equal to minimum because minimum is the largest t-norm.

So, now; that means, definitely this is less than or equal to x now this is less than or equal to y. No matter what the T is. In fact, since x is less than or equal to y T of x 1 is equal to x is less than or equal to y. So, this implies 1 itself belongs to A x y this implies a x y is. In fact, 0 1 implies supremum of A x y is equal to 1 plus I_T of x y for any t norm T. Of course please note we are operating under the assumption x is less than or equal to y.

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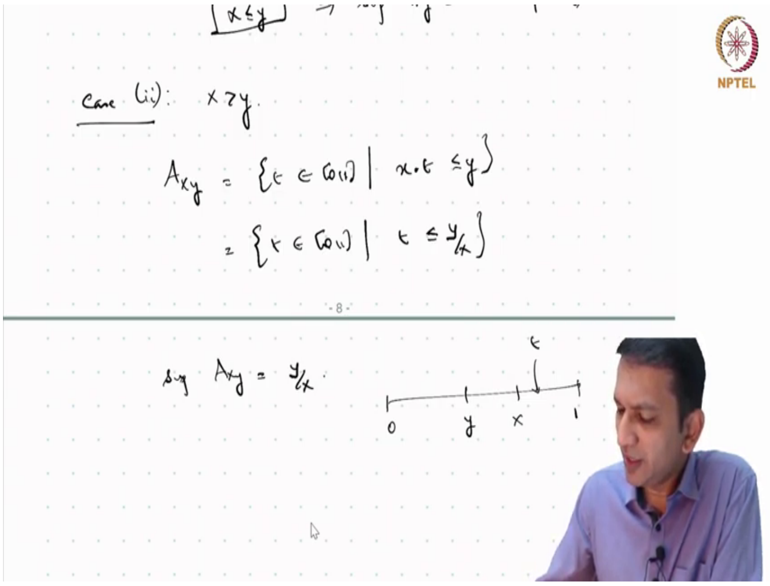

Now, consider the case x greater than y. So, in this case let us look at A x y the set of all t element of [0, 1] and for the product t norm it is x t less than or equal to y. Note that x is greater than y and now we want the maximum such t supremum of all those t such that x t is less than or equal to y. This could also be written equivalently as t less than or equal to y by x. Note that x is greater than y which means y by x the value it can take is strictly less than 1 and from here it is clear that supremum of a x y is actually equivalent to y by x.

If you are going to look at it like this we have 0 1 here and y here and x here if you are asking the question what is that t that we can go up to such that t is actually that value maximum of all the t s such that T(x t), T(x, t) is less than or equal to y. So, clearly for the product - norm the maximum t that you can get is y by x.

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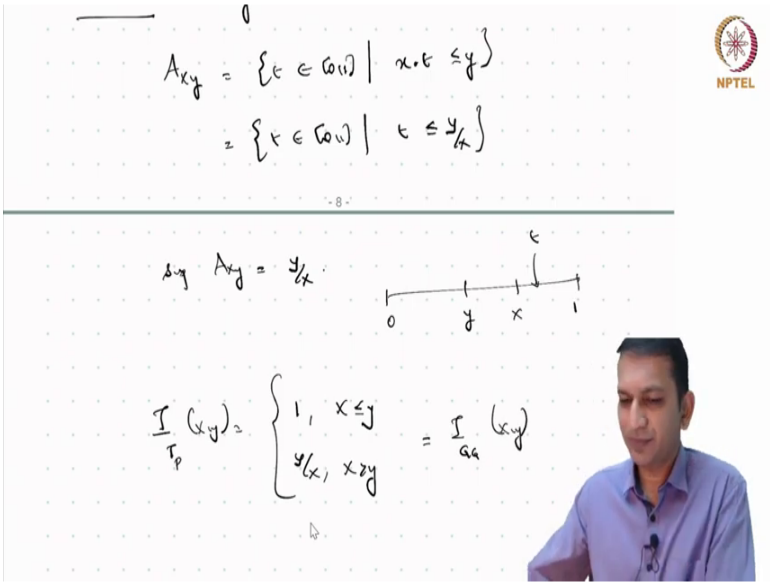

So, if you were to put these two together for the product t norm what we have is this 1 if x is less than equal to y and y by x if x is greater than y. Now, this is nothing but the Goguen implication as we know.

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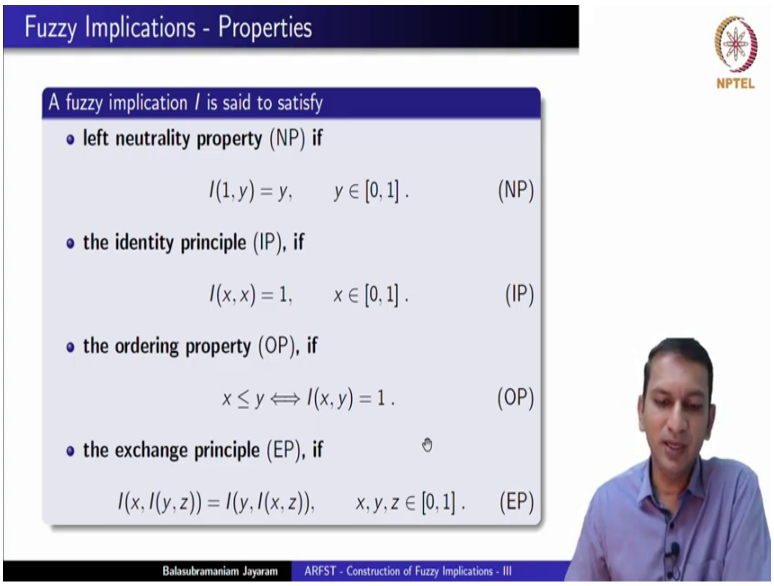

Well let us look at discussing the desirable properties of the family of R-implications. Once again let us recall that we have these four properties the left neutrality property or just the neutrality property which talks about if x is equal to 1 then the implication should take the value that is in the second argument; that means, I of 1, y should be equal to y. Geometrically we know that it is restricting the right boundary to that of the identity function.

The identity principle talks about what happens to the graph of the implication fuzzy implication when you walk along the diagonal. The ordering property talks about when it can be 1 it can be 1 because of the monotonicity of fuzzy implication if it has IP. Then whenever x is less than strictly less than y also it is one, but ordering property precludes it from having the value 1 when x is greater than y. And finally, the exchange principle as you all now know it allows you to exchange the arguments x and y in this equation.

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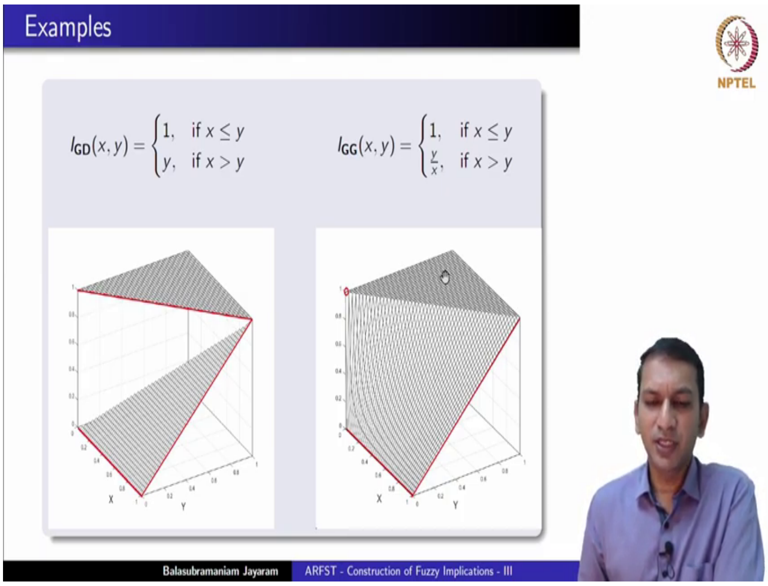

Let us look at the graphs of some R implications and also discuss the desirable properties if you look at the Godel implication this is what we have obtained it is the R implication obtained from the minimum t norm we see that it has the neutrality property it has IP. In fact, it has OP ordering principle and it can also be shown that it does have the exchange principle if you look at the natural negation it is the smallest fuzzy negation at 0 it is 1 and at every other point it is 0.

So, is the case with Goguen implication which as we have just seen is the R implications what obtained from the product t norm its natural negation is the smallest fuzzy negation it has MP it has IP and also OP it can also be shown that it satisfies the exchange principle.

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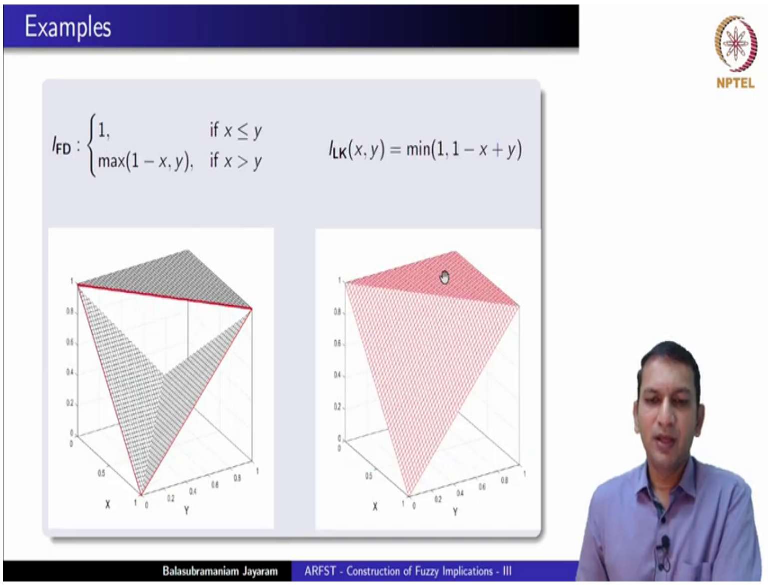

These are again our basic fuzzy implications the Fodors and Lukasiewicz in fuzzy implications. Once again both of them satisfy OP and hence IP also they satisfying neutrality property and interestingly their natural negation as you can see is actually 1 minus x. Please recall that both the Fodor and Lukasiewicz fuzzy implications are also SM implications. However, neither the Godel nor the Goguen implication is an SM implication.

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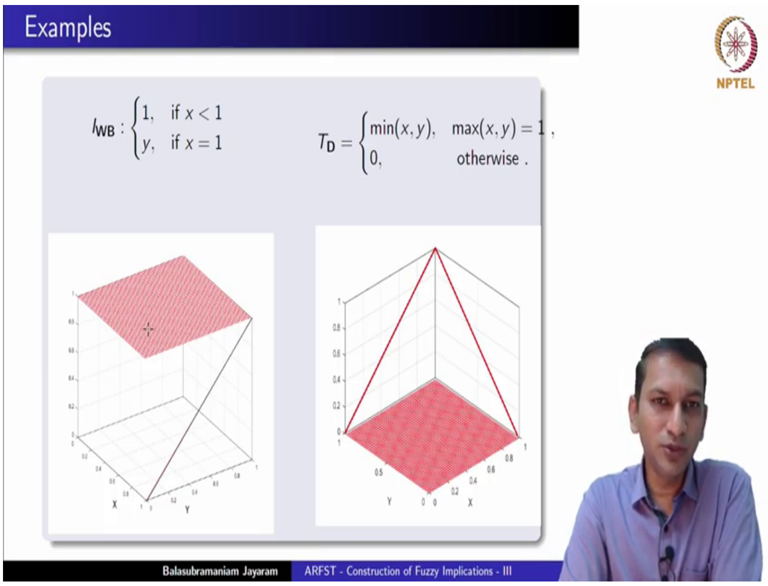

This is the Weber implication you will see that it has a neutrality property it is natural negation of the largest fuzzy negation. It has IP, but not OP it does not have the ordering property because even when x is greater than y it has the value 1. Now if you are wondering to which t norm or from which t norm can we obtain this as an R implication it is the drastic t norm.

If you take the drastic t norm and use the formula I T the corresponding R implication that you would obtain will be the Weber implication. Well let us and geometrically we have seen when a fuzzy implication has an IP or OP especially R implication.

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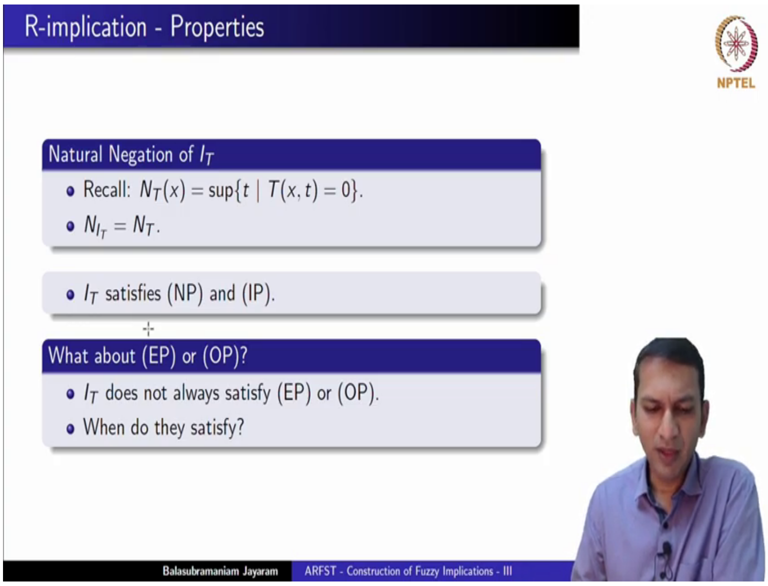

Now, let us look at it a little more in detail. What is the natural negation of an R implication? Recall given a t norm this is how we define the natural negation of a t norm.

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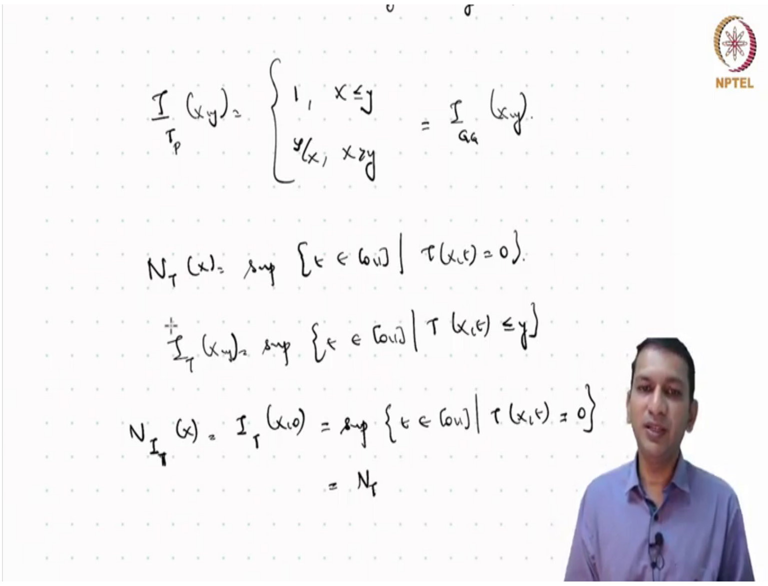

The natural negation of a T-norm is defined as all those t element of [0, 1] such that T of x, t is equal to 0. Now, $I_T(x,y)$ is defined as supremum of all those t element of [0, 1] such that T of x, t is less than or equal to y. And we know that the natural negation of an implication is nothing but I of x, 0.

So, if it is I_T then it is I_T of x, 0 and what we get here is supremum of all those t element of 0 1 such that t of x, T is less than or equal to 0 we know that T is a mapping from 0 1 square to 0 1. So, the smallest value I can take is. In fact, 0 which means this can only be equal to 0 which essentially says. This is nothing but the natural negation obtained from the corresponding T.

Now, what about the other properties? It can be easily shown that it satisfies both the neutrality property and the identity principle why so?

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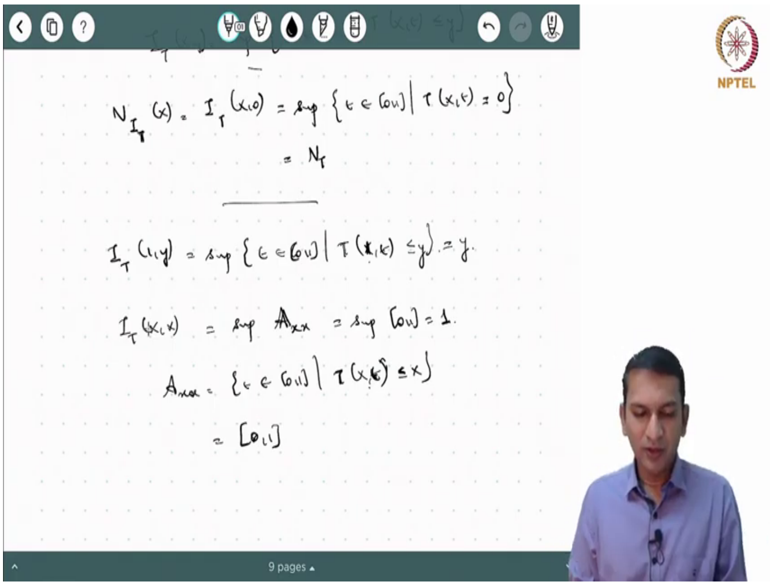

Let us look at I_T of 1, y. Now t is any t norm and this is the definition of an R implication. So, you will see it is nothing, but supremum of t element of [0 1] such that T of x, 1, t is less than or equal to y we know that 1 is the neutral element of a t norm; that means, if you want this to be less than or equal to y the maximum value the supremum that T can attain is in fact, t or in fact, y which means this is equal to y.

Now, what happens to the case I t of x, x? Now this is nothing but looked at another way supremum of A xx. Now A xx is equal to set of all those t element of [0 1] such that T of xx x, t is less than or equal to x. Once again making use of the fact that t the 1 is the neutral element of a t norm we have seen earlier too that the maximum value that t can go up to s 1. So, this is actually equal to [0 1].

So, talking about supremum it is supremum of 0 1 and it is equal to 1. From these two it is very clear that any R implication satisfies both neutrality property and the identity principle. The question is what about exchange principle of ordering property? Well not all R implications do satisfy this.

We have seen only the basic fuzzy implications examples from basic fuzzy implications of course, all of them satisfy EP, but as you have seen Weber does not satisfy OP. Now the question is when do they satisfy this? In the case of exchange principle the jury is still out there people still work on it is not really nailed down the properties on a t norm such that I T the R implication obtained from T does have the exchange principle we know that those obtained from left continuous t norms definitely satisfy the exchange principle. But that is not necessarily the case ok.

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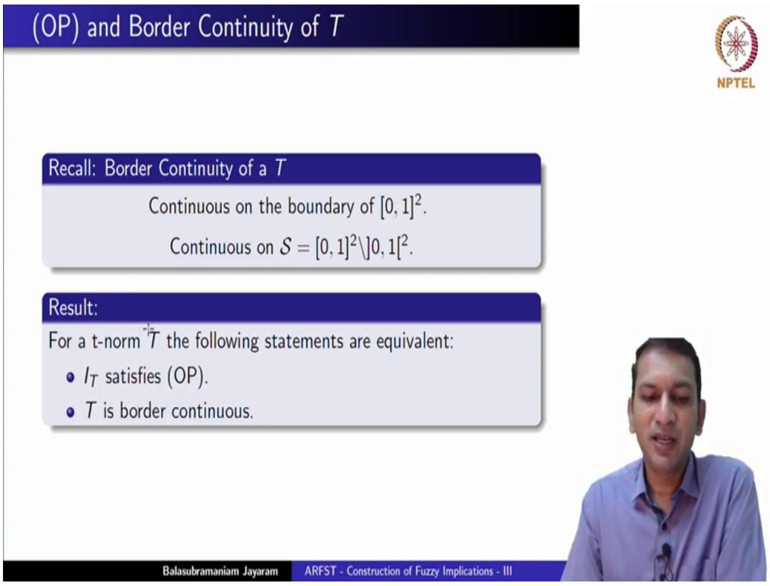

With respect to the ordering property let us recall what we defined as the border continuity of a T norm t we said a t norm is border continuous if it is continuous on the boundary of the 0 1 square on the unit square. Essentially it means it should be continuous on this set. It can be proven that an R implication satisfies ordering property if and only if the t norm from which it is generated is border continuous. Now without going into the proof of this research which is quite involved.

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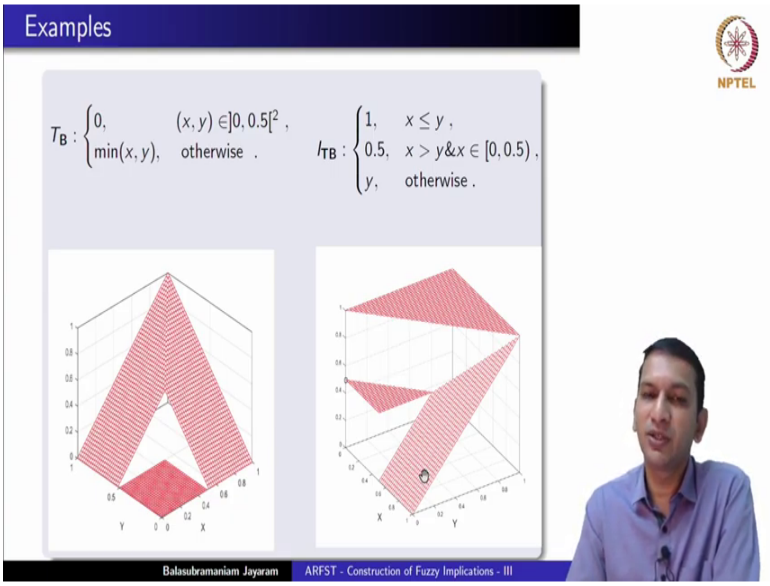

Let us look at some examples you might recall that this t norm is actually border continuous of course, clearly it is not continuous there is a huge gap here it is border continuous, but not continuous. Before looking at the corresponding R implication let us only look at what its natural negation is. So, for a t norm the natural negation is the boundary of its 0 region. So, you see that at 0 it is 1 between 0 and 0.5 it is actually 0.5 and from 0.5 to 1 it is 0.

So, if you were to draw this is how it would look like this is the natural negation of this t norm. Now if you look at its corresponding R implication we see that clearly it has OP and what is interesting is the natural negation of that is at 0 it is 1 between 0 and 0.5 it is 0.5 and from 0.5 to 1 it is actually 0; that means, the natural negation of the R implication natural negation obtained from the R implication coincides with the natural negation obtained from the t norm.

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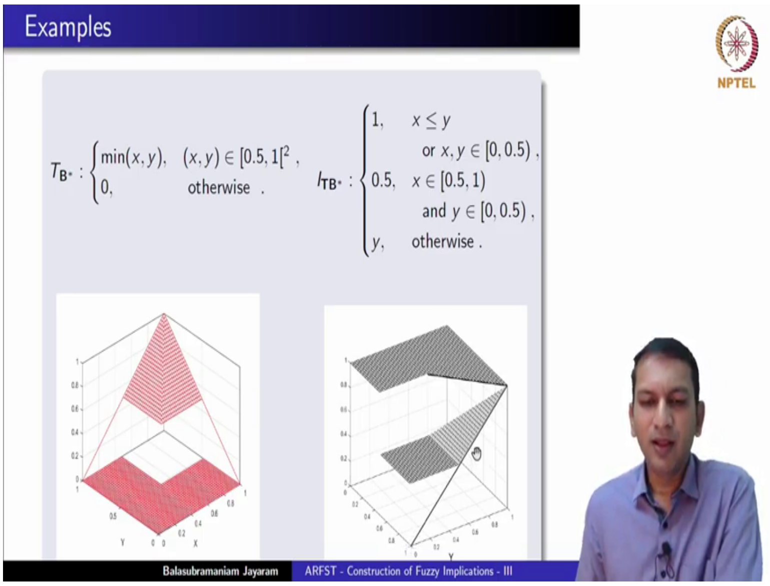

Now, this is a t norm which is not even border continuous as you can see here. However, notice the natural negation of this t norm is between 0 and 0.5 it is 1 and after that it is 0.5 at 1 it is 0. If you look at the corresponding R implication this is how it looks like since it is not border continuous we do not expect it to be having the ordering property which is also true as you can see it takes the value 1 here even though it is greater than y; however, what is interesting is the natural negation of this R implication it is 1 till 0.5 and after that it is 0.5 and at 1 it is 0.

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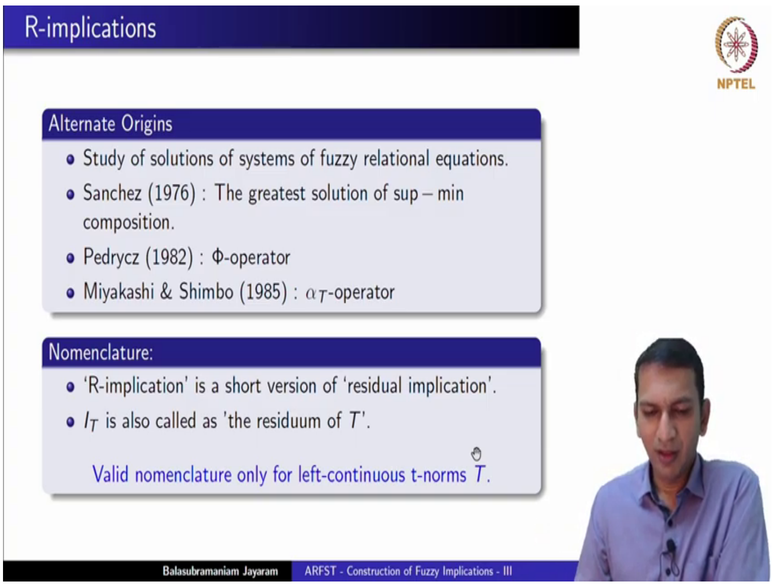

Well, as was mentioned R implications have had their origins at different source points in the study of solutions of systems of fuzzy relational equations R implications did emerge. Fuzzy relations and fuzzy relational equations will be taken up in the lectures given in the next week.

It is sufficient to know that R implications or have also had their origins the study of such systems. In fact, a Sanchez way back in 1976 showed that it is essentially the greatest solution of the sup min composition that is where an R implication will raise its hood. It had appeared as the phi operator in the work of Pedrycz later on in 1982 and also as the alpha t operator in the work of Miyakashi and Shimbo in 1985.

So, you can see that over a period of a decade from 76 to 85 different researchers have found that R implications emerge in different contexts. Finally, a word about the nomenclature the term R implication itself is a shorter version of residual implication, but we should be very careful in using this I T is also called as the residuum of T.

But this nomenclature is valid only if the t norm is a left continuous t norm the reason for this will be explained going forward because only if the t norm is a left continuous t norm can we ensure that the t norm T and the R implication it they satisfy what is called the residual residuation principle or residuated property from which we get this term residuum or residual implication whose shortened version is actually the R implication.

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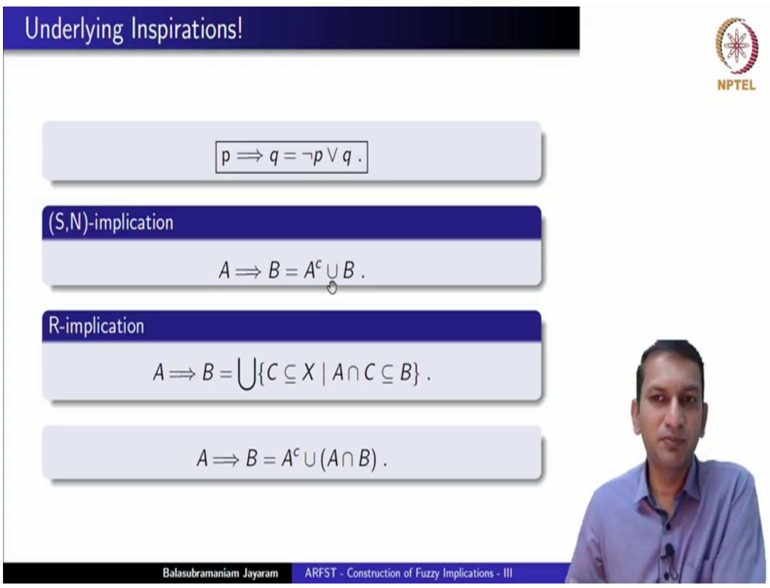

Well, let us get back to what we started with. We know that p implies q can be represented as negation p or q in the classical setting using this we move into the set theoretic framework and say that ok it could be also looked at as A complement union B later on we further gave another equivalent definition in terms of finding the largest subset C of x whose intersection with A is contained in B these 2 gave rise to the families of S N implications and R implications looking at this formula can also be equivalently written like this as A complement union A intersection B.

Now, this form the inspiration for yet another class of implications at least in the fuzzy logic framework it is called the QL implication.

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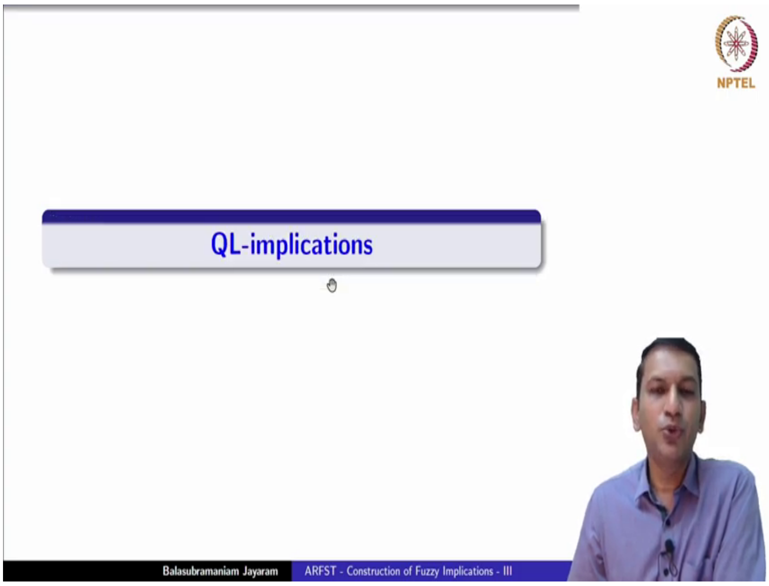

Of course these are generalizations coming from implications which are used in quantum logic.

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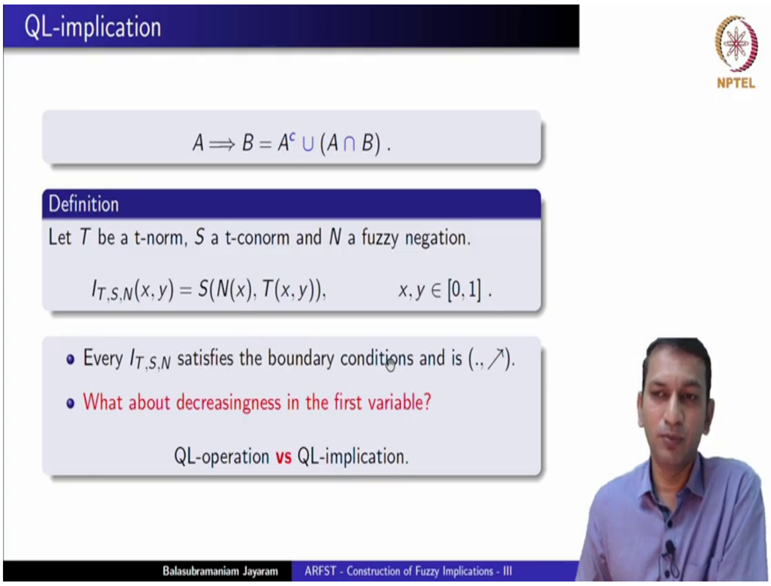

So, now you see here there are three different operations involved a complement, union and the intersection which are typically generalized in the form of negation t-conorm and at t norm respectively. So, we start with these three operations A t norm the t-conorm and a fuzzy negation and define a function I like this S of N x, T of x y and since this function is based on these three operators TS and n we often denote it as I T S N.

While every I T S N satisfies the boundary conditions of fuzzy implication and is also increasing in the second variable the decreasingness of this function. So, defined in the first variable is not guaranteed. For instance that it is increasing in the second variable can be seen quite immediately if you fix an x N x is fixed x is fixed. So, as y increases T of x, y increases S of N x, T xy will also increase. So, clearly it is increasing in the second variable that this function satisfies the boundary conditions of a fuzzy implication can be quickly verified, but if you ask about the decreasingness of this in the first variable well that cannot be ensured.

So, this leads us to discussing this function in two different ways as a QL operation alone versus a QL implication. What is the difference? QL operation does not satisfy decreasingness in the first variable if you obtain an implication using this formula we will call it a QL implication if you only obtain an operation which is increasing in the second variable and satisfies the boundary conditions or fuzzy implication, but not the monotonicity then we say it is only a QL operation.

Essentially this always ensures QL operation, but what we are interested is in a QL implication.

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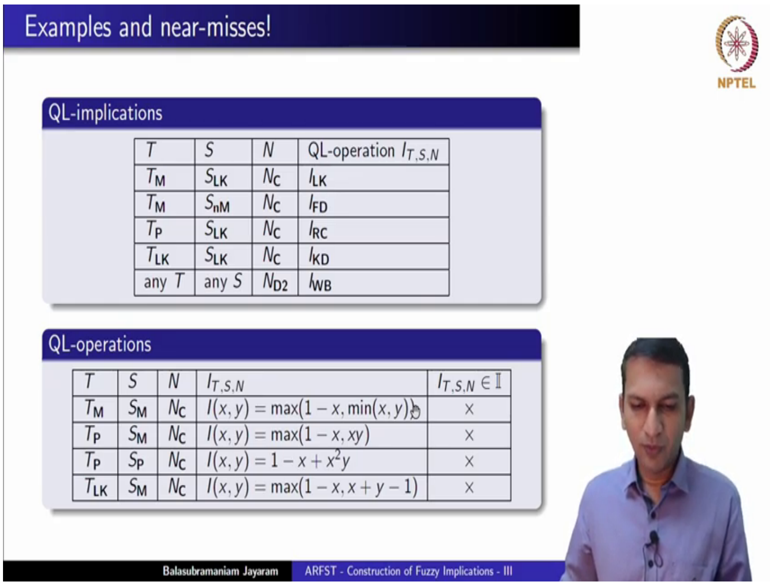

Once again quite interestingly you find that five among the nine basic implication that we have seen are. In fact, QL operations some of them are also S N implication. In fact, all these five are indeed S N implications. Let us look at some near misses; that means, those that give only QL operations and are not QL implications.

So, if you take this triple minimum maximum and the usual classical negation the function you obtain is this max of 1 minus x, min x y. Unfortunately this does not turn out to be fuzzy implication.

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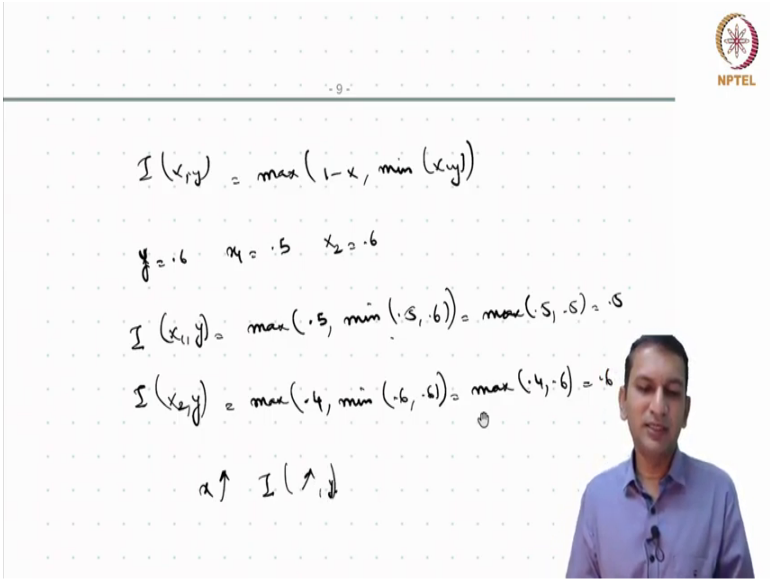

Why so? Let us look at this formula is equal to max of 1 minus x, minimum x, y. So, let us fix a y let us fix y to be 0.6 and take two values for x, x 1 is equal to 0.5 and x 2 is equal to 0.6. Now, if you look at x 1, y, this is max of 1 minus x 1 this I am going to write it directly 1 minus 0.5 is 0.5, minimum of 0.5, 0.6 is equal to max of 0.5, minimum of this is 0.5 max of these two is 0.5. If you look at I of x 2, y. This is max of 1 minus x which is 0.4, minimum of x is 0.6. Now x 2 0.6, 0.6. So, this is max of 0.4, 0.6 which is 0.6.

As you can see as x increases for this particular function I also increases in the first variable for a fixed y clearly; that means, this function is not an implication not a fuzzy implication and hence we do not get a QL implication from this triple of functions.

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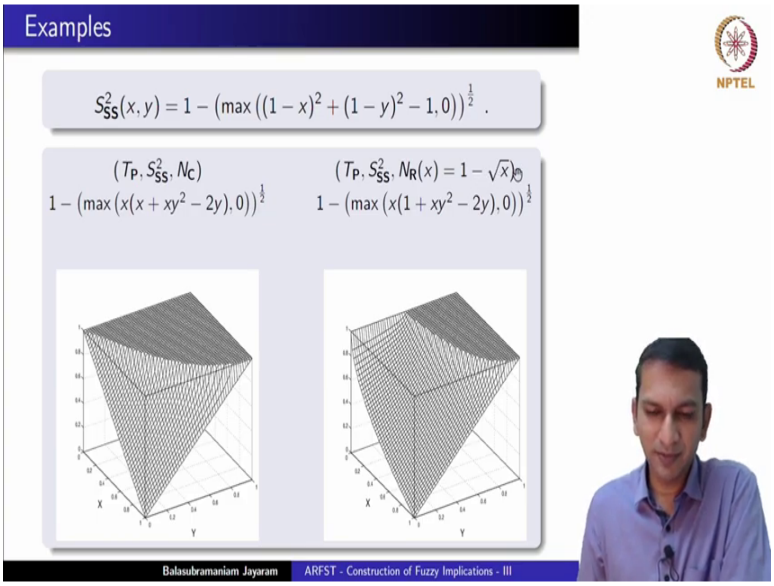

Let us look at some examples where we get QL implications, it is also very illustrative for the kind of implications that this family can throw up and that we will see from its geometry; that means, by looking at the 3D plots. So, if you take this t-conorm and the product t norm under the classical negation 1 minus x it does lead us to a fuzzy implication. So, this triple does indeed give a QL implication the graph of it looks like this. As you can see it has the neutrality property its natural negation is 1 minus x it appears it has identity property.

So, if you put y is equal to 0 here then what you get is 1 minus x square 1 minus x square and under root x square is under root. So, you will get 1 minus x which is what is the natural negation of this fuzzy implication.

If you take the same t-conorm and the product T norm, but change the negation to 1 minus root x this is the function you get which is again a fuzzy implication and the natural negation indeed will give 1 minus root x itself.

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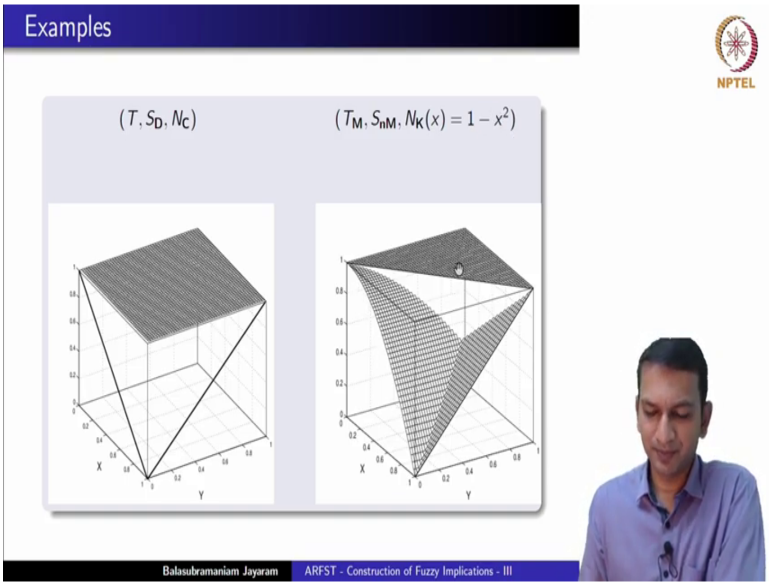

If you consider any t norm and the drastic t-conorm with the largest t-conorm and the classical negation this is the implication we would get as you can see 1 minus x is the natural negation it has a neutrality property it is almost a Weber implication except that the natural negation is redrawn to make it 1 minus x.

If you consider the minimum t norm the nilpotent minimum t-conorm and the negation 1 minus x square we get a QL implication that looks like this. Once again neutrality property is given 1 minus x square is a natural negation and what is interesting is it also has OP.

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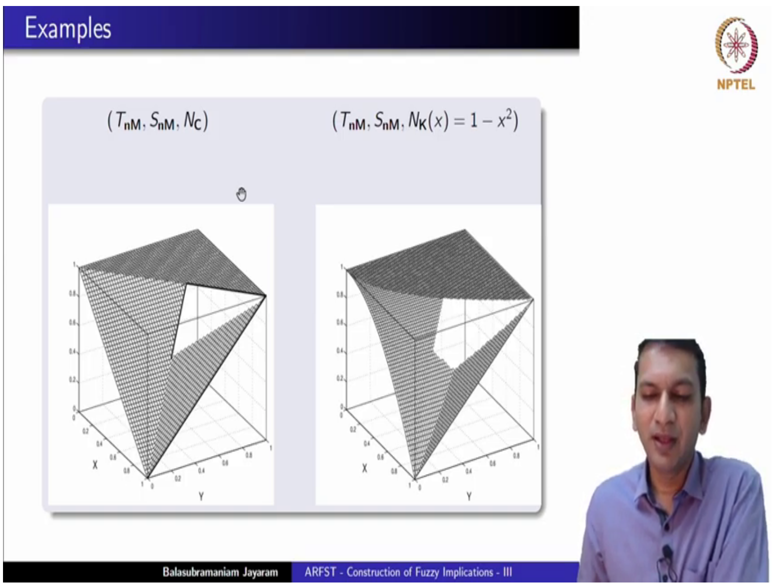

If you take this triple nilpotent minimum t norm nilpotent minimum max nilpotent maximum t-conorm and then classical negation you get such a fuzzy implication once again it has ordering property. And similarly is the case with this of course, it does not have the ordering property it has IP, NP and the natural negation is 1 minus x square.

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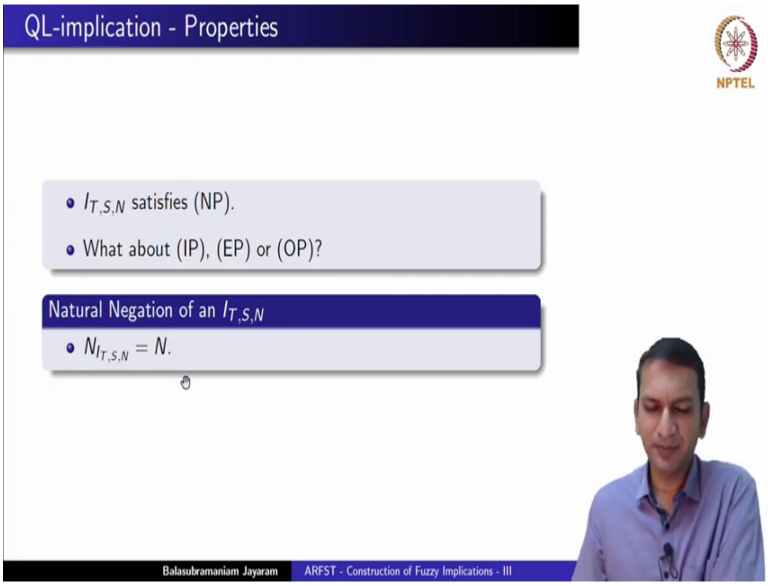

So, if you were to ask in general what are the properties it satisfies? It is clear that I_(T,S,N) will always satisfy NP what about the other properties identity exchange and ordering principle once again these are very tough questions there are some sufficient conditions and necessary conditions such results do exist, but a complete characterization of QL implication satisfying these properties is as yet unknown. And it is also clear that the natural negation that you obtain from a QL operation is in fact, the negation that you use to obtain the QL operation in the first place.

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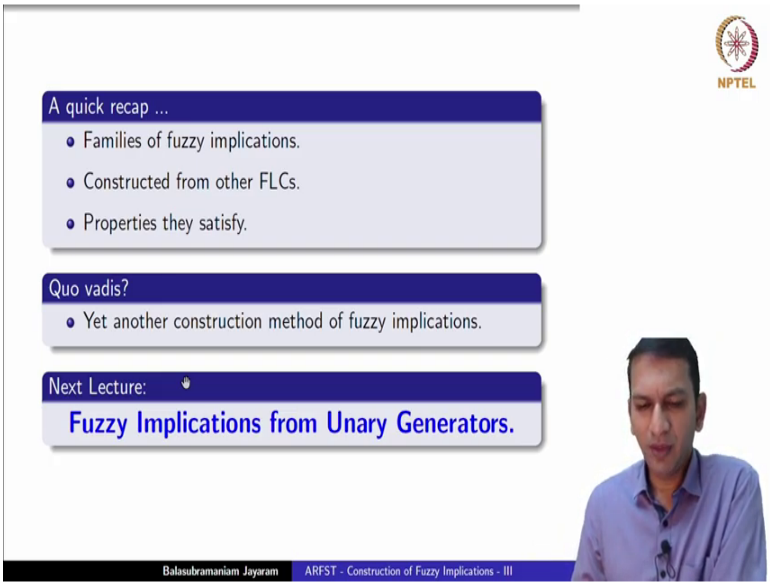

Well, with this we will come to the end of discussing constructions of fuzzy implications from other fuzzy logic connectives. At least in this course these are the three families that we are going to see in depth as being constructed from other fuzzy logic connectives.

We have seen three such families that of (S,N), R and QL implications and we have discussed the properties that they satisfy in the next lecture we will look at yet another construction method of fuzzy implications that of obtaining them from unary generators.

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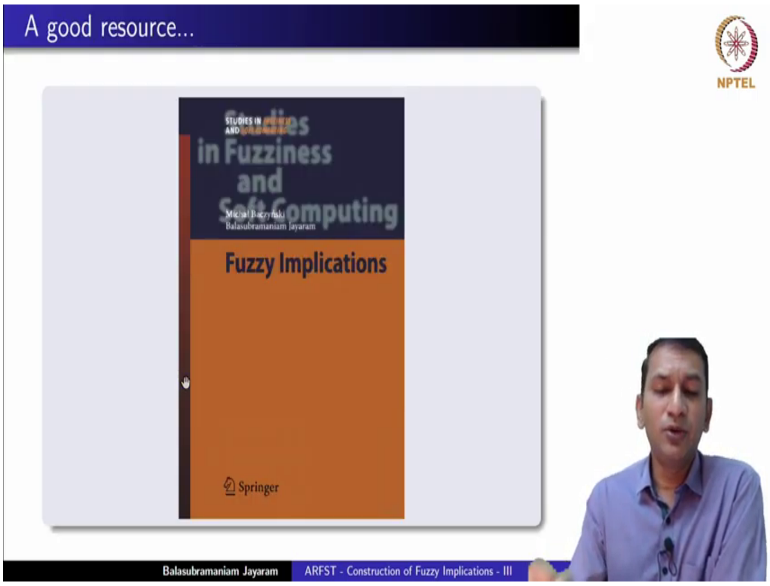

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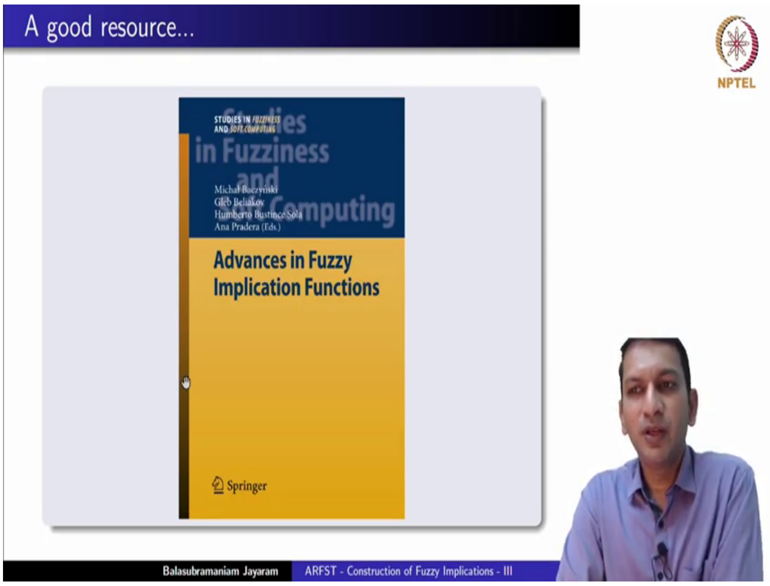

Once again the good resource for the topics covered in this lecture is that of the book on Fuzzy Implications. You could also refer to a more recent collection of research articles titled under Advances and Fuzzy Implication Functions which also discusses some of the interesting problems related to R implications and QL implications some open problems and problems that have been solved and problems that are yet to be solved.

Glad that you could join us for this lecture. I am looking forward to seeing you again in the next lecture.

Thank you again.